A Control-theoretic View on Incentives*

YU-CHI HO,† PETER B. LUH‡ and GEERT JAN OLSDER§

A unifying framework exists for several results about Stackelberg games including incentive problems with special strategy spaces and a relationship with social choice theory.

Key Words—Social and behavioral sciences; team theory; stochastic control; multivariable control systems; game theory; cybernetics; hierarchical systems; incentives.

Abstract—The idea of declaring a reward (punishment) for a decision maker according to his particular choice of action in order to induce certain 'desired' behavior on the part of the decision maker is known as incentive (threat). This practice is age old. However, only in recent years have the notions been formalized. In the development of a control-theoretic view on incentives, we first investigate the deterministic version of the incentive problem. This reveals the basic simple idea behind the problem. It also illustrates the different possibilities introduced by the presence of dynamics and multi-follower nature of the problem. This is followed by two variants of the stochastic version of the problem where we concentrate on the role of uncertainties. Relationship to economic literature is also discussed.

1. INTRODUCTION

The idea of declaring a reward (punishment) for a decision maker according to his particular choice of action in order to induce certain 'desired' behavior on the part of the decision maker is known as incentive (threat). This practice is age old. However, only in recent years have the notions been formalized. Before going into details, let us pursue a bit further the above intuitive motivations.

The following are the minimal ingredients for the incentive problem.

(1) There are at least two decision makers; the leader who declares the incentives, the follower who chooses his acts based on the declared incentives.

(2) The leader's decision is denoted as $u_0$; the follower's, $u_1$; and the incentive, $\gamma$: $u_0 \to u_1$.

(3) From the leader's viewpoint, there is a desired act, $u_1^d$ for the follower. It is possible but not necessary that $u_1 = u_1^d$ minimizes some $L_0(u_1)$ which is the payoff for the leader.

(4) The follower is an optimizer who is interested in minimizing some $L_1(u_0, u_1)$ knowing $\gamma$. Note that dependence of $L_1$ on $u_0$ is necessary for the incentive problem to be meaningful.

In (1)–(4) we have been deliberately vague concerning the specifications of the domains and ranges for $L_0, L_1, u_0$ and $u_1$; the information $z_0, z_1$ available to the leader and the follower respectively when they choose their decisions; and the presence of uncertainties and dynamics.

Roughly speaking, the literature on the subject can be divided into two groups. From the control theory side, emphasis has been almost exclusively on deterministic problems involving dynamics, see for instance Başar and Selbuz (1979a); Tolwinski (1981); Papavassilopoulos and Cruz (1979, 1980). Some stochastic problems have been treated in Başar (1980b). In other words, $u_0$ and $u_1$ in the above problem represent controls applied to dynamical systems in time. Subject to causality constraints, a variety of dependences of $u_0$ on $u_1$ can be studied. More will be said in Section 2. On the mathematical economic front, a great deal of literature exists concerning uncertainties in incentive problems (Groves and Loeb, 1979; Dasgupta, Hammond and Maskin, 1979). Briefly, the follower is supposed to possess information $z_1$ about the state of nature, $\xi$ which is unknown but useful to the leader. The leader must induce the follower to reveal $z_1$ via $u_1$ by proper choice of the incentives $u_0 = \gamma_0(u_1) = \gamma_0(\gamma_1(z_1))$ where $\gamma_1$ is the strategy chosen by the follower. Allocation of public good is a prime example of such problems.1

In Section 2 we shall first study the deterministic version of the incentive problem. This will reveal the basic simple idea behind the...
problem. It will also illustrate the different possibilities introduced by the presence of dynamics and multi-follower nature of the problem. This will be followed by two variants of the stochastic version of the problem in Sections 3 and 4 where we shall concentrate on the role of uncertainties. Relationship to economic literature will be discussed in Section 5.

2. DETERMINISTIC INCENTIVE PROBLEMS

2.1. The basic approach

Let \( u_0 \) and \( u_1 \) take values in \( U_0 \subseteq R^n \) and \( U_1 \subseteq R^n \), respectively; \( L_0: R^n \rightarrow R; L_1: R^n \rightarrow R \). We define the desired choice for the leader as

\[
(u_0^d, u_1^d) = \arg \min_{u_0, u_1} L_0(u_0, u_1). \tag{1}
\]

The incentive problem can then be simply stated as:

(P-1). Find \( \gamma_0: U_1 \rightarrow U_0 \), \( \gamma_0 \in \Gamma_0 \) such that

\[
\arg \min_{u_1} L_1[\gamma_0(u_1), u_1] = u_1^d \tag{2a}
\]

\[
\gamma_0(u_1^d) = u_0^d \tag{2b}
\]

where \( \Gamma_0 \) is the class of admissible incentives.

Note that (2a) and (2b) require choosing a set of \( n_0 \) functions \( \gamma_0: U_1 \rightarrow U_0 \) to satisfy \( n_0 + n_1 \) equations. If the set of \( n_0 \) functions has \( n_0 + n_1 \), or more undetermined parameters then we might in general accomplish this by choosing the parameters appropriately. However, before proceeding on this task, let us dispose of one naive approach, i.e. using an ‘infinite threat’. Take the simple example of \( L_0 = u_0^2 + u_1^2 \), \( L_1 = (u_0 - 1)^2 + (u_1 - 1)^2 \) where \( u_0 \) and \( u_1 \) are scalars. By inspection, \( u_1^d = u_0^d = 0 \). Consider \( u_0 = ku_1 \) with \( k \) approaching infinity as a possible incentive mechanism. The idea is that any choice of \( u_1 \neq 0 \) will make \( L_1 \) approach infinity and thus force \( u_1 \) to approach \( u_1^d \) in his own interest. However by substituting \( u_0 = ku_1 \) into \( L_1 \), it is easily shown that \( u_1 = (k + 1)/(k^2 + 1) \) and \( u_0 = ku_1 = (k^2 + k)/(k^2 + 1) \). \( u_0 \) approaches \( 0 \) (\( = u_1^d \)), \( u_0 \) approaches \( 1 \) (\( \neq u_0^d \)) as \( k \) approaches infinity, and (2b) is violated. Consequently ‘infinite threat’ as described above is not generally feasible. More elaborate examples can be constructed to show that infinite threat will not always work. Furthermore, such a threat may not be credible in practice. Similarly incentives such as

\[
u_0 = u_0^d \text{ for } u_1 = u_1^d
\]

\[
u_0 = \text{infinite for } u_1 \neq u_1^d
\]

can be ruled out if we suitably restrict the class of admissible incentives, e.g. \( \Gamma_0 \) must contain only continuous maps.

Returning to the problem of choosing \( \gamma_0 \) to satisfy (2), let us consider an incentive \( \gamma_0 \) of the form

\[
u_0 = u_0^d + g(u_1, u_1^d)
\]

where \( g(u_1^d, u_1^d) = 0 \). In particular, let \( g = k(u_1 - u_1^d) \). Formula (3) automatically satisfies (2b), thus we only need to choose \( g \) or \( k \) to satisfy (2a). For this example equation (2a) reduces to \( (k + 1)/(k^2 + 1) = 0 \) or, equivalently, \( k = -1 \). With this choice of \( k \), we note that \( L_1(ku_1, u_1) = 2u_1^2 + 2 = L_0(u_0^d, u_1) + 2 \). In other words, by this choice of incentive, we have simply made the objectives of the follower and leader essentially the same thus fulfilling the old adage

“If you wish other people to behave in your interest, then make them see things your way”.

(4)

This self-evident truth is the heart of the incentive mechanism. We shall come back to it many times. Following (2a) we shall in fact generalize (2a) as

\[
\arg \min_{u_1} L_1[\gamma_0(u_1), u_1] = \arg \min_{u_1} L_0(u_0^d, u_1). \tag{2a’}
\]

Note that (2a’) does not necessarily require

\( L_1[\gamma_0(u_1), u_1] \) and \( L_0(u_0^d, u_1) \) to be identical within linear transformation as in the above example. It should also be clear from the above discussion that there is considerable (almost unlimited) freedom in the choice of \( \gamma_0 \) that will satisfy (2a’). This non-uniqueness can also be visualized graphically. Let us return to the example mentioned above. In Fig. 1(a) the \( L_1 \) contours have been drawn. The desired point \( (u_0^d, u_1^d) \) is the origin. By announcing \( u_0 = \gamma_0(u_1) = -u_1 \), the leader ensures that the solution \( (u_0, u_1) \) will lie on the line \( u_0 = -u_1 \) in the \( (u_0, u_1) \) space independent of the action of the follower. Being rational, the follower will choose the point on this line which minimizes his cost function; such a choice is \( u_1 = 0 \). We shall say that the problem is incentive controllable (i.e.) since \( (u_0^d, u_1^d) \) can be realized.

Note that the line \( u_0 = -u_1 \) is not the only
curve which realizes \((u^d_0, u^d_1)\); any curve \(u_0 = \gamma_0(u_t)\) with \(0 = \gamma_0(0)\) and with its graph outside the \(L_1\) contour through \((0,0)\), will do the same. Since in the above problem an affine \(u_0 = \gamma_0(u_t)\) exists for which \((u^d_0, u^d_1)\) can be achieved, we call it linearly incentive controllable (l.i.c.). The problem depicted in Fig. 1(b) is i.c., but not l.i.c.; that is the curve \(u_0 = \gamma_0(u_t)\) through \((u^d_0, u^d_1)\), and with its graph outside the \(L_1\) contour through this point, can not be linear. In Fig. 1(c) a problem has been depicted that it is not even continuously i.c. (a problem is continuously i.c. if a continuous curve \(u_0 = \gamma_0(u_t)\) exists such that \((u^d_0, u^d_1)\) can be achieved). In Fig. 1(d), the problem is continuously i.c. but not l.i.c. The problem is e.l.i.c. in the sense that the leader can not force the joint decision to \((u^d_0, u^d_1)\) by an affine \(\gamma_0\), however, he can get arbitrarily close to it by an affine \(\gamma_0\). An example of this will be presented in Section 2.2. In Section 5, a problem will be encountered which is not i.c. at all according to its definition given above; it is, however, mixed i.c. (by announcing a 'mixed' strategy \(\gamma_0\) \((u^d_0, u^d_1)\) can be achieved).

2.2. Extension to multi-stage case

Consider a \(T\)-stage problem with decision variables \(u_{00}, u_{01}, \ldots, u_{0{T-1}}; u_{10}, u_{11}, \ldots, u_{1{T-1}}\). The first index indicates decision maker and the second index refers to time. The idea is that at time \(t\), the leader can choose his decision \(u_{0t}\) based on past decisions of the follower, thus incentives can be computed to all \(u_{1t}\), except the last \(u_{1{T-1}}\). For the same reason \(u_{00}\) cannot be used to provide incentives. Thus without loss of generality we shall assume \(u_{00}\) and \(u_{1{T-1}}\) to be constants or fixed strategies determined by considerations outside the incentive problem. Now let \(\gamma^d_0 = (\gamma^d_0, \ldots, \gamma^d_{T-1})^T\) and \(\gamma^d_1 = (\gamma^d_1, \ldots, \gamma^d_{T-2})^T\) be the desired sequences of strategies which are obtained as

\[
(\gamma^d_0, \gamma^d_1) = \arg \min_{\gamma_0, \gamma_1} L_0(\gamma_0, \gamma_1).
\]

Let \(u^d_{0t}, u^d_{1t}\) be the corresponding desired sequences of decisions; and \(L_0^d\) be the associated desired cost.* Because of the requirement of causality, the general form of incentives is

\[
u_{0t} = \gamma_0(u_{1t})
\]

\[
u_{02} = \gamma_0(u_{10}, u_{11})
\]

\[
\vdots
\]

\[
u_{0{T-1}} = \gamma_0(u_{10}, u_{11}, \ldots, u_{1{T-2}}).
\]

A simple special case of (6) is

\[
u_{0t} = \gamma_0(u_{1t-1}) \quad \text{for } t \text{ from } 1 \text{ to } T-1
\]

which is simply a concatenation of single-stage problems. Another simple example would be to require

\[
u_{0t} = \text{constant} \quad \forall i \neq T-1
\]

\[
u_{0{T-1}} = \gamma_0(u_{10}, u_{11}, \ldots, u_{1{T-2}})
\]

which has the interpretation that all rewards (punishments) are given at the end based on the entire decision history of the follower.

A more usual situation is to use the concept of 'state' which summarizes past decision histories of the system. Let \(x_t\) be the state at time \(t\),

\[
x_t = f_{t-1}(x_{t-1}, u_{0t-1}, u_{1t-1})
\]

then \(x_t\) is a function of initial state \(x_0\) and all past decisions. We shall examine two variations of (6).

Variation 1. Suppose we have \(\gamma^d\) and \(\gamma^d_1\) as the desired sequences of strategies which generate the desired sequences of decisions \(u^d\) and \(u^d_1\). In the spirit of (3), let

\[
u_{0t} = \gamma^d_{0t}(x_t) + k_t \cdot (x_t - \bar{x}_t)
\]

for \(t\) from 1 to \(T-1\)

where \(\bar{x}_t = f_{t-1}(x_{t-1}, \gamma^d_{0t-1}(x_{t-1}), \gamma^d_{1t-1}(x_{t-1}))\) (i.e. the state at time \(t\) if both the decision makers
used the desired strategies at \( t - 1 \).\(^*\) Note that \( x_i \) is a function of \( u_{ij} \), \( j < t \) for fixed \( \gamma_i^d \), thus formula (9) is a special case of formula (6). If we can find \( k_j \) for all \( j \) such that the follower’s optimal strategies, taking into account formula (9), is \( \gamma_i^{d,0}, \ldots, \gamma_i^{d, t-1} \), then \( u_{0t} = u_{0t}^d \) and \( u_{1t} = u_{1t}^d \) for all \( t \). The desired sequences of decisions can thus be realized.

In Başar and Selbuz (1979a), the authors derived a set of sufficient conditions for the existence of \( k_i \), which are assumed to be constants, for linear system with quadratic criteria. The advantage of formula (9) is that since \( k_i \) is a constant, \( u_{0t} \) is affine in its information. The disadvantage is that if \( u_{1t} \) is not \( \gamma_i^{d, t} \) (\( x_i \)) at any \( t \), then in general we have \( x_j \) which is not \( \gamma_i^{d, t} \) for \( j > t \). That is, the follower will be punished forever once a deviation, however unintentional, is observed, even if he returns to \( \gamma_i^{d, t} \) afterwards [also see Tolwinski (1981)].

**Example 1.** Consider the two-person three-stage dynamic game described by the state equations

\[
\begin{align*}
    x_1 &= x_0 + u_{0,0} + u_{1,0} \\
    x_2 &= x_1 + u_{0,1} + u_{1,1} \\
    x_3 &= x_2 + u_{0,2}
\end{align*}
\]

and the quadratic cost functions

\[
\begin{align*}
    L_0 &= x_0^2 + 2u_{0,0} + 2u_{0,1} + 2u_{0,2} + u_{1,0}^2 + u_{1,1}^2 \\
    L_1 &= x_1^2 + u_{0,2} + 2u_{0,3} + bu_{1,1}^2 + 2u_{1,0}^2
\end{align*}
\]

where \( b \) is some constant, \( b > 0 \) and \( b \neq 1 \). The desired solution for the leader can easily be obtained

\[
\begin{align*}
    \gamma_{0,0}^d &= -\frac{1}{2} x_0, \\
    \gamma_{0,1}^d &= -\frac{1}{2} x_1, \\
    \gamma_{0,2}^d &= -\frac{1}{2} x_2, \\
    \gamma_{1,0}^d &= -\frac{1}{2} x_0, \\
    \gamma_{1,1}^d &= -\frac{1}{2} x_1.
\end{align*}
\]

We now define \( \tilde{x}_i \) as the state of the system at stage \( i \) when \( \gamma_0 \) and \( \gamma_1 \) follow the desired closed-loop strategies above; \( \tilde{x}_0 = \frac{1}{2} x_0, \tilde{x}_1 = \frac{1}{2} x_1, \tilde{x}_2 = \frac{1}{2} x_2 \). The desired sequence of decisions can be realized by the following incentives

\[
\begin{align*}
    u_{0,1} &= -\frac{1}{2} x_1 + k_1 (x_1 - \tilde{x}_1) \\
    u_{0,2} &= -\frac{1}{2} x_2 + k_2 (x_2 - \tilde{x}_2)
\end{align*}
\]

where \( k_1 = (14 - b) / 5 (b - 1), k_2 = b - \frac{1}{2} \). Note that

if \( u_{1,0} = -\frac{1}{2} x_0 + e (e \neq 0) \), then \( u_{0,1} \) and \( u_{0,2} \) will not be equal to \( -\frac{1}{2} x_1 \) and \( -\frac{1}{2} x_2 \), respectively, even if \( u_{1,0} = -\frac{1}{2} x_1 \).

Now let us consider the case when \( b = 1 \), for which the above incentive mechanism is not well defined. Some rather straightforward, but extensive manipulations show that for \( k_j \) to be \( \gamma_i^{d, t} \), the leader approaches his desired cost \( L_0^d \). We have a situation as depicted in Fig. 1(d); i.e., the problem is only e-i.c. In Fig. 1(d) the problem is continuously i.c., and it turns out to be the case here too. If the leader chooses

\[
\begin{align*}
    \gamma_{0,1} &= -\frac{1}{2} x_1 + c |x_0| \sqrt{|x_1 - \tilde{x}_1|} \quad \text{sgn} (x_1 - \tilde{x}_1) \\
    \gamma_{0,2} &= -\frac{1}{2} x_2 - \frac{1}{2} (x_2 - \tilde{x}_2)
\end{align*}
\]

where \( c \) is a constant such that \( c^2 > 652 / 675 \), he will obtain his desired cost \( L_0^d \).

**Variation 2.** Tolwinski (1981) used the following incentive mechanism:

\[
\begin{align*}
    u_{0,1} &= \gamma_i^{d, t} (x_i) + g_i (x_i - \tilde{x}_i) \quad \text{for} \ t \text{ from} \ 1 \text{ to} \ T - 1
\end{align*}
\]

where \( g_i \) is a nonlinear function with \( g_i (0) = 0 \) and \( \tilde{x}_i = \text{f}(x_{i-1}, u_{0,i-1}, \gamma_i^{d, i-1}, (x_{i-1})) \). Note the difference in the definition of \( \tilde{x}_i \) between variations 1 and 2. In this case, so long as \( u_{1,0} = \gamma_i^{d, i-1} (x_{i-1}) \) then \( \tilde{x}_i = x_i \) regardless whether or not \( u_{0,i-1} = \gamma_i^{d, i-1} (x_{i-1}) \). Thus if the follower acted improperly for whatever reason at \( t - 2 \) but resumes the correct decision at \( t - 1 \) then the leader will only react (punish) at \( t - 1 \) for one step. Thereafter beginning at \( t \), the system is ready to resume the desired sequence of decisions again from whatever the resultant state \( x_n \) and the strategy \( (\gamma_0, \gamma_1) \) is still a solution for the problem considered on the interval \([t, T]\). In fact, it is easy to devise variations of mechanism (10) where the punishment for deviation may last over one, two, three, ..., \( T - 1 \) stages.

The above discussion hopefully makes clear that an enormous range of possibilities for incentive exists as special cases of (6). Mechanisms (9) and (10) are the most obvious cases. We should explore others and demand additional properties such as noise immunity, linearity, complexity, uniqueness, etc. to be satisfied with any incentive mechanism. The surface has only been scratched. In fact, it is generally agreed that deterministic formulation of the closed-loop Stackelberg problem is impractical. The slightest bit of noise will destroy the solution. The main reason for expounding this case is to clarify the underlying concept involved which will then be extended to the more realistic stochastic cases.
2.3. Extension to many-follower case

When there are two or more followers in the problem, the relationship among the followers must be specified. We shall illustrate a few of them, assuming coalitions among followers are not allowed. First, let \( \gamma_i \) be the desired strategies for decision maker \( i \) for \( i \) from 0 to \( m \). An incentive mechanism \( \gamma_0 \) is said to induce a dominant strategy solution if

\[
\arg \min \ L_i = \gamma_i^d \quad \text{with arbitrary } \gamma_i^d,
\]

\[
\forall \ j, j \neq i; \quad i = 1, \ldots, m.
\]  \hfill (11)

For example with \( u_0 = (u_{01}, u_{02}) \), let \( L_0 = (u_1 + u_2)^2, L_1 = (u_1 - 1)^2 + u_{01}, L_2 = (u_2 - 1)^2 + u_{01} \), then the incentive mechanism \( u_{01} = 2u_1 \) and \( u_{02} = 2u_2 \) will induce \( u_i = 0 \) regardless of \( u_1 \), and similarly for \( u_2 \). Dominant strategy solution is the most desirable result since it effectively decouples the followers from each other. However, such solution is very difficult to realize since in general the leader is not that powerful. In the Nash equilibrium solution concept, we only require

\[
\arg \min \ L_i = \gamma_i^d \quad \text{with } \gamma_i = \gamma_i^d,
\]

\[
\forall \ j, j \neq i; \quad i = 1, \ldots, m
\]  \hfill (12)

i.e. each agent will behave desirably conditioned on the fact that others will do so. In economics literature, a distinction is made between definition (12) and

\[
\arg \min \ L_i = \gamma_i^d \quad \text{with } u_i = u_i^d
\]

\[
\forall \ j, j \neq i; \quad i = 1, \ldots, m
\]  \hfill (12')

where the latter is called the Nash solution while definition (12) is called the Bayes solution. This difference becomes significant in the stochastic case. Nash equilibrium [equation (12)] has been studied in Başar and Selbuz (1979a) under dynamic system setup, where the leader's strategy is again of the form (9).

A particular case occurs if the incentive mechanism \( \gamma_0 \) can be chosen such that the cost function of all the followers become identical. Then the followers face a team problem, which has only one 'reasonable' solution. As an example, consider the following example.

**Example 2**

\[
L_0 = u_{01}^2 + u_{02}^2 + u_1^2 + u_2^2
\]

\[
L_1 = u_{01} - 3u_{02} + (u_1 - 1)^2 + (u_2 - 1)^2
\]

\[
L_2 = u_{01} + u_{02} + (u_1 + 1)^2 + (u_2 + 1)^2
\]

The minimum of \( L_0 \) is 0 which occurs when all the decisions are zero. If the leader announces

\[
\begin{align*}
    u_{01} &= \frac{1}{3}(u_1 - 1)^2 + \frac{1}{2}(u_2 - 1)^2 + \frac{1}{2}(u_1 + 1)^2 \\
    + \frac{1}{2}(u_2 + 1)^2 - 2
    \\
    u_{02} &= \frac{1}{4}(u_1 - 1)^2 + \frac{1}{2}(u_2 - 1)^2 - \frac{1}{4}(u_1 + 1)^2 \\
    - \frac{1}{4}(u_2 + 1)^2
\end{align*}
\]

then the followers face the problem

\[
\min \ u_i \left( (u_1 - 1)^2 + (u_2 - 1)^2 + (u_1 + 1)^2 + (u_2 + 1)^2 - 2 \right) \]

\[
\min \ u_i \left( (u_1 - 1)^2 + (u_2 - 1)^2 + (u_1 + 1)^2 + (u_2 + 1)^2 - 2 \right)
\]

which is a team problem, of which the solution is \( u_1 = u_2 = 0 \). Substituting these values into \( u_{01} \) and \( u_{02} \) yields \( u_{01} = u_{02} = 0 \). In this example with two followers, another incentive mechanism exists which makes the followers face a zero-sum game, which also has only one 'reasonable' solution. If the leader announces

\[
\begin{align*}
    u_{01} &= -\frac{1}{4}(u_1 - 1)^2 + \frac{1}{2}(u_2 - 1)^2 - \frac{1}{2}(u_1 + 1)^2 \\
    - \frac{1}{2}(u_2 + 1)^2 + 2
    \\
    u_{02} &= -\frac{1}{4}(u_1 - 1)^2 + \frac{1}{2}(u_2 - 1)^2 - \frac{1}{2}(u_1 + 1)^2 \\
    + \frac{1}{2}(u_2 + 1)^2
\end{align*}
\]

then the followers face the problem

\[
\min \ u_i \left( -(u_1 - 1)^2 + (u_2 - 1)^2 + (u_1 + 1)^2 + (u_2 + 1)^2 + 2 \right) \]

\[
\min \ u_i \left( -(u_1 - 1)^2 + (u_2 - 1)^2 + (u_1 + 1)^2 + (u_2 + 1)^2 + 2 \right)
\]

for which the saddle-point solution is \( u_1 = u_2 = 0 \). Substituting these values into \( u_{01} \) and \( u_{02} \) leads to \( u_{01} = u_{02} = 0 \).

In contrast to the Nash equilibrium concept, there may exist among the followers certain additional levels of hierarchy. For example, DM1 may announce his strategy before the rest of the followers, knowing the leader's strategy. DM1 can thus also implement a kind of incentive mechanism of his own on the rest of the followers. A problem of this sort is said to have multi-levels of hierarchy, and has been studied in Başar (1980b) and Tolwinski (1980). Here we shall assume there are three decision makers. DM0 announces his strategy first as a function of the decisions of DM1 and DM2, i.e. \( u_1 \) and \( u_2 \), respectively. Then DM1 announces his strategy
as a function of $u_i$. A set of sufficient conditions can be stated such that DM1 and DM2 are induced to help DM0 to minimize $L_0$; also DM2 is induced to help DM1 in minimizing $L_1$. For any given $\gamma_0$ and $\gamma_1$, let $\bar{L}_1(u_0, u_1, u_2)$ be derived from $L_1(u_0, u_1, u_2)$ with $u_0$ being replaced by $\gamma_0(u_1, u_2)$; and $\bar{L}_2(u_2; \gamma_1, \gamma_0)$ be derived from $L_2(u_0, u_1, u_2)$ with $u_0$ being substituted by $\gamma_0(u_1, u_2)$ and $u_1$ by $\gamma_1(u_2)$. Define

$$
(u_0^0, u_1^0, u_2^0) = \arg \min_{u_0, u_1, u_2} L_0(u_0, u_1, u_2)
$$

$$
[u_1^4(\gamma_0), u_2^4(\gamma_0)] = \arg \min_{u_1, u_2} \bar{L}_1(u_1, u_2; \gamma_0)
$$

$$
u_t^4(\gamma_1; \gamma_0) = \arg \min_{u_2} \bar{L}_2(u_2; \gamma_1, \gamma_0).
$$

Then a set of sufficient conditions for $(\gamma_0, \gamma_1^*)$ to achieve the team optimum is

$$
u_t^4(\gamma_1^*; \gamma_0^*) = u_t^0
$$

$$
u_t^4(\gamma_0^*) = u_t^0
$$

and

$$
\gamma_0^*(u_t^0, u_2^0) = u_0^0.
$$

The desired decision $(u_0^0, u_1^0, u_2^0)$ can thus be realized. In Tolwinski (1980) some extra conditions are imposed so that DM1’s equilibrium strategy $\gamma_1^*$ is optimal for him for any choice of $u_t \in U_t$.

In a three-person game, the relationship among the decision makers can be illustrated graphically. Figure 2(a) shows the one-leader, two-follower model, with the leader being at the top of the structure. The incentive mechanism $\gamma_0$ may induce either a dominant strategy solution or a Nash equilibrium solution between the followers. Figure 2(b) shows the case with multi-levels of hierarchy. The conventional Nash solution concept (with no leader, and all the decision makers in symmetric positions) can be depicted by Fig. 2(c). A problem can also be formulated where there are two leaders playing Nash and one follower being dictated by them, as shown by Fig. 2(d).

When coalitions among followers are allowed, the problem becomes more difficult to handle. Even if a dominant strategy exists, the followers may deviate from the desired strategies by forming coalitions. The well-known prisoners’ dilemma is an excellent example. We shall not pursue it further here.

### 3. STOCHASTIC INCENTIVE PROBLEMS I (NESTED CASE)

A natural generalization of the problem in Section 2 is to introduce uncertainties represented by the state of nature, $\xi$, where $\xi \in \Xi$ and $\Xi \subseteq \mathbb{R}^n$. Let the distribution of $\xi$, $p(\xi)$, be known to the decision makers. We introduce

$$
J_0 = E[L_0(u_0, u_1, \xi)]
$$

$$
J_1 = E[L_1(u_0, u_1, \xi)]
$$

as the criteria for the leader and the follower, respectively. The follower is given certain information (measurement, observation) $z_t \in Z_t \subseteq \mathbb{R}^m$, where

$$
z_t = \eta_t(\xi).
$$

The leader in addition to $u_t$ also has available information $z_0 \in Z_0 \subseteq \mathbb{R}^m$. Thus his information structure $\eta_t$ consists of $u_t$ and $z_0 = \eta_0(\xi)$. For this section we shall assume

$$
(A1).
$$

i.e. the follower’s information is nested in that of the leader.

Given definition (13), the desired solution is now given not by $(u_0^*, u_1^*)$ but by strategies

$$
[\gamma_0^d(z_0, u_1), \gamma_1^d(z_1)] = \arg \min_{\gamma_0, \gamma_1} E[L_0(u_0 = \gamma_0(z_0, u_1), u_1 = \gamma_1(z_1), \xi)].
$$

Equation (16) denotes a decentralized statistical decision or team problem. Because of (A1), it is reasonable to suppose that $\gamma_0^d$, $\gamma_1^d$ can be determined (Ho and Chu, 1972). The stochastic version of (P-1) can be stated as

$$
(P-2). \text{ Find } \gamma_0: \mathbb{R}^m \cdot U^1 \rightarrow U^0 \text{ with } \gamma_0 \in \Gamma_0
$$

such that
arg min \(E/\eta[L_1(\gamma_0(z_0, \eta), \gamma_1, \xi)] = \gamma_1^d(z_1)\) (17a)  
\[ \gamma_0(z_0, \gamma^d_1(z_1)) = \gamma_0^d(z_0, \gamma^d_1(z_1)). \] (17b)

Note that we require (17a) and (17b) to be satisfied for all \(z_0\) and \(z_1\), i.e. they are identities. Since \(\gamma_0(z_0, u_i)\) is a function of two variables, it is not unreasonable to expect that it can be chosen to satisfy the two identities (17a) and (17b). In the case when \(z_0\) and \(z_1\) take on discrete values, identities (17a) and (17b) are equivalent to a system of (2a) and (2b) each indexed by a particular pair of \((z_0, z_1)\) values. The analogs of formulas (3) and (2a) are

\[ u_0 = \gamma_0^d(z_0, u_i) + g[z_0, u_i, \gamma^d_1(z_1)] \] (18)

where \(g(z_0, u_i, u_i^d) = 0\), and

arg min \(E[L_1(\gamma_0(z_0, \eta), \gamma_1, \xi)] \)
\[ = \arg \min E[L_0(\gamma_0^d(z_0, \eta), \gamma_1, \xi)]. \] (17a')

If we convert (17a') to its equivalent extensive form then we have

arg min \(E[z_1[L_1(\gamma_0(z_0, u_i), u_i, \xi)] \)
\[ = \arg \min E[z_1[L_0(\gamma_0^d(z_0, u_i), u_i, \xi)]. \] (17a'')

which is equivalent to require

arg min \(h_1(z_1, u_i) = \gamma^d_1(z_1)\)
\[ = \arg \min h_0(z_1, u_i) \forall z_1 \] (19)

where \(h_i = \arg \min h_i(z_1, u_i), i = 0, 1\), is given by the obvious definition. Equation (19) says that we require the minimizing function \(u_i = \gamma_i(z_i)\) to be identical for both \(h_0\) and \(h_1\). This requirement can be given a different characterization which is useful in certain situations.

Definition.* Two functions \(h_0\) and \(h_1\) are said to be independently person-by-person monotonic (IPM) iff \(\forall z_1 \in Z_1, z_1' \in Z_1\) and \(u_i' \in U_i\), if \(h_i[\gamma^d_i(z_1), z_1] \leq h_i(u_i', z_1')\) implies \(h_i[\gamma^d_i(z_1), z_1] < h_i(u_i', z_1')\), then \(u_i' \neq \gamma^d_i(z_1)\). The function \(\gamma^d_i(z_1)\) is here defined by \(\gamma^d_i(z_1) = \arg \min h_0(z_1, u_i)\).

Suppose that for a given \(z_1\), the desired solution is \(\gamma^d_1(z_1)\). Consider now another \(u_i' \in U_i\), where \(h_i[\gamma^d_i(z_1), z_1] \leq h_i(u_i', z_1)\). What IPM says is that if instead of \(z_1\), the state of nature is \(z_1'\) and if the follower has \(h_i[\gamma^d_i(z_1), z_1] \leq h_i(u_i', z_1)\), then \(u_i'\) will not be the desired solution for the leader when the state of nature is \(z_1'\).

Theorem 1. (i) If identity (19) holds then \((h_0, h_1)\) satisfies IPM. (ii) If \((h_0, h_1)\) satisfies IPM, then

\[ \arg \min h_1(z_1, u_i) = \arg \min h_0(z_1, u_i) \forall z_1 \]
\[ U_i^d \]

where \(U_i^d\) is the range of \(\gamma^d_i(z_1)\), and \(U_i^d \subseteq U_i\). In particular, if \(U_i^d = U_i\), then identity (19) holds.

Proof: [also reference Theorem 4.3.1 of Dasgupta, Hammond and Maskin (1979)].

(i) Suppose identity (19) holds. Let \(\gamma^d_1(z_1) = \arg \min h_i(z_1, u_i) \forall z_1 \in Z_1, z_1' \in Z_1\) and \(u_i' \in U_i\), if \(h_i[\gamma^d_1(z_1), z_1] \leq h_i(u_i', z_1')\) implies \(h_i[\gamma^d_1(z_1), z_1] < h_i(u_i', z_1')\), then we must have \(u_i' \neq \gamma^d_1(z_1)\).

For otherwise from identity (19), \(\gamma^d_1(z_1) = \gamma^d_1(z_1') = u_i'\), will contradict \(h_i[\gamma^d_1(z_1), z_1'] < h_i(u_i', z_1')\).

(ii) Suppose \((h_0, h_1)\) satisfies IPM. \arg \min h_0(z_1, u_i) = \gamma^d_1(z_1)\) by definition. Let \(u_i' = \arg \min h_i(z_1, u_i)\). If identity (20) does not hold, then there exists \(z_1 \in Z_1\) such that with \(u_i = \gamma^d_1(z_1)\) and \(u_i' = \gamma^d_1(z_1)\), \(h_i(u_i, z_1) < h_i(u_i', z_1)\). On the other hand since \(u_i' \in U_i^d\), there must exist \(z_1' \in Z_1\) such that \(\gamma^d_1(z_1') = u_i'\). Thus IPM implies that \(u_i' \neq \gamma^d_1(z_1)\), a contradiction.

Remark: In certain situations it is easy to prove \((h_0, h_1)\) not satisfying IPM by finding counter examples. If \((h_0, h_1)\) does not satisfy IPM, then identity (19) will not hold. A sufficient and often convenient condition for (19) is, of course, to have \(h_0 = h_1\) or \(L_0(\gamma_0, \gamma_1, \xi) = L_1(\gamma_0, \gamma_1, \xi)\), i.e. making the payoff function of the leader and the follower identical. But this is not necessary. IPM captures the essence of the requirement that the optimum of \(h_0\) and \(h_1\) are identical.

Example 3. [Example 1 of Ho, Luh and Muralidharan (1981).] Let

\[ J_0 = E[L_0] = E[\gamma^d_0 = u_0 + u_0 u_1 - u_1^2 + \xi_1 u_0 + \xi_2 u_1] \]
\[ J_1 = E[L_1] = E[-2u_1^2 + (\xi_1 + \xi_2) u_1 + b u_0 - u_1] \]

where \(\xi_1\) and \(\xi_2\) are independent zero-mean Gaussian random variables and \(b \neq 0\). Let the information structure be

\[ \eta_0: u_1, \xi_1, \xi_2 \quad \eta_1: \xi_2. \]

The team solution is \(\gamma^d_0 = u_1 + \xi_1, \gamma^d_1 = \xi_2\). With \(\gamma_0 = (1 - b + 3\xi_2)(u_1 - 1)/b + \xi_1 + u_1\), we have

*The concept of IPM naturally extends to the case of many followers. In fact this is the case in the economics literature where IPM was first introduced.
\[ J_1 = E[-2u_t^2 + (\xi_1 + \xi_2)u_t + (1 - b) + 3\xi_2(u_t - 1) + b\xi_1 + bu_t - u_t] = E[-2u_t^2 + 4\xi_tu_t + \xi_tu_t + \text{terms not involving } u_t] \]

and

\[ E[\xi_t|L_1] = E[-2u_t^2 + 4\xi_tu_t + \text{terms not involving } u_t]. \]

Thus, \( \arg \min_{u_t} E[\xi_t|L_1] = \xi_t \). On the other hand, for \( u_t = \eta_0 = u_t + \xi_t \), we have

\[ J_0 = E[-\frac{1}{2}u_t^2 + (\xi_1 + \xi_2)u_t] \]

and

\[ E[\xi_t|L_0] = E[-\frac{1}{2}u_t^2 + \xi_tu_t + \text{terms not involving } u_t]. \]

Thus \( E[\xi_t|L_0] \) and \( E[\xi_t|L_1] \) are obviously IPM since they satisfy identity (19).

4. STOCHASTIC INCENTIVE PROBLEMS \( \xi \) (NONNESTED CASE)

The only difference between problems treated in this Section and that of Section 3 is the fact that \( \eta \not\subset \eta_0 \), i.e. the follower possesses private information not known to the leader. To simplify the discussion, let us assume that \( z_t = \xi \) and \( z_o = \phi \). However, we encounter an immediate problem when we attempt to calculate the desired solution \( \gamma_t(u_t) \) and \( \gamma_t(z_t) \) using the analog of (16). We now face a dynamic team problem. Since \( \eta \not\subset \eta_0 \), in general we do not know the optimal team solution. Nevertheless, it is possible to ask what should be the solution if the leader knows \( z_t \). Denote this 'first best' solution, which is assumed to be unique, by \( \gamma_t^t(u_t, z_t) \) and \( \gamma_t^t(z_t) \), where the superscript \( t \) represents 'team'. We define

(P-3). Find \( \gamma_t: U^1 \rightarrow U^\theta, \gamma_t \in \Gamma_\theta \) such that

\[ \arg \min_{\gamma_t} E[L_1(\gamma_t, \gamma_t, \xi_t)] = \gamma_t^t(z_t) \quad (21a) \]

\[ \gamma_t(\gamma_t(z_t), 1) = \gamma_t^t(z_t) \quad (21b) \]

But (P-3) is generally infeasible. Unlike the nested case in Section 3, \( \gamma_t \) is now only a single variable function. Definition (21b) completely specifies \( \gamma_t \) leaving no degrees of freedom to satisfy identity (21a). To illustrate this, consider the following.

Example 4. The cost functions are

\[ L_0 = u_t^2 + u_t^2 + u_t^2 + \xi_t u_t + \xi_t u_t \]

\[ L_1 = 2u_t^2 + u_t^2 + 2u_t^2 + b\xi_t u_t \]

where \( b \) is a constant known to the decision makers. The information structure is

\[ \eta_0: u_t \quad \eta_1: \xi_t \]

\( \xi_t \) is a zero-mean Gaussian random variable. The team solution, in which the leader also knows \( \xi_t \), is

\[ \gamma_t = \frac{1}{2}u_t, \quad \gamma_t = -\frac{1}{2}\xi_t. \quad (22) \]

Condition (21b) now reads

\[ \gamma_t(\gamma_t(z_t)) = \gamma_t(-\frac{1}{2}\xi_t) = \frac{1}{2}\xi_t \]

eliminating \( \xi_t \) from the above identity, this requirement becomes

\[ \gamma_t(u_t) = -\frac{1}{2}u_t. \quad (23) \]

Thus \( \gamma_t \) is completely specified by (23) and no freedom is left to satisfy identity (21a). It is easy to verify that (21a) cannot be satisfied if \( \gamma_t \) is given by (23) unless \( b = \frac{1}{2} \). Thus the problem is not (linearly) i.e. for \( b \neq \frac{1}{2} \). Note also that if this is not i.e., then it will not be i.e. (continuously or not) at all!

In order to make (P-3) feasible, additional restriction will be imposed on \( L_0 \) and/or \( L_1 \).

(A2). \( L_0 \) is independent of \( u_0 \).

Assumption (A2) not only eliminates identity (21b) completely but also makes unambiguous that \( u_t = \gamma_t(z_t) = \arg \min_{u_t} L_0(\gamma_t(z_t), \xi_t) \) is the desired solution for the leader. From economics point of view, (A2) can also be justified as will be shown later in Section 5 when we discuss the allocation of public goods. Under (A2), (P-3) then becomes

(P-3'). Find \( \gamma_t: U^1 \rightarrow U^\theta \) such that

\[ \arg \min_{\gamma_t} E[L_1(\gamma_t, \gamma_t, \xi_t)] \]

\[ = \arg \min_{\gamma_t} E[L_0(\gamma_t, \xi_t)]. \quad (24) \]

Again defining \( E[\xi_t|L_0(u_t, \xi_t)] = h_0(u_t, \xi_t) \) and \( E[\xi_t|L_1(\gamma_t(u_t), \xi_t)] = h_1(u_t, \xi_t) \), identity (24)
becomes

$$\arg \min_{u_i} h_i(u_i, z_i) = \arg \min_{u_i} h_i(u_i, z_i) \quad \forall \ z_i$$

$$\arg \min_{u_i} h_i(u_i, z_i) \quad \forall \ z_i$$

(25)

**Theorem 2.** (i) If identity (25) holds then $(h_0, h_i)$ satisfies IPM. (ii) If $(h_0, h_i)$ satisfies IPM, then

$$\arg \min_{u_i} h_i(u_i, z_i)$$

$$\arg \min_{u_i \in U_i^d} h_i(u_i, z_i) \quad \forall \ z_i$$

(26)

where $U_i^d$ is the range of $\gamma_i^d$, $U_i^d \subseteq U_i$. In particular, if $U_i^d = U_i$, then identity (25) holds.

**Proof:** exactly the same as that of Theorem 1.

In (P-3'), the follower's act $u_i$ determines $L_i$. However, suppose instead the leader controls not only $u_0$ but also $u_i$, and the sole function of the follower is to report the value of $\xi$, then the leader can achieve the desired solution provided the follower inform the true value of $\xi$. Thus instead of (P-3'), we can consider the so-called direct incentive problem

(P-3'). Find $u_0 = \gamma_0(\gamma_1(\hat{\xi}))$ and $u_i = \gamma(\hat{\xi})$ such that

$$\arg \min_{\xi} E[L_i(\gamma_0(\gamma_1(\hat{\xi})), \gamma(\hat{\xi}), \xi)] = \xi.$$  

(27)

In (P-3'), the follower's decision $\hat{\xi}$ is simply to report the value of $\xi$ which he alone knows. The leader's decision is to find $\gamma_0$ and $\gamma_1$ such that the follower reports truthfully. It is clear that if $\gamma_0$ solves (P-3') then $\gamma = \gamma_0 \cdot \gamma_1^l$ and $\gamma_1^l$ constitutes a solution of (P-3'), i.e.

**Theorem 3.**† To every (P-3') that admits a solution $\gamma_0$ there is an equivalent (P-3') that has a solution $u_0 = \gamma_0(\gamma_1^l(\hat{\xi}))$, $u_i = \gamma_1^l(\hat{\xi})$ and $\hat{\xi} = \xi$.

---

*There may be other solutions to (P-3'). But we assume that if 'truth' is one of the solution then it will be chosen, see Dasgupta, Hammond and Maskin (1979).
†Reference Theorem 4.1.1 of Dasgupta, Hammond and Maskin (1979) for the multi-follower case.
‡In previous sections we started with cost functions $L_i(u_0, u_i, \ldots, u_m)$, where the contours are curves of indifference. In general, a preference ordering over A is more basic, since there is no topological structure as there is in the case when preferences are given through cost functions.
§In terms of notations of previous sections, $L_0$ maps the followers' payoff functions (individual preference orderings) $L_0, \ldots, L_m$ into the leader's payoff function (social ordering) $L_0$. However in social choice theory $L_0$ in general is not given. Instead one tries to construct a $L_0$ having several desirable properties. The impossibility theorems in social choice theory are important to us in the sense that they state that no $L_0$ exists which satisfies certain reasonable properties.

**Example 5.** The cost functions are

$$L_0 = \frac{1}{2} u_i^2 + u_i \xi$$

$$L_i = u_i + \frac{1}{2} u_i^2 + 2bu_i \xi$$

where $b$ is a positive constant known to the decision makers. The information structure is

$$\eta_0: u_i \quad \eta_1: \xi$$

where $\xi$ is a random variable. If the leader announces $u_0 = (b - \xi)u_i^2$, then $L_i = 2bL_0$ and he will obtain his team solution. This indirect incentive problem can be converted into a direct one by defining

$$u_0 = (b - \xi)\hat{\xi}^2, \quad u_i = -\hat{\xi}$$

where $\xi$ is the follower's reported value of $\xi$. This direct incentive problem is cheap proof in the sense that the best thing the follower can do is to report the truth.

5. RELATIONSHIP TO ECONOMIC LITERATURE

5.1. Relation with social choice theory

In this subsection various concepts of the social choice theory will be introduced and their relations with previous sections will be indicated. In economics, social choice theory deals with incentives and particularly with incentives to the correct revelation of private information for public use. Though such problems were known to exist for a long time, social choice theory received its main impetus when Arrow formulated his famous impossibility theorem (Arrow, 1951).

In social choice theory it is assumed that there is a finite number $(m)$ of agents, followers in our terminology; and a set $A$ of alternatives, of which a typical member is indicated by $a$. In the context of this paper, $a = (u_0, u_1, \ldots, u_m)$. For the sake of simplicity, we shall assume

(A3). Each agent $i$ $(i = 1, \ldots, m)$ has a strict ordering $P_i$ over $A$. Where an ordering is a complete, transitive, asymmetric, binary relation. Note that many results are also valid for indifferences.

Let $\Sigma(A)$ denote the class of admissible orderings, then $P = (P_1, \ldots, P_m) \in [\Sigma(A)]^m$ is called a preference profile. Let $S(A)$ be the class of allowed social orderings on $A$. A social welfare function (SWF) is a mapping from $[\Sigma(A)]^m$ into $S(A)$; it will be indicated by $L_0$. The SWF assigns a social ordering to any allowed preference profile.

We will mention four possible properties of
SWF. An SWF satisfies the assumption of universal domain (UD) if \( \Sigma(A) \) is the class of all possible orderings on \( A \). An SWF satisfies the pareto optimality (PO) if all agents prefer \( a_1 \) to \( a_2 \), then the social ordering also prefers \( a_1 \) to \( a_2 \); formally: \( a_1 \triangleleft a_2 \forall i \) implies \( a_i \triangleleft (a_1 \triangleleft (a_2) \). An SWF satisfies the independence of irrelevant alternatives (IIA) if the ranking of two alternatives by \( \triangleleft_l(P) \) only depends on the ranking of these alternatives by the agents. That is, for arbitrary \( a_1, a_2 \) and \( P, P_i \) such that, \( a_i \triangleleft (a_2) \) if and only if \( a_i \triangleleft (P) \) if and only if \( a_i \triangleleft (P_i) \). In other words, it should not matter how many other alternatives are between the two. 'Intensities' \( (a_1 \) is a little bit preferred to \( a_2 \) or \( a_1 \) is by far preferred to \( a_2 \) can not be taken into account if IIA holds. Lastly, an SWF is called dictatorial if there exists an agent \( i \) whose \( P_i \) determines the social ordering \( \triangleleft_l(P) \).

Arrow's impossibility theorem states that any SWF which satisfies UD, PO and IIA must be dictatorial, provided that set \( A \) contains at least three elements.

From the decision making's point of view, SWF is just an intermediary step used to define an optimal social alternative from \( A \). This leads naturally to the concept of social choice function. Given the preferences of the agents, an SCR assigns an alternative to any allowed preference profile \( P \), i.e. it maps \( \Sigma(A)^n \) into \( A \). SCR corresponds to the function \( \gamma(\cdot) \) of Section 4 or \( \gamma^{d}(\cdot) \) of Section 2. For SCR there is also an impossibility theorem which will be formulated after having introduced a number of SCR properties.

An SCR \( \gamma(\cdot) \) satisfies citizen sovereignty (CS) if for every \( a \in A \) there exists a preference profile \( P \) such that \( \gamma(P) = a \). For direct incentive problems, an SCR is manipulable means that some agents can be better off by not reporting their true preferences (they lie instead). An SCR is truthfully implementable in dominant strategies (TID) if there is no preference profile at which it is manipulable. This property is sometimes called 'strongly individually incentive compatibility'. An SCR is dictatorial if there is an agent \( i \) such that for any \( P \in \Sigma(A)^n \) his reported alternative is always the social choice.

In the spirit of Theorem 3, we can see that to every dominant strategy there exists an equivalent direct strategy satisfying TID. Thus from now on we shall consider direct incentive problems only, with SCR corresponding to \( \gamma^{d}(\cdot) \). The Gibbard–Satterthwaite impossibility theorem (Gibbard, 1973; Satterthwaite, 1975) then states that if the range of an SCR has at least three alternatives, and this SCR satisfies CS, TID and UD, it is dictatorial. We introduce the leader to be the person who chooses \( \gamma^d(\cdot) \), then a possible goal of the leader is to choose \( \gamma_0 \) in such a way it satisfies the reasonable assumptions of being cheat proof (i.e. TID), nondictatorial, UD and CS. The Gibbard–Satterthwaite impossibility theorem then says that such a choice does not exist. Note that such a \( \gamma_0 \) does not exist, in spite of the fact that no \( \gamma^d(\cdot) \) was given at the outset of the problem as was the case in Section 4. Thus the result of the Gibbard–Satterthwaite impossibility theorem is even stronger. Also note that currently there are preference profiles, whereas in Section 4 the preference profiles already had a lot of structures, the only unknowns being the 'state of nature' \( \xi \) which were random variables.

In search for positive results, various suggestions can be made, such as

(a) restrictions on the domain of preferences;

(b) introduction of a mixed (random) social choice strategy;

(c) weaken the requirement of dominant strategy.

Various other possibilities for positive results have been mentioned in Green and Laffont (1979).

5.2 Positive and negative results

In this subsection we shall give two classes of problems which can be truthfully implemented in dominant strategies by restricting the domains of preferences (Examples 6 and 7) and one class for which this is not possible (Example 8). The last example in this subsection deals with a problem which is truthfully implementable with mixed \( \gamma_0 \), however, not through any pure \( \gamma_0 \) (Example 9).

Example 6. We are given \( m \) followers of whom the payoff functions (to be maximized) are

\[ L_i = u_i(u_0) + u_{0i}, \quad i = 1, \ldots, m. \]  

(28)

The leader chooses both \( u_0 \) and \( u_{0i}, i = 1, \ldots, m \), and he wants to maximize

\[ L_0 = \sum u_i(u_0). \]  

(29)

Note that the decisions \( u_{0i}, i = 1, \ldots, m \) do not enter in the leader's criterion, which is in agreement with assumption (A2) in Section 4. The decision \( u_0 \) is usually thought of as a public good, which affects all the followers; \( u_{0i} \) is thought of as a personal reward (or penalty if it is negative). The leader does not know the (util-
ity functions $v(\cdot)$. Each follower reports a function $\delta_i(\cdot)$, not necessarily the true one, to the leader. Let the leader base $u_0$ and $u_{0i}$ on the reported $\delta_i$ in the following way

$$u_0 = \arg \min_{u_0} \{ \Sigma \delta_i(u_0) \} \tag{30a}$$

$$u_{0i} = \Sigma \delta_i(u_0). \tag{30b}$$

Now it easily follows that it is in the interest of each follower to report the true $v(\cdot)$, independent of what the other followers will do (truth or not). Thus the problem can be truthfully implemented in dominant strategies.

This problem has been extensively studied by Groves (1973) and (30a), (30b) is known as the Groves mechanism.

Example 7. In this example we assume the followers' payoff functions are single peaked. These payoff functions are given by $L_i(u_0)$, $i = 1, \ldots, m$. For the sake of simplicity we assume $u_0$ to be a scalar [for extensions see Sen (1970)]. In this case, the function $L_i$ being single peaked is identical to $L_i$ being unimodal. The leader decides $u_0$ based upon the reports from the followers regarding the optimal $u_0^i$ that maximizes $L_i(u_0)$. That is, the leader not knowing $L_i(\cdot)$, $i = 1, \ldots, m$, bases his $u_0$ on the reported $u_0^i$. The leader does not have a payoff function himself; he simply wants to construct a direct incentive mechanism $u_0 = \gamma_0(u_0^i, \ldots, u_0^n)$ which has many desirable properties. It easily follows that if $u_0^i$ is chosen according to the 'median voter rule', i.e. he chooses $u_0$ such that there are as many $u_0^i$ to the left of him as there are $u_0^i$ to the right, then this SCR is TID, PO and non-dictatorial. The verification of this explanatory statement is left as an exercise for the reader.

The reason why the Gibbard–Satterthwaite impossibility theorem does not apply to Examples 6 and 7 is that the assumption UD is violated. Both functions (28) and the single peaked payoff functions are small subclasses of all preference orderings.

The following example (Dasgupta, Hammond and Maskin, 1979) resembles the previous one in certain aspects, but the result is negative: no 'reasonable' direct mechanism exists, even though the assumption on UD is also not satisfied.

Example 8. Consider an economy with $n$ goods and $m$ decision makers, all being followers. We assume $n \geq 2, m \geq 2$. There are fixed positive stocks $(w_1, \ldots, w_n)$ of these $n$ goods and the task of the leader is to devise a partitioning of the available goods among the followers. We assume that the payoff function of each follower is given by a strictly convex and strictly monotonic function

$$L_i(u_{0i}), \ i = 1, \ldots, m$$

where $u_{0i} = (u_{0i1}, \ldots, u_{0in})$ is follower $i$'s share of the goods. Each follower is interested in maximizing his own payoff function. $u_{0i1}, \ldots, u_{0in}$ will be chosen by the leader, after the followers have revealed their payoff functions $L_i$; i.e. $u_0 = (u_{0i1}, \ldots, u_{0in}) = \gamma_0(L_1(\cdot), \ldots, L_m(\cdot))$. The reported $L_i(\cdot)$ is of course not necessarily the true $L_i(\cdot)$.

Thus formulated, the leader faces a direct incentive problem. His goal is to distribute the goods in such a way that the pareto optimality condition is satisfied. (Note that the leader does not have his own payoff function. We could think of $\Sigma \lambda_i L_i$ for different combinations of nonnegative $\lambda_i$s as the leader's payoff and require the social choice to be pareto optimal.) The function $\gamma_0$ satisfies the pareto optimality if $L_i(u_0) \geq L_i(u_0^i)$ $\forall i$, with at least one strict inequality sign, then $u_0^i \neq \gamma_0(L_1, \ldots, L_m)$. Theorem 4.4.1 of Dasgupta, Hammond and Maskin (1979) now states that if $\gamma_0$ yields a pareto optimal outcome and if $\gamma_0$ can be truthfully implemented in dominant strategies, then $\gamma_0$ must be dictatorial. A dictatorial mechanism $\gamma_0$ means here that one of the followers gets all the goods. Though this is a pareto solution, it is clear that it is a very unsatisfactory one. The proof of this result can be found in Dasgupta, Hammond and Maskin (1979).

We will conclude this subsection with an example in which a mixed $\gamma_0$ is truthfully implementable. Random social choice procedures have been suggested before (Green and Laffont, 1979), but in a different context. As a possible procedure for a direct incentive scheme it was suggested in Green and Laffont (1979) to choose a follower at random and using his announced preferences as the social ordering. Example 9, which deals with an indirect incentive, is different in nature.

Example 9 (mixed strategies). Let

$$L_0 = u_0^2 + u_1^2$$

$$L_1 = (u_0 - \xi_i)^2 + (u_1 + 1)^2$$

*This lack of knowledge may sometimes be characterized alternatively by a random variable $\xi$ with $v(\xi, \xi)$. Then Example 6 essentially becomes the (P-0)* version of Section 4.

†In this example we shall ignore the 'strict ordering' part of (A3), and allow indifferences among alternatives.
with the following information structure

\[ \eta_0: u_1 \quad \eta_1: \xi \]

where \( \xi \) is a zero-mean Gaussian random variable with unit covariance. By depicting the contours for the possible \( L_1 \) cost function, it is easily shown that any continuous \( u_0 = \gamma_0(u_1) \) will not lead to \( L_0 = 0 \). Suppose that the leader announces

\[
\begin{align*}
u_0 &= 0 \text{ for } u_1 \geq 0 \text{ and } u_0 = -Nu_1 \text{ for } u_1 < 0 \\
u_0 &= 0 \text{ for } u_1 \geq 0 \text{ and } u_0 = +Nu_1 \text{ for } u_1 < 0
\end{align*}
\]

both with chance \( \frac{1}{2} \). Here \( N > 1 \) is a constant. It is easily shown that the best the follower can do is to choose \( u_1 = 0 \); another choice of \( u_1 \) will lead to higher average costs (higher than \( L_1 = 2 \)) for the follower. Therefore the problem is i.c. in mixed strategies.

5.3. Other related problems

So far we have mainly dealt with problems in which the leader was able to obtain his team solution. However, what can the leader achieve if he knows that the team solution is beyond reach? Consider Example 9 and let us restrict this example to pure strategies. The leader can not obtain his team solution, though he can get arbitrarily close. Other problems can be devised where he can not ever get \( \epsilon \) close (consider Example 5 with \( b < 0 \). The leader can not obtain his team payoff by the method described above and probably can not achieve it at all.) In its generality, this problem has not (yet) been solved. Only a few special problems have been successfully treated (Mirrlees, 1971; Shavell, 1979). In Tolwinski (1980) some results in this vein are given for the deterministic case.

Let us give a brief description of a problem considered in Spence (1977). An individual's preference is characterized by two parameters, the marginal valuation of income, denoted by \( \lambda \), and the evaluation of the good relation to income, denoted by \( \phi \). The parameters \( \lambda \) and \( \phi \) are jointly distributed in the population according to the distribution \( h(\lambda, \phi) \). If an individual \( i \) with parameters \( \lambda \) and \( \phi \) buys \( u_i \) units of the good for \( \gamma_0(u_i) \) dollars, the social gain is

\[ L_i = \lambda[v(u_i, \phi) - \gamma_0(u_i)] \]

where the function \( v(u_i, \phi) \) can be thought of as the dollar value of \( u_i \) units of good to this individual. The leader, who knows \( h(\lambda, \phi) \), but does not know the \( \lambda, \phi \) values for each individual, wants to choose \( \gamma_0(\cdot) \) in such a way to maximize

\[ L_0 = \int \int \lambda[v(u^*, \phi) - \gamma_0(u^*)] h(\lambda, \phi) \, d\lambda \, d\phi \]

where \( u^* \) is determined by maximization of \( L_i \), expressed in terms of \( \lambda \) and \( \phi \).

The solution to this problem has been obtained in Spence (1977) given that the function \( v(u_i, \phi) \) satisfies some reasonable differentiability requirements. Crucial in the determination of the solution is that \( u_i \) is a scalar (calculus of variations techniques are used in the derivation, in which \( u_i \) is treated as an independent variable).

Another related problem is the one of the principal and agent, considered in Shavell (1979). The principal is the leader and his payoff is given by \( L_0[x(u_i, \xi) - u_0(x)] \), where \( x \) is a function of \( u_i \), the agent's decision, and \( \xi \), a random variable; \( u_0 \) is the payment of the principal to the agent. The agent's payoff is \( L_i(u_0, u_i) \). As a specific application, one can think of \( u_0 \) as the fee which the principal (client) pays his agent (lawyer); \( u_i \) is the lawyer's effort in the court case, of which the outcome is \( x(u_i, \xi) \). The principal and agent are each assumed to act so as to maximize their expected utilities

\[ J_0(\gamma_0, u_i) = \int L_0[x(u_i, \xi) - \gamma_0(x(u_i, \xi))]p(\xi) \, d\xi \]

\[ J_i(\gamma_0, u_i) = \int L_i[\gamma_0(x(u_i, \xi)), u_i]p(\xi) \, d\xi \]

where \( p(\xi) \) is the density function of \( \xi \). The major difference between this problem and other problems considered before is that \( u_i \) is not directly observable by the principal. The effort of the agent is seen through the function \( x(u_i, \xi) \) (corrupted measurement). The discussion in Shavell (1979) centers around the question of how the fee \( u_0 = \gamma_0(x) \) is related to the outcome \( x \) and, in addition, what happens if the principal would have some information (which may be imperfect) about the agent's effort \( u_i \) directly.

6. CONCLUSION

A unification of some recent results in Stackelberg problems has been provided. A link with incentive problems in the economic literature has been made. Of all incentive problems and corresponding solutions, only a very small portion, like the tip of an iceberg, has been scratched.

Acknowledgements—The research reported in this paper was supported by the Department of Energy under Contract
REFERENCES


Michel R. Gevers was born in Antwerp, Belgium in 1945. He graduated from Louvain University, Louvain, Belgium in 1968 as an electrical engineer. He obtained a Ph.D. degree from Stanford University, California, in 1972. In 1969 he was awarded a Harkness Fellowship, and in 1971 an ESRO/NASA International Fellowship. He is also a fellow of the Belgian American Educational Foundation. In 1972 he joined the Laboratoire d'Automatique et d'Analyse des Systèmes, Louvain University, Louvain-la-Neuve, Belgium, where he is now professor. From 1976 to 1980 he was head of this control laboratory. In 1980 he spent a sabbatical year at the University of Newcastle, NSW, Australia. His research interests are in stochastic systems, system identification, multivariable systems and adaptive systems. He has applied various identification techniques to biomedical, industrial and water resource problems.

Y. F. Huang was born in Taiwan on 8 June 1954. He received the B.S. degree from the National Taiwan University, in 1976, the M.S. degree from the University of Notre Dame in 1979, and the M.A. degree from Princeton University, in 1980, respectively. Since September 1979, he has been a research assistant in the Department of Electrical Engineering and Computer Science, Princeton University, Princeton, NJ, where he is currently completing his doctoral dissertation.

Yu-Chi Ho received his S.B. and S.M. degrees in electrical engineering from the Massachusetts Institute of Technology, U.S.A. After graduation, he worked on numerical control systems at the Research Laboratory Division of the Bendix Corporation, in Michigan. He returned to Harvard and obtained his Ph.D. degree in applied mathematics in 1961. Since then Professor Ho has been on the faculty of Harvard University and is presently Gordon McKay Professor of Engineering and Applied Mathematics. He is a consultant to various industrial and research organizations and co-inventor of four U.S. patents on various aspects of numerical and digital control systems. He was a member of the Army Scientific and Advisory Panel from 1968–1974, Editorial Board of the IEEE Proceedings from 1965–1969, IEEE Publication Board from 1966–1971 and the Editorial Board IEEE Press from 1971–1973. In 1969, he was the Chairman of the First International Conference on the Theory and Applications of Differential Games and was Chairman, International Conference on Directions in Large Scale Systems: Decentralized Control and Many-Person Optimization, September 1975; editor (with S. K. Mitter, M.I.T.) of the proceedings. In 1970 he was a Guggenheim Fellow at Imperial College and Cambridge University, U.K.; in 1973 he was a U.S.A.–U.S.S.R. Senior Exchange Fellow; and in 1970 and 1977 he was a U.K. SRC Senior Fellow. His current interests are information in large decision and control systems, and manufacturing automation.

Peter B. Luh received his B.S. degree in electrical engineering from National Taiwan University (1973), his M.S. in aeronautics and astronautics from the Massachusetts Institute of Technology (1977), and Ph.D. in applied mathematics from Harvard University (1980). From 1975 to 1977, Dr. Luh served as a research assistant at the Wright Brothers Wind Tunnel and the Aerophysics Laboratory, M.I.T. He performed wind tunnel instrumentation and testing, designed digital control system for a magnetic model suspension and balance system, and implemented a prototype control by using a microprocessor. He served as a teaching fellow (1977) and research assistant (1978–1980) at the Decision and Control Group, Division of Applied Sciences, Harvard University. He performed research in the areas of optimal control theory, many-person optimization problems, and peak load pricing in electric energy systems. In the summer of 1978, he held a position at Bell Laboratories, Holmdel, New Jersey. While at Bell Laboratories, he designed microprocessor software in a multi-processor environment. From 1978 to 1980, he also served as a consultant to Bell Bernack and Newman Inc., Cambridge, for a project on variable pricing and load management. Since October 1980, Dr. Luh has been with the Electrical Engineering and Computer Science Department, University of Connecticut, Storrs. He is presently teaching courses, performing and supervising research in the areas of optimal control, decentralized hierarchical control, game theory, and pricing and load management for electric energy systems. Dr. Luh is a member of IEEE and Sigma Xi.

Hidenori Kinura was born in Tokyo in 1941. He received the B.S., M.S. and Ph.D. degrees from the University of Tokyo in 1965, 1967 and 1970, respectively. He joined the Faculty of Engineering Science, Osaka University in 1970 as a research assistant, where he has been an associate professor at the Department of Control Engineering since 1973. During 1974–1975 he was at Warwick University and University of Technology, U.K. His current research interests lie in multivariable control theory, mathematical system theory and digital signal processing. He is a member of the Society of Instrumentation and Control Engineers, the Japan Association of Automatic Control, the Institute of Electronics and Communication Engineers, the Mathematical Society of Japan and the Institute of Electronics and Electrical Engineers.

Andrea Holmberg was born in Helsinki, Finland in 1949. She received the master of science and licentiate of technical sciences degrees in electrical engineering from Helsinki University of Technology in 1974 and 1977, respectively. Since 1973 she has been with the Systems Theory Laboratory, Helsinki University of Technology as teaching and research assistant. Currently she is a research assistant fellow of the Academy of Finland. From November 1979 to March 1980 she was visiting scholar, lecturing in control theory, at the Institution of Control Theory, Chalmers University of Technology, Gothenburg, Sweden. Her research interests are in modelling and identification of biotechnical and ecological systems. She is a member of the Finnish Society of Automatic Control.
D. Q. Mayne was born in South Africa. During 1950–1959 he worked as a lecturer at the University of Witwatersrand and as a research and development engineer at the British Thomson Houston Company, Rugby, U.K. From 1960 he has worked at Imperial College becoming a reader in 1967 and a professor in 1971. He was a visiting research fellow at Harvard University in 1971 and has spent several summers at the University of California, Berkeley in collaborative research. He was awarded a Senior Fellowship of the Science Research Council (U.K.) in 1979. His major research interests are optimization and computer-aided design.

Jukka Ranta was born in Helsinki, Finland in 1949. He received his M.Sc. degree from Helsinki University of Technology, Systems Theory Laboratory in 1973 and the licentiate of technical sciences degree from the same university in 1976. From 1973–1980 he held various teaching and research positions at Helsinki University of Technology, Systems Theory Laboratory and was research assistant fellow of the Academy of Finland. Since 1980 he has worked as a research engineer at Technical Research Centre of Finland, Electrical Engineering Laboratory with its Automation Systems Section. His current interests are man-machine systems, human aspects of automation, artificial intelligence, pattern recognition and control problems of microbiological and ecological systems. He is a member of Finnish Society and Automatic Control and Association of Electrical Engineers in Finland.

Geert Jan Otter, born in 1944, received his M.Sc. and Ph.D. degrees (cum laude) in applied mathematics at the University of Groningen, The Netherlands (1968, 1971). Since 1971 he has been employed at Twente University of Technology, where he teaches graduate courses in optimal control and system theory. He spent the academic years 1972–1973 and 1979–1980 at the Department of Aeronautics and Astronautics, Stanford University and at the Division of Applied Sciences, Harvard University, respectively. Since 1981 he has been working part time at Hollandse Signaalapparaten, an electronics firm. Currently he is chairman of the Mathematics of Control Committee of IFAC. His main interests are in the fields of optimal control, system and differential game theory.

Masatoshi Sakawa was born in Matsumoto, Japan on 11 August 1947. He received the B.E., M.E., and D.E. degrees in applied mathematics and physics from Kyoto University, in 1970, 1972, and 1975, respectively. Since 1975 he has been with Kobe University where, since 1981, he is an associate professor in the Department of Systems Engineering. From March 1980 to April 1980, he joined the International Institute for Applied Systems Analysis, Laxenburg, Austria in the system and decision sciences area, and worked in decision analysis and planning theory and its application to resources and environmental planning. His main research interests include optimization theory for large-scale systems, multiobjective optimization and decision theory, and its applications to environmental systems, and reliability design. Dr Sakawa is a member of the Japan Association of Automatic Control Engineering, the Operations Research Society of Japan, the Institute of Electronics and Communication Engineerings of Japan, and the Society of Instrument and Control Engineerings of Japan.

Elijah Polak was born in Poland in 1931. He received the Ph.D. degree in electrical engineering, from the University of California, Berkeley, in 1961. Since 1958, he has been employed by the University of California where he is at present a professor of electrical engineering and computer sciences. He was a Guggenheim Fellow in 1968–1969, and a U.K. Science Research Council Senior Post-doctoral Fellow, at Imperial College, London in 1972, 1976 and 1979, respectively. His research interests lie in the area of optimization-based computer-aided design (theory and software), with applications to electronic circuit design, control system design and structural design; optimization algorithms and systems theory. He is a fellow of the Institute of Electrical and Electronic Engineers, a member of the Society of Industrial and Applied Mathematics and a member of the Mathematical Programming Society.

Fumiko Soto was born in Osaka, Japan. She received the B.A. degree and the Ph.D. degree in 1967 from the University of Tokyo, Japan, both in economics. At present she is an associate professor at Kyoto University, Japan, and is working with the Kyoto Institute of Economic Research, Regional Science Research Unit. From April 1979 to May 1980, she joined the International Institute for Applied Systems Analysis Laxenburg, Austria, in the system and decision sciences area, and is working on the task of decision analysis and planning theory along with its application to resources and environmental management. Since 1970, her research activities have centered primarily on environmental systems planning with multiobjective optimization and decision analysis. Dr Soto is a member of the Regional Science Association, American Economic Association, Japan Association of Automatic Control Engineers, and many other related associations.