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A NEW PERSPECTIVE IN DESIGNING DELAYED FEEDBACK CONTROL FOR THERMO-ACOUSTIC INSTABILITIES (TAI)

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This article suggests the deployment of a unique mathematical tool for assessing the thermo-acoustic instability (TAI) in a Rijke tube and proposes an analytical design strategy for its feedback control. A widely accepted characteristic of TAI is its time-delayed dynamics, which originate from the regenerative acoustic coupling terms. Linear systems theory has also evolved on similar classes of problems especially in recent years. This document offers a bridge between the two veins of research. We first review the analytical model of the TAI phenomenon, which renders a set of delayed differential equations. Then, we apply a new mathematical tool called the cluster treatment of characteristic roots (CTCR) paradigm on this dynamics. CTCR provides non-conservative and exhaustive stability predictions for this class of systems. This capability is employed for both uncontrolled and feedback-controlled Rijke tube structures. The findings are unique from two angles: (i) stability declarations are made in the parametric space of the system, such as geometric dimensions (much differently from the peer studies that are at best point-wise evaluations), and (ii) these declared sets of stable operating parameters are exhaustive (i.e., for a given system no other parametric selection can provide stability). These capabilities become crucial when designing thermoacoustically stable combustors as well as determining their operating conditions. As a highlight contribution in this article, for those operating conditions that induce instability, we offer a methodology to synthesize a feedback control law that can recover stability, again utilizing the CTCR paradigm. Example case studies and analytical justifications of these novelties are provided.

Keywords: Active control; Cluster treatment of characteristic roots (CTCR); Thermo-acoustic instability; Time delay

1. INTRODUCTION

“Thermo-acoustic instability” (TAI), a.k.a. “Rayleigh instability” has a long history and a special place in modern combustion science. It has been a focal point of research by many investigators especially over the last few decades (Candel, 2002; Dowling and Morgans, 2005; McManus et al., 1993; Raun et al., 1993). The complexity of the subject dynamics emanates from the interaction between acoustic and thermal events in an enclosure (i.e., the combustor). As first described by Rayleigh (1878), a confined unsteady...
heat release emerging from the flame in a combustion chamber drives acoustic waves. These waves, after reflecting from the ends of the chamber, influence their creator (the heat release). This “regenerative” phenomenon may lead to sustained pressure fluctuations in the combustion chamber. A point of importance is that this property is inherent to the overall dynamics, and it is not induced by an exogenous feedback control structure. Additionally, modern combustors feature many advanced operational characteristics, such as diffusion flames, bluff-body wake control, added turbulence for mixing, and so on. Due to these complexities, the conditions for instability are mostly detected empirically and typically after the construction and testing of the combustor when the remedial options are few and costly as noted by (Cazalens et al., 2008). In this article, we wish to establish novel analytical prediction and control mechanisms as contributions.

The TAI phenomenon is also observed in experiments that deploy an electrical resistance (Bittanti et al., 2002; Gelbert et al., 2012; Matveev, 2003) as the heat source instead of a flame caused by a combustible in a chamber. Therefore, it is safe to claim that the primary reasons of TAI are not those particularities related to the complex dynamics of a flame and its shape, although undoubtedly they play some role. Culick (2006) elaborates on the influence of flame-acoustic interactions, while Candel (2002) puts “combustion-centered” analysis of TAI into perspective and stresses the importance of flame dynamics in this approach.

This article presents a novel treatment to the problem especially using the strengths of a recent mathematical paradigm. This interesting interplay between the “combustion instability” and the recent discoveries in “mathematics of time-delayed systems (TDS)” is expected to open a broad range of future research directions. The mathematical paradigm is called the cluster treatment of characteristic roots (CTCR). It is a unique concept that can declare the stability boundaries of linear systems, which are influenced by rationally-independent multiple time-delays. Furthermore, it can determine these boundaries non-conservatively and exhaustively.

In order to bring these two seemingly decoupled research frontiers (TAI and CTCR) together one needs to simplify the governing equations of TAI in a linear time invariant (LTI) class. Admittedly, a thorough theoretical treatment of acoustic-flame coupling is unrealistic at the present for a modern combustor, where the flame is strongly vortical and turbulent (Lieuwen, 2002; Wu et al., 2001). As an important progression we wish to follow existing and recognized research findings, restricting the investigation to a simple case of a Rijke tube (Dowling, 1997; Kopitz and Polifke, 2008; Rijke, 1859) with the assumptions that the flow is laminar and the dynamics are governed by an unsteady heat release.

It has been shown that these simplifications result in yet another complexity, namely, the declaration of stability in the parameter space for LTI systems with multiple rationally-independent delays. For this purpose we utilize the CTCR paradigm, which constitutes the primary scientific contribution of the present work. Furthermore, we use the same mathematical tool to investigate the potential of feedback stabilization for existing TAI. Some simple but promising findings in this front are included in this text, which broadens the range of earlier investigations (see Table 1 in Dowling and Morgans, 2005).

The text is constructed in the following segments: Section 2 is a problem statement containing the simplifying assumptions and the first principles. The same section proposes a feedback control strategy to stabilize an existing TAI. In Section 3, we describe the pillars of the CTCR paradigm. Section 4 is devoted to the adoption of CTCR into the TAI problem and example case studies. In Section 5, we provide some conclusions on the present work.
2. PROBLEM DESCRIPTION AND DERIVATION OF GOVERNING EQUATIONS

In this section we present a brief overview of broadly accepted findings of Dowling, (1997), Evesque (2000), and Evesque et al. (2003), as the groundwork towards the main contributions of this article. We wish to start with a crucial premise for clarity: Any causal nonlinear functional relationship can be linearized around some operating conditions that pose an equilibrium. Furthermore, for systems that are represented by such dynamics, the occurrence of instability is always initiated by a linear progression even when the core dynamics are nonlinear.

The physics behind TAI is summarized next, starting with the underlying first principles relevant to several segments of the dynamics: thermo-fluidic structure, acoustic behavior, and thermo-acoustic interface.

2.1. Thermo-Fluidic Structure

In a Rijke tube (Rijke, 1859) we take a control volume, which is sandwiched in Figure 1 between cross-sections ① and ②. Some simplifying assumptions are made on this structure first. The heat release zone (a.k.a. heating zone) is assumed to be a thin cross-section of the tube, and the thermo-fluid dynamic features, such as flow velocity and temperature, are taken as uniform across any cross-section along the tube. Therefore, these dynamics are considered to be a 1-D event.

The underpinning physical first principles of such a thermo-fluidic event are stated in the following equations:

Continuity\[ \rho_2 u_2 = \rho_1 u_1 \] (1)

Conservation of momentum\[ p_2 + \rho_2 u_2^2 = p_1 + \rho_1 u_1^2 \] (2)

Conservation of energy\[ \left( c_p T_2 + \frac{1}{2} u_2^2 \right) \rho_2 u_2 A = \left( c_p T_1 + \frac{1}{2} u_1^2 \right) \rho_1 u_1 A + \Delta Q \] (3)

where \( \rho_1, \rho_2 \) are the specific masses at the two cross-sections, \( u_1, u_2 \) are the air velocities, \( c_p \) is the isobaric specific heat capacity, \( p_1, p_2 \) are the pressures, and \( T_1, T_2 \) are the absolute temperatures at the two cross-sections. \( Q \) represents the thermal power injections within the control volume and \( A \) represents the cross-sectional area of the heating zone.

We consider the flow to be laminar, inviscid, and the substance air to be an ideal gas. Accordingly, one can write:

The ideal gas law\[ p = \rho R_s T \] (4)

Property relations\[ \frac{c_p}{c_v} = \gamma, \quad c_p - c_v = R_s, \quad c_p = \frac{\gamma}{\gamma - 1} R_s \] (5)

where \( R_s \) is the specific gas constant, \( c_v \) is the constant volume specific heat, and \( \gamma \) is the heat capacity ratio. After proper mathematical simplifications among Eqs. (1) to (5), one obtains:
Notice that these equations represent the relations between the cross-sectional pressures and the velocities, which are moderated by $Q/A$, power density in the cross-section. In essence, we interpret these relationships as the creators of $(u_2, p_2)$ departing from $(u_1, p_1)$ with the dynamic excitation of $Q$. They are nonlinear in nature and causal (i.e., small fluctuations in $u_1, p_1$ and $Q$ cause small fluctuations in $u_2, p_2$). This observation triggers a meaningful next step, which is the local linearization of (6) and (7). That is presented as:

$$p_i = \tilde{p}_i + \bar{p}_i, \quad u_i = \bar{u}_i + \tilde{u}_i, \quad \rho_i = \bar{\rho}_i + \tilde{\rho}_i \quad (i = 1, 2)$$

Figure 1 Rijke tube depiction, the traveling pressure waves, and a control loop.
where “−” notation depicts the nominal (mean) variables, and “∼” is for the respective fluctuating (variational) quantities.

### 2.2. Acoustic Behavior

The physics associated with TAI have been clear from Rayleigh’s (1878) revelation onwards that there is a strong regenerative acoustic component moderating the thermo-fluidic events in a Rijke tube. Pressure perturbations at the heating zone travel acoustically through the tube and reflect back from both ends. The reflected waves influence the local velocity at the heating zone, imposing a heat release fluctuation (so called “unsteady heat release” in the literature, e.g., Dowling, 1997; Culick, 2006; Nagaraja et al., 2009).

In this study we focus on Rijke tubes without forced airflow where the mean flow velocity \( u \) is negligibly small compared to the average speed of sound \( c \) (\( u \ll c \)). Then the acoustics of the tube are governed via the one-dimensional sound wave equation:

\[
\frac{\partial^2 \tilde{p}}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \tilde{p}}{\partial t^2} = 0
\] (9)

which describes the propagation of the sound waves with the speed of sound, \( c \). In (9) \( x, t \) represent the spatial coordinate (see Figure 1) and the time, respectively. Equation (9) is a hyperbolic partial differential equation for which there exists a D’Alembert solution that is exact and in the form of:

\[
\tilde{p}(x, t) = f(t - \frac{x}{c}) + g(t + \frac{x}{c})
\] (10)

where \( f \) and \( g \) are arbitrary functions, which satisfy the boundary and initial conditions. It suffices to state that the pressure fluctuation, \( \tilde{p}(x, t) \), can be expressed as:

\[
\tilde{p}_u(x, t) = f\left(t - \frac{x}{c_1}\right) + g\left(t + \frac{x}{c_1}\right) \quad \text{for} \quad -x_u < x < 0 \quad (11)
\]

\[
\tilde{p}_d(x, t) = h\left(t - \frac{x}{c_2}\right) + j\left(t + \frac{x}{c_2}\right) \quad \text{for} \quad 0 < x < x_d \quad (12)
\]

where \( h \) and \( j \) are similar functions to \( f \) and \( g \) satisfying the respective boundary and initial conditions (see Figure 1). \( f \) and \( g \) represent the propagation in the upstream side, while \( h \) and \( j \) are for the downstream side. \( c_1, c_2 \) represent the average speeds of sound in upstream and downstream sections of the tube, respectively.

### 2.3. Thermo-Acoustic Interface

In order to represent the interconnection between the thermo-fluidic modeling and acoustics (in Sections 2.1 and 2.2), we revisit the particle physics from the first principles next. The variational form of the conservation of momentum for a moving particle is:

\[
\frac{\partial \tilde{u}}{\partial t} + \frac{\partial \tilde{p}}{\partial x} = 0
\] (13)
Let us focus on the upstream side of the tube. From (11) using the chain rule one can write:
\[ \frac{\partial \tilde{p}_u}{\partial x} = \frac{1}{c_1} \left[ g' \left( t + \frac{x}{c_1} \right) - f' \left( t - \frac{x}{c_1} \right) \right] \] (14)
where “’” denotes derivatives with respect to the complete argument (i.e., \( t + \frac{x}{c_1} \) or \( t - \frac{x}{c_1} \)).
Substituting (14) into (13) and integrating over time, we arrive at:
\[ \tilde{u}_u(t, x) = \frac{1}{\rho u c_1} \left[ f \left( t - \frac{x}{c_1} \right) - g \left( t + \frac{x}{c_1} \right) \right] \] (15)
Similar treatment can be done on the downstream side of the tube as well. After further manipulations using the identity \( p_i \gamma = c_i^2 p_i \) for \( i = 1, 2 \) and truncating the higher-order terms of \( \tilde{p}, \tilde{u}, \) and \( \tilde{\rho} \), the dynamics can be recast in a matrix form:
\[ X \left( \begin{array}{c} g \\ h \end{array} \right) = Y \left( \begin{array}{c} f \\ j \end{array} \right) + Z \] (16)
where the constant matrices \( X, Y \in \mathfrak{R}^{2 \times 2} \) (Evesque, 2000) are given in Appendix A and \( Z = \left( \begin{array}{cc} 0 & \tilde{Q} \\ \tilde{Q} / (A \tilde{\gamma}_1) & 0 \end{array} \right) \) with \( \tilde{Q} = Q - \bar{Q} \), representing the variations in the heat release.
Meanwhile, the boundary conditions at \( x = -x_u \) and \( x = x_d \) dictate that pressure variations at these points are both zero, if there is no external manipulation (such as a loudspeaker-induced pressure). Introducing the acoustic reflection conditions at the two ends of the tube (\( R_u \) and \( R_d \) both being < 1), one can write:
\[ f(t) = -R_u g(t - \tau_u) \quad j(t) = -R_d h(t - \tau_d) \] (17)
where \( \tau_u = 2x_u/c_1, \tau_d = 2x_d/c_2 \) are upstream and downstream round-trip travel delays of the acoustic waves, respectively.

2.4. Incorporation of Feedback Control

When a feedback-controlled loudspeaker pressure output \( l(t) \) is exerted, as shown in Figure 1, the acoustic boundary condition changes at the lower end of the tube. Then (17) should be rewritten as:
\[ f(t) = -R_u g(t - \tau_u) + l(t - \tau_u/2) \quad j(t) = -R_d h(t - \tau_d) \] (18)
We adopt a proportional control logic for the speaker output with a delay as in:
\[ l(t) = -Kp_{mic}(t - \tau_c) \quad \rightarrow \quad L(s) = -K e^{-\tau_c s} P_{mic}(s) \] (19)
where \( K \) is the control gain, \( \tau_c \) is the artificially added control delay, \( L(s) = \mathcal{L}[l(t)] \) is the Laplace domain representation, and
\[ p_{mic}(t) = h \left( t - \frac{x_{mic}}{c_2} \right) + j \left( t + \frac{x_{mic}}{c_2} \right) \] (20)
is the pressure variation measured with a microphone that is located $x_{mic}$ away from the heat release zone as shown in Figure 1. Defining $\tau_{mic} = x_{mic}/c$, and using (17), the Laplace transform of (20) can be rewritten as:

$$P_{mic}(s) = \left( e^{-\tau_{mic}s} - R_d e^{-(\tau_d - \tau_{mic})s} \right) H(s)$$

(21)

where $H(s) = \mathcal{L}[h(t)]$. Substituting (21) in (19), one can write the pressure waveform created by the loudspeaker as:

$$L(s) = -K \left( e^{-(\tau_{mic} + \tau_c)s} - R_d e^{-(\tau_d - \tau_{mic} + \tau_c)s} \right) H(s)$$

(22)

Notice that the control logic proposed in (19) closely resembles a number of earlier instability control efforts, which also suggested a phase-shift or time-delay controllers [see Table 1 in Dowling and Morgans (2005)]. We quote from the cited document “... the controller parameters were obtained, usually empirically, for a single operating condition. Such approaches are unlikely to be adequate for practical combustion systems, where there may be multiple instability modes.” These observations are also reported in other studies (Banaszuk et al., 2004; Fleifil et al., 1998; Seume et al., 1998). Furthermore, in order to assign a phase shift the control designer should know the target frequency (or the frequency of a particular mode). In order to avoid such occurrences of instability, a distinct feature is introduced in this article from the control logic design perspective. It is a holistic treatment on the entire dynamics as opposed to some selected modes only. We present here a complete stability map of the closed-loop system with respect to the control parameters, the gain $K$ and the delay $\tau_c$. This stability map displays all possible sets of $(K, \tau_c)$ exhaustively and non-conservatively in the domain of these parameters. The control parameters should be selected in these regions to assure the system stability. We show that some of the unstable regions on our maps, involve multiple modes of oscillations, which are driven unstable concurrently by the improper selection of $(K, \tau_c)$. This capability allows the synthesis of the control logic deterministically where earlier mentioned problems related to time-delay (or phase shift) controllers are avoided. We will return to this point later in the text.

### 2.5. Heat Release Dynamics and Complete System Model

The heat release fluctuation $\tilde{Q}$ is causally related to the velocity fluctuations at the heating zone. That is, as explained by Evesque (2000), fluctuations in $\tilde{u}$ directly influence the instantaneous heat release fluctuations $\tilde{Q}$. Lieuwen (2002) also remarks on this topic pointing to the complexities in determining a flame transfer function for industrial combustors and identifies shortcomings in the state of art. Noiray et al. (2008) proposes some nonlinear flame describing functions in an effort to achieve more accurate modeling of heat release dynamics. The exact nature of this causal relation has been and still is a central research issue in combustion science. The unsteady heat release, indeed, is part of the solution; hence, it is not known a priori (Wu et al., 2001). Therefore, as a common practice, investigators empirically predict this relationship for modern combustors, while for simple heater configurations some analytical or computational methods also exist. We denote this connection in Figure 2 as $\tilde{Q}(s) = \phi(s)\tilde{U}_1(s)$, where $\tilde{Q}(s) = \mathcal{L}[Q(t) - \overline{Q}]$ and $\tilde{U}_1(s) = \mathcal{L}[\tilde{u}_1(t)]$. It is important to note that this transfer function needs to be proper or strictly proper in nature (i.e., higher or equal degrees of $s$ terms in the denominator than the numerator) due to causality reasons.
As explained earlier, the relationship between $\tilde{u}_1$ and the pressure waves $f$, $g$ can be expressed using (15) at $x = 0$ as:

$$\tilde{u}_1 = \frac{f(t) - g(t)}{\rho_1 c_1}$$

(23)

Converting this relation into Laplace domain and using (18), one obtains:

$$\tilde{U}_1(s) = \frac{-R_u e^{-\tau s}}{\rho_1 c_1} G(s) + e^{-\tau s/2} L(s)$$

(24)

where $G(s) = \mathcal{L}[g(t)]$. For the present analysis we choose heat release dynamics in the form of a first-order transfer function, although the procedure we follow is general enough to handle more complicated transfer functions:

$$\phi(s) = \frac{a}{bs + 1}$$

(25)

This model has been explored extensively in the past and seems to yield accurate results both for electrical resistance heaters (Gelbert et al., 2012; Kopitz, 2008) as well as flames (Fleifil et al., 1996). Incorporating (24) and (25) as in Figure 2, we obtain a relationship declaring a second regeneration mechanism (i.e., thermal regenerative effect) in TAI as:

$$\tilde{Q}(s) = \frac{a}{bs + 1} \left[ -\frac{R_u e^{-\tau s}}{\rho_1 c_1} G(s) + e^{-\tau s/2} L(s) \right]$$

(26)

Combining Eqs. (18), (22), and (26) and substituting in (16), the governing equations for the complete system can be expressed in a matrix form as:

$$M(s) \begin{pmatrix} G \\ H \end{pmatrix} = 0$$

(27)

where $M$ is a $(2 \times 2)$ transfer matrix as given in Appendix A.
3. MAIN CONTRIBUTIONS: CONTROL DESIGN FOR TAI AND THE CTCR PARADIGM

3.1. Stability Analysis of the Dynamics

The linearized closed-loop dynamics as given in (27) depict an unforced delay differential equation (DDE) in Laplace domain. The stability of this system is analyzed through its characteristic equation:

$$\det(M) = 0$$  \hspace{1cm} (28)

which involves transcendental terms due to the two acoustic delays and one control delay. This class of expressions is known as “exponential polynomials” or “quasi-polynomials” in the literature. Such characteristic equations possess infinitely many roots, all of which have to lie in the left half of the complex plane as the necessary and sufficient condition for stability.

The TAI analysis can be performed in two tracks, one for the uncontrolled system, which is obtained by setting \(K = 0\) in Eq. (27), and the other for the feedback controlled dynamics. The corresponding characteristic equations are naturally different but their general features are the same (they are both exponential polynomials). Therefore, the same mathematical tools could be used for both.

The stability analysis of the uncontrolled system facilitates the prediction of TAI. The main contribution and focus of this study, however, is to design a stabilizing feedback control law for a scenario when the open-loop system is unstable (i.e., the undesirable TAI is occurring). A significant amount of literature exists, which offers numerous methodologies to achieve this (Evesque, 2000; Dowling and Morgans, 2005 and the extensive Table 1 within; Gelbert et al., 2012). Most of these procedures propose some feedback constructs, which use the downstream pressure measurements with an imposed phase shift. The assurance of stability is addressed by various techniques, such as LQR (linear quadratic regulator), \(H_\infty\), LMS (least mean squares) only for some specific set-points, not in a parametric space. We follow a similar feedback structure here except the stabilizing parametric selections are made using our CTCR procedure, which allows a complete stability display in the parametric domain. A comparable earlier stability investigation (McManus et al., 1993) is based on proposed modal solutions to TAI dynamics, which result in a transcendental equation to be solved numerically. Therefore, a complete stability declaration in the parametric space is computationally prohibitive in such methods.

The question of stability assessment of the exponential polynomial type characteristic equation is where CTCR methodology appears. The systems and mathematics literature has strong evidence of an increasing research activity on the stability of DDEs a.k.a. “time-delayed systems” (Bellman and Cooke, 1963; Niculescu, 2001). We wish to cite Richard (2003) for a comprehensive review on the advances in time-delayed systems. Recently, several numerical methods have also been proposed to approximate the characteristic roots of LTI-TDS (Breda et al., 2005; Engelborghs et al., 2002; Vyhlidal and Zitek, 2009). Lyapunov–Krasovskii approaches have also been popular for stability analysis (Fridman, 2001; Gu and Niculescu, 2006; Mazenc et al., 2012). However, these treatments are conservative and strongly dependent on the selection of the Lyapunov–Krasovskii functionals. They provide point wise stability assessment for a given set of delay values. Therefore, in order to generate stability maps in a broader parameter space (such as delays), this operation
has to be repeated at grid points dense enough to be meaningful causing a high computational demand. The CTCR paradigm has recently emerged from the research group of the authors bringing rule-based capabilities to alleviate this specific shortfall (Olgac and Sipahi, 2004, 2005). The method creates an intriguing output, an exhaustive and non-conservative declaration of stability regions in the domain of some system parameters. For instance, of particular interest here are the control parameters \((K, \tau_c)\).

The conceptual development here is executed on the rudimentary combustion setting of Rijke tube. It is, however, presented with a much broader scope in mind for more complicated combustion chambers, and even for modern day aerospace or land-based gas turbines. Next, we describe the pillars of the CTCR paradigm.

### 3.2. The CTCR Paradigm

Regardless of the nature of the transfer function \(\phi(s)\), the characteristic equation (28) represents a very small but important subclass of TDS, called the “Neutral TDS” (Hale and Verduyn Lunel, 2002; Niculescu, 2001). The neutrality attribute comes from the fact that the highest degree \(s\) term (which represents the time derivative operation in Laplace domain) appears jointly with a transcendental multiplier, such as \(e^{-\tau s}\). This property declares that the highest order derivative operator in TAI contains time delays in its argument. Such features have also been observed in combustion literature, for instance, modal equation (6) in (Durox et al., 2002). Surprisingly, however, the investigators failed to take note of the mathematical progress available for the very same stability problem, such as that given in Olgac et al. (1997), which could have considerably enlightened the venue.

The neutral class of TDS (NTDS) offers some challenging and very specific features. As the TAI dynamics belong in this class, we will present the highlights of the CTCR paradigm particularly relevant to NTDS. For readers who are interested in the background of the paradigm, we suggest the works of Olgac and Sipahi (2004, 2005), which treat NTDS cases with single delay only and Sipahi and Olgac (2005, 2006) on retarded systems with multiple delays. In order not to divert the mainstay of the discussions far into the mathematics, we will revisit the highlights of CTCR next.

The characteristic equation (28) is an exponential polynomial in the form:

\[
CE(s, \tau_u, \tau_d, \tau_c, \tau_{mic}, K) = \left(\chi_0 + \sum_{i=1}^{\chi_i e^{-\tau_i s}}\right) s + \eta_0 + \sum_{j=1}^{\eta_j e^{-\tau_j s}}
\]

where \(\chi_i, \eta_j\) are functions of system parameters (e.g., \(K, R_u, R_d, p_1\), etc.) and \(\tau_i, \tau_j\) are various linear combinations of the time-delays \(\tau_u, \tau_d, \tau_c,\) and \(\tau_{mic}\):

\[
\tau_k = \kappa_{k0}\tau_c + \kappa_{k1}\tau_u + \kappa_{k2}\tau_d + \kappa_{k3}\tau_{mic}, \quad k = i, j
\]

Notice the transcendental factor of \(s\) in (29), asserting the neutral nature of the system. On this system, we will give a series of key theorems from the NTDS literature leaving the proofs to the above mentioned references. Theorem 1 is a precondition for Theorem 2, which is nothing other than the restatement of the earlier mentioned necessary and sufficient conditions for stability.
Theorem 1. A necessary (but not sufficient) condition for the stability of (29) is that the associated difference equation has to be stable.

\[ D(s, \tau_u, \tau_d, \tau_c, \tau_{mic}, K) = \left( \chi_0 + \sum_{i=1} \chi_i e^{-\tau_i s} \right) = 0 \]  

(31)

In other words, the infinitely many roots (spectrum) of (31) should lie in the left-half complex plane, \( \mathbb{C}^- \). If not, the original system given in (29) is guaranteed to be unstable. This theorem is also known as “the strong stability (or delay-stabilizability) condition” in the literature (Avelar and Hale, 1980; Hale and Verduyn Lunel, 1993; Olgac and Sipahi, 2004).

Theorem 2. The necessary and sufficient condition for the closed-loop system to be stable is that the spectrum of (29) must lie in \( \mathbb{C}^- \). It can be shown that this condition predicates Theorem 1.

A unique practical procedure to resolve the relevant stability map assuring both Theorems 1 and 2 resides in the CTCR paradigm. It is obvious that there is no need to perform the steps of CTCR, if the conditions in Theorem 1 are not fulfilled. If they are fulfilled, however, some compositions of control parameters and delays may render stability. That is the knowledge we are targeting here. For a fixed set of these parameters, one can also utilize the general Nyquist stability criterion (Dowling and Morgans, 2005; Kopitz and Polifke, 2008; Morgans and Stow, 2007), however, we wish to caution the reader on two important shortfalls of this process:

i. Nyquist criterion is valid only for a single parameter composition (i.e., it is a point-wise stability method). Therefore, one needs to deploy the process repeatedly over a sufficiently dense 2D grid in order to obtain a complete stability map. That is, the space of two of the control parameters, such as \((\tau_u, \tau_d)\) or \((K, \tau_c)\), has to be screened point by point, which brings an insurmountable computational load.

ii. It is a geometric methodology and often yields obscurity as to the counts of encirclements around a particular point (Dowling and Morgans, 2005).

The essence of the CTCR procedure, on the contrary, is to generate an efficient and complete stability map in the parametric space, as described comprehensively by Olgac and Sipahi (2004, 2005) and Sipahi and Olgac (2006). We provide here a brief review, just to convey the key ideas and definitions.

It is known that for a linear system with multiple delays to switch the stability posture, the characteristic roots should cross the imaginary axis, say at \( \omega_i \) (notice that \( i = \sqrt{-1} \) refers to the imaginary unit in the remainder of the text, not to be confused with index variable \( i \)). Thus, for a successful stability analysis, one needs to detect exhaustively all the potential imaginary root crossings for all combinations of delays \( \tau \in \mathbb{R}^{n+} \). Let us denote the complete set of such crossing frequencies with \( \Omega \), and the corresponding root set with \( S_\Omega \):

\[ \Omega = \{ \omega \mid CE(s = \omega i, \tau) = 0, \tau \in \mathbb{R}^{n+}, \omega \in \mathbb{R} \} \quad S_\Omega = \{ s = \omega i \mid \omega \in \Omega \} \]  

(32)

We use \((\tau, \omega)\) notation to indicate the causality relation between \( \tau \) and \( \omega \) (i.e., \( \tau \) yields \( \omega \)). It is trivial to observe from (29) that an imaginary root \( s = \omega i \) with \((\tau, \omega)\) correspondence
will be repeated infinitely many times for the mesh points with equidistant grid size, \(2\pi/\omega\), as:

\[
\tau_{ij} = \tau_i + \frac{2\pi}{\omega}j_i, \quad j_i = 0, 1, \ldots \quad i = 1, 2, \ldots, n
\]

(33)

These trajectories continuously partition the \(\tau\) domain into encapsulated regions in which the number of unstable roots, \(NU\), remains fixed. Consequently, any stability reversal can only occur at the boundaries of these regions.

The above argument brings us to a complicated problem of determining exhaustively the boundaries of these infinitely many regions. Interestingly, however, there is a discipline in this chaotic looking picture owing to two intriguing propositions.

**Proposition I.** It is proven by Sipahi and Olgac (2005, 2006) that there is only a *manageably small number* of hypersurfaces in \(\tau\) space called the “kernel hypersurfaces” defined by:

\[
\varphi_0 = \{ \tau | \langle \tau, \omega \rangle, \tau \in \Re^{n^+}, \omega \in \Omega, 0 \leq \tau_i \leq \frac{2\pi}{\omega}, i = 1, 2, \ldots, n \}
\]

(34)

where \(\langle \tau, \omega \rangle\) correspondence creates the complete set of \(\Omega\). Notice that for all \(\omega \in \Omega\) values, \(\varphi_0(\tau)\) represents the loci of the corresponding smallest positive \(\tau\) combinations. All remaining hypersurfaces are created from this set, \(\varphi(\tau)\), utilizing the point-wise translation property Eq. (33) for \(j_i > 0\). Obviously, all of these hypersurfaces possess the identical set of imaginary roots, \(S_\Omega\). They are called the “offspring hypersurfaces.” The complete set of all the hypersurfaces is denoted by \(\varphi(\tau)\).

Any kernel point on the trajectories of \(\varphi_0(\tau)\) defined by \(j_i = 0\) imposes its \(\omega\) variations identically onto its offspring \((j_i > 0)\). Thus, \(\Omega\) remains invariant from kernel hypersurfaces to offspring. The kernel and the offspring hypersurfaces constitute the complete (and exhaustive) distribution of \(\tau\) points where the characteristic equation \(CE(s, \tau)\) has root sets containing at least one pair of imaginary roots. And more interestingly, there exists no point in \(\tau \in \Re^{n^+}\) space, outside the loci set \(\varphi(\tau)\), which renders imaginary characteristic roots.

**Proposition II.** A critical property is the directional “root tendency” along \(\tau_i\) at the crossing points of \(s = \omega i\). It is defined by:

\[
RT|_{s=\omega i} = \text{sgn} \left[ \frac{\text{Re} \left( \frac{\partial s}{\partial \tau_i} \right) \bigg|_{s=\omega i} \right]
\]

(35)

\(RT\) has a very interesting feature: At \(s \in S_\Omega\) as \(\tau_i\) increases at a point on a kernel hypersurface and its corresponding offspring, the root tendency \(RT|_{s=\omega i}^{\tau_i}\), *remains unchanged* so long as all other delays in \(\tau\) except \(\tau_i\) are kept fixed. This so called “root tendency invariance” property has recently been recognized in mathematics literature (Sipahi and Olgac, 2005, 2006). This feature, in essence, declares the stabilizing (or destabilizing) transitions along the regional boundaries defined by \(\varphi(\tau)\).

Using the two properties above, one can establish the complete stability maps of the system against delay variations by performing the following steps of the CTCR algorithm.
Without loss of generality, the procedure is described for a case where stability is assessed with respect to only two delays \((\tau_u, \tau_d)\), as for instance in an uncontrolled Rijke tube.

1. Determine exhaustively the kernel and offspring hypersurfaces, \(\varphi(\tau)\).
2. Starting from non-delayed system, \(\tau = 0\), evaluate the number of unstable roots, \(NU(0)\), which is a trivial task.
3. Following line segments in \((\tau_u, \tau_d) \in \mathbb{R}^2\), which are parallel to the coordinates \(\tau_u\) and \(\tau_d\), connect the origin \((\tau = 0)\) to a point of interest \(\tau_0\).
4. As this path crosses the kernel and offspring hypersurfaces, advance \(NU\) by +2 (or -2) for the invariant \(RT = +1\) (or -1), according to the D-subdivision method of El’sgol’ts and Norkin (1973).
5. Exhaustively identify the regions in \((\tau_u, \tau_d)\) space where \(NU = 0\) as “stable” and the others \((NU > 0)\) as “unstable.”

Note that the same procedure is applied to design control parameters for an unstable Rijke tube setting. In such a case, \((\tau_u, \tau_d, \tau_{mic})\) have fixed values while the above described steps are followed in the space of the parameters \((K, \tau_c)\).

Step 1 is the crucial operation from which CTCR departs. Among many possible methodologies (Ergenc et al., 2007; Sipahi and Olgac, 2006) we deploy in this work spectral delay space (SDS) approach to assess the stability of uncontrolled dynamics. Appendix B provides the highlights of the method of which the details can be found in Fazelinia et al. (2007). Once the kernel and offspring are determined the root tendency invariance property is executed, which results in the complete and exact stability outlook of the system in the space of the rationally-independent delays (or control parameters) and this representation is the unique contribution of CTCR. Such stability outlook enables the system designer (i.e., the combustion specialist) to search for the stable operating conditions for a given combustor, or to synthesize a stabilizing control law for a Rijke tube where TAI is observed.

Once again, as the uncontrolled and controlled dynamics fall into the similar general mathematical category of neutral time-delayed systems, CTCR applies in the same fashion, as we will see next over example cases.

4. EXAMPLE CASE STUDIES

We present two case studies in this section, one for prediction of TAI and the other for feedback stabilization of it. The capabilities of CTCR are displayed in constructing the parametric stability pictures (i.e., the stability maps) for such systems just to give the reader an idea of the main strengths of the mathematical tool. In essence, these strengths could enable combustion specialists to (a) predict TAI at the design stage, (b) adjust the operational parameters accordingly to prevent it, or (c) implement a stabilizing feedback control mechanism.

Before moving forward with the stability analysis for the uncontrolled Rijke tube, it is assumed that the speed of sound does not show a discernible variance between the two ends of the heating zone since this feature has little effect on the outcome of TAI (Heckl, 1988). Taking the effect of this assumption on matrices \(X\) and \(Y\) (see Appendix A) into account and setting \(K = 0\) in (28), the simplified characteristic equation for the uncontrolled system becomes:
CE(s, τ_u, τ_d) = 2Ab\bar{\gamma}^2{\mathbf{\bar{p}}}_1 (R_uR_de^{-(\tau_u+\tau_d)s} - 1) s - R_da (1 - \gamma) e^{-\tau_d s} - R_da (\gamma - 1) e^{-\tau_u s} + R_uR_d (2\sqrt{\bar{\gamma}}{\mathbf{\bar{p}}}_1 + a\gamma - a) e^{-(\tau_u+\tau_d)s} + a - a\gamma - 2\bar{\gamma}^{1/2}\bar{p}_1 = 0 \tag{36}

which makes the dynamics “neutral.” From the highest order s term in Eq. (36), the associated difference equation is:

\[ R_uR_de^{-(\tau_u+\tau_d)s} - 1 = 0 \tag{37} \]

From the nature of reflection coefficients \( R_u \) and \( R_d \), one can claim \( |R_uR_d| < 1 \). Equation (37) yields:

\[ |e^{-(\tau_u+\tau_d)s}| = \frac{1}{|R_uR_d|} > 1 \tag{38} \]

which implies that all the infinitely many roots of Eq. (37) have Re(s) < 0; therefore, it represents a stable operator and the necessary strong stability precondition (delay stabilizability) of Section 3.2 is satisfied. We continue performing the further steps of the CTCR paradigm on Eq. (36), as summarized in the previous section.

Using the suggestions by Gelbert et al. (2012) for this case study, the parameters of the heat release dynamics in Eq. (36) are taken as \( a = 200 \) and \( b = 0.002 \). These parameters \( a \) and \( b \) are related to the heating element and flow characteristics. The interested reader can refer to the cited document for the suggested correlations. The remaining parameters are given in Table 1. Implementing the CTCR paradigm by following the steps explained in the previous section, one can obtain the stability map of the system in the space of \((\tau_u, \tau_d) \in \mathbb{R}^2 \) as seen in Figure 3. The determination of the kernel hypersurface, which lies in the first step of the methodology, is briefly explained in Appendix B.

There is only one kernel (shown in green) which creates the entire stability table in Figure 3. Substituting \( \tau_u = \tau_d = 0 \) in Eq. (36), it is determined that the origin is stable (i.e., \( NU = 0 \)). Then following steps 3 and 4 in Section 3.2, \( NU \) in distinct regions separated by the hypersurfaces are evaluated. Finally, as defined in step 5, the stable operating regions (\( NU = 0 \)) are shaded in Figure 3. It is clear that the heating zone at the mid-point of the tube renders stability (\( \tau_u = \tau_d \)). This feature has been observed by experimentalists as well (Carvalho et al., 1989; Matveev, 2003; Raun et al., 1993). Figure 3 also declares that the heating zone being very close to the end points of the tube would also invite stable operating conditions. Furthermore, one can state that the heating zone being in the lower half (i.e., \( \tau_u/\tau_d < 1 \)) introduces instability much more readily than otherwise (i.e., \( \tau_u/\tau_d > 1 \)), which

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_u )</td>
<td>0.93</td>
<td>—</td>
</tr>
<tr>
<td>( R_d )</td>
<td>0.93</td>
<td>—</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>1.4</td>
<td>—</td>
</tr>
<tr>
<td>( \Lambda )</td>
<td>0.00075</td>
<td>m²</td>
</tr>
<tr>
<td>( \bar{\gamma}_1 )</td>
<td>1.2</td>
<td>kg/m³</td>
</tr>
<tr>
<td>( \tau_1 )</td>
<td>340</td>
<td>m/s</td>
</tr>
</tbody>
</table>
has also been observed in the literature (Matveev, 2003; Rayleigh, 1878). As validation of these findings, we provide two time traces for the two different operating points: A (unstable with $NU = 2$) and B (stable) as the insets of Figure 3.

Next, we wish to demonstrate a feedback control operation as a means of eliminating existing TAI in a Rijke tube. Using a proportional control strategy with artificial time delay as described in Section 2.4, we repeat the same assumptions as in the uncontrolled case and obtain the characteristic equation for the closed loop system from (28) as follows:

$$CE(s, \tau_u, \tau_d, \tau_{mic}, \tau_c, K) = 2A\vec{e}_1^2 \bar{\rho}_1 (KR_d e^{-(\tau_d + \tau_u + \tau_{mic}/2) s}) - Ke^{-(\tau_c + \tau_{mic} + \tau_u/2) s}$$

$$+ R_u R_d e^{-(\tau_d + \tau_u + \tau_{mic}/2) s} - 1) s + a - a \gamma - 2A\vec{e}_1^2 \bar{\rho}_1 + (\gamma - 1)a R_d e^{-\tau_d s} + (1 - \gamma)a R_u e^{-\tau_u s}$$

$$+ (\gamma - 1)a R_u R_d e^{-(\tau_u + \tau_d + \tau_{mic}/2) s} + 2K(1 - \gamma)ae^{-(\tau_c + \tau_{mic} + \tau_u/2) s} + 2K(\gamma - 1)a R_d e^{-(\tau_d + \tau_u + \tau_{mic}/2) s}$$

$$+ 2A\vec{e}_1^2 \bar{\rho}_1 (KR_d e^{-(\tau_c + \tau_u - \tau_{mic} + \tau_u/2) s} + R_u R_d e^{-(\tau_u + \tau_d + \tau_{mic}/2) s} - Ke^{-(\tau_c + \tau_{mic} + \tau_u/2) s}) = 0$$

(39)

Again the system is an NTDS. First, the strong stability condition should be checked. The associated difference equation can be written using the highest order $s$ term of Eq. (39) as follows:

$$KR_d e^{-(\tau_d + \tau_u - \tau_{mic} + \tau_u/2)s} - Ke^{-(\tau_c + \tau_{mic} + \tau_u/2) s} + R_u R_d e^{-(\tau_u + \tau_d) s} - 1 = 0$$

(40)

Treating these three clusters of delays as rationally independent, the following expression must hold in order to satisfy the strong stability condition (Hale and Verduyn Lunel, 2002):

$$|K| + |KR_d| + |R_u R_d| < 1$$

(41)
which simply declares bounds for the proportional feedback gain, $K$ as follows:

$$|K| < \frac{1 - |R_u R_d|}{1 + |R_d|}$$

(42)

The main objective now reduces to synthesize control parameters $(K, \tau_c)$ for a Rijke tube configuration where the present tube length and heater location cause unstable operation. We take the parameters $(\tau_u, \tau_d)$ in Eq. (39) as known and fixed for such a system. In addition, we treat the position of the feedback sensor (e.g., microphone) as a fixed parameter. That is, the corresponding parameter $\tau_{mic}$ remains fixed. The characteristic equation (39) then becomes a function of $(s, K, \tau_c)$, which can be simplified as:

$$CE(s, K, \tau_c) = A(s) + Ke^{-\tau_c s}B(s)$$

(43)

where $A(s)$ and $B(s)$ are exponential polynomials in $s$. The control parameters $(K, \tau_c)$ at stability boundaries are solved from Eq. (43) by substituting $s = \omega i$ and using magnitude and phase conditions:

$$\tau_c(\omega) = \frac{1}{\omega} \left[ i\pi + \angle B(\omega i) - \angle A(\omega i) \right],$$

$$K(\omega) = (-1)^{i+1} \left| \frac{A(\omega i)}{B(\omega i)} \right|, \quad i = 0, 1, 2, \ldots$$

(44)

This approach is identical to the procedure that has been used in earlier studies to obtain parametric stability charts for the tuning of delayed-resonator vibration absorbers (Filipovic and Olgac, 2002).

Substituting the system parameters of Table 1 into Eq. (42), the strong stability bound on gain amplitude is calculated as $|K| < 0.07$. Case-specific time delays are chosen as $\tau_u = 2.2$ ms, $\tau_d = 3.8$ ms, and $\tau_{mic} = 0.41$ ms, which correspond to a 101.6-cm-long tube where the heat source is placed at 36.8 cm from the lower (i.e., upstream) end and the microphone sensor is positioned at $x_{mic} = 13.9$ cm.

The resulting $(K, \tau_c)$ sets calculated from Eq. (44) can now be sketched to reveal a graphical representation of stable operation zones. The resulting stability chart is given in Figure 4. Green curves in Figure 4 correspond to kernel set and blue curves are the offspring for both $K > 0$, and $K < 0$. Note that the origin $(K, \tau_c) = (0, 0)$ in Figure 4 represents an unstable uncontrolled system due to the selection of $(\tau_u, \tau_d) = (2.2$ ms, $3.8$ ms). This fact can be deduced from Figure 3, as the said delay composition falls into the same region as point A where $NU = 2$. Calculating the $RT$s on the kernel set and using the $RT$ invariance property, the stable regions where $NU = 0$ are revealed and shaded in grey on Figure 4. Therefore, by selecting the feedback control parameters $(K, \tau_c)$ inside these shaded regions, stability can be recovered. This outcome essentially answers the quest this article is built on. In the following discussions we provide some deeper insight to the methodology.

To give a better understanding of $RT$ invariance property, the locus of (which happens to be) the only unstable root is provided in Figure 5. Note that complex characteristic roots appear in conjugate pairs. The root transition depicted in Figure 5 also has a conjugate partner, which is not shown. This root locus corresponds to a segment of the control delay $\tau_c$ as it is varied in the interval depicted by a black dashed line in Figure 4. The control parameters
Figure 4 Stabilizing \((K, \tau_c)\) compositions (shaded) for given \((\tau_\text{in}, \tau_\text{dl}, \tau_\text{mic})\). Light curves (green when viewed in color) are kernel and dark (blue in color) are offspring. Notice dense unstable regions at higher \(K\) values due to increased density of higher modes. Gray-shaded regions are stable. The dashed-dotted window is detailed in Figure 6.

Figure 5 Movement of the unstable characteristic root corresponding to the change of control delay \(\tau_c\) while \(K = 0.035\), as depicted in Figure 4.

start from \((K, \tau_c) = (0.035, 2 \text{ ms})\) and end at \((K, \tau_c) = (0.035, 12 \text{ ms})\). Note that imaginary root crossings occur at \((K, \tau_c) = (0.035, 3.47 \text{ ms})\) and \((K, \tau_c) = (0.035, 9.38 \text{ ms})\), which are marked with black squares; both carry the feature of \(RT = -1\) as expected.
from Proposition II. This means that during this transition $NU$ decreases by 2 (remember the conjugate root pair); therefore, $NU = 0$, and the system becomes stable. On the other hand, after passing through the points $(K, \tau_c) = (0.035, 5.06\,\text{ms})$ and $(K, \tau_c) = (0.035, 10.87\,\text{ms})$ (marked with black circles), $NU$ increases by 2 and the stability is lost again, since at those points $RT = +1$ is destabilizing.

In order to validate the stability declarations in Figure 4, some simulations are presented at several points inside a rectangular segment with magnified view for the details (Figure 6). As declared, the system is unstable at points C and D, whereas stable at point E. In addition, the characteristic root distributions of the uncontrolled system (i.e., the origin of Figure 4) along with the feedback controlled cases at points C, D, and E are given in Figure 7, in the upper half of the complex plane. Dominant system roots are evaluated using a numerical approximation code, quasi-polynomial mapping-based root finder (QPmR) that is developed by Vyhlidal and Zitek (2009). The instabilities in the uncontrolled case (Figure 7a) and at point C (Figure 7b) are caused by characteristic roots having imaginary parts 170.6 Hz and 168.1 Hz, respectively. For point D (Figure 7c), however, the imaginary part of the unstable root is 502.0 Hz. Figure 8 depicts a zoomed region of Figure 7 to observe the characteristic roots in detail. Here, it is seen that imaginary parts of the roots (shown as $f_1, f_2, \ldots$) are very close to the frequencies corresponding to acoustic modes of the tube ($f^{\text{a}}_1 = 167.3\,\text{Hz}, f^{\text{a}}_2 = 334.6\,\text{Hz}, f^{\text{a}}_3 = 501.9\,\text{Hz}, \ldots$) calculated as $f^{\text{a}}_i = i/(\tau_u + \tau_d)$. Connecting the characteristic roots with the respective modes, one can say that the control parameters at point D invite instability in the third mode while stabilizing the first mode. This addresses to an earlier observed problem in the literature; while trying to stabilize one mode of the system, another mode may get into instability (Dowling and Morgans, 2005). Note that the offspring hypersurfaces corresponding to higher modes of instability are repeated more frequently in Figure 4. This observation can easily be
Figure 7 Characteristic root distributions for: (a) the uncontrolled system, (b) point C, (c) point D, and (d) point E as marked on Figure 6. The light (green in color) dots denote unstable roots, whereas the dark (blue) dots are stable roots.

Figure 8 A zoomed-in version of Figure 7c, where the frequencies corresponding to the characteristic roots are displayed.
explained through Eq. (33), which declares the distance between two adjacent offspring being inversely proportional to the crossing frequency, $\omega$. Another observation is that the real part of the unstable root in Figure 7c is relatively close to the imaginary axis, therefore, the growth rate of instability is small. This is in accordance with the inset simulation for point D in Figure 6. Although the system seems to be stable at the beginning of the time trace, the unstable growth becomes evident after $t = 1$ s.

An important message to take away from this case study is that the ability to detect instabilities caused by higher modes is critically important for infinite dimensional system, and the CTCR paradigm handles it well. As discussed in Section 2.5, the absence of such a capability forced the researchers to move away from time-delay control laws; however, a systematic controller design procedure involving the CTCR methodology as we describe in this article may reverse the trends.

5. CONCLUSIONS

This article considers the complex dynamics of thermo-acoustic instability on a classical Rijke tube. It starts with an overview on the thermo-acoustic model of TAI departing from the first principles, which arrives at a variational form of the dynamics (i.e., linearized governing equations). This process evolves into a representative characteristic equation involving multiple rationally-independent delays. The ensuing regenerative dynamics is shown to be in the class of “neutral time-delayed system,” stability of which is notoriously known in the mathematics and systems communities.

On the representative dynamics, we utilize the recently developed cluster treatment of characteristic roots paradigm. We first demonstrate the stability assessment of the uncontrolled Rijke tube in its parameter space. It primarily declares the position of the heating zone to avoid TAI for a particular Rijke tube operation. As the main contribution of the article, we propose a stabilizing proportional-delayed feedback control law based on stability maps generated via CTCR. This analysis considers the system as infinite-dimensional as opposed to the current trend in the literature, where analyses are mostly truncated into finite number of acoustical modes. As a result the new method is exhaustive and it reveals instabilities due to multiple modes of oscillation. This feature distinguishes our approach from the state of the art, where similar control strategies are deployed resulting in undesired phenomena, such as secondary mode instabilities. This is the first utilization of CTCR in the literature for the control synthesis of thermo-acoustic instability. Undoubtedly, future research directions are wide open as we touched upon within the text.

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APPENDIX A: MATRICES IN GOVERNING EQUATIONS

\[
X = \begin{bmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{bmatrix} = \begin{bmatrix}
-1 & 1 \\
\frac{1}{\gamma - 1} & \frac{1}{\gamma_1} \frac{1}{\gamma - 1}
\end{bmatrix} \quad Y = \begin{bmatrix}
Y_{11} & Y_{12} \\
Y_{21} & Y_{22}
\end{bmatrix} = \begin{bmatrix}
1 & -1 \\
\frac{1}{\gamma - 1} & \frac{1}{\gamma_1} \frac{1}{\gamma - 1}
\end{bmatrix}
\]

\[
M = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}
\]

where

\[
M_{11} = X_{11} + R_u e^{-\tau_us} Y_{11}
\]
\[
M_{12} = X_{12} + R_u e^{-\tau_ds} Y_{12} + K e^{-(\tau_c + \tau_u/2)} Y_{11} \left( e^{-\tau_m s} - R_d e^{-(\tau_d - \tau_m)} \right)
\]
\[
M_{21} = X_{21} + R_u e^{-\tau_us} Y_{21} + \phi(s) (R_u e^{-\tau_us} + 1)/(A \rho_1 \gamma_1^2)
\]
\[
M_{22} = X_{22} + R_u e^{-\tau_ds} Y_{22} + K e^{-(\tau_c + \tau_u/2)} \left( e^{-\tau_m s} - R_d e^{-(\tau_d - \tau_m)} \right) \left[ Y_{21} + \phi(s)/(A \rho_1 \gamma_1^2) \right]
\]

APPENDIX B: CALCULATION OF THE KERNEL HYPERSURFACE FOR UNCONTROLLED RIJKE TUBE

For clarity and convenience, we present a brief summary on the evaluations of the kernel hypersurface for Eq. (36) in Section 4. As mentioned under Section 3.2, several methods are available to achieve this task and new approaches are currently being examined. In this study we utilize the SDS method combined with Rekasius’ holographic mapping procedure (Fazelinia et al., 2007).

As the first step in this method, the acoustic travel delay-induced transcendental terms in Eq. (36) are replaced using Rekasius transformation (Rekasius, 1980):

\[
e^{-\tau_i s} = \frac{1 - T_i s}{1 + T_i s}, \quad T_i \in \mathbb{R}, \quad i = u, d \tag{B.1}
\]

Note that in contrast to Padé approximants, this substitution is exact for \( s = \omega i \) and under the following restriction:

\[
\tau_i = \frac{2}{\omega} \left[ \tan^{-1}(T_i \omega) + k\pi \right], \quad k = 0, 1, 2, \ldots \tag{B.2}
\]

After the substitution (B.1) is applied, the transcendental characteristic equation (36) reduces to a multinomial in the form of:

\[
\bar{C}E(s, T_u, T_d) = \sum_{k=0}^{N} C_k(T_u, T_d) s^k \tag{B.3}
\]
The problem is now recast into the exhaustive determination of all \((T_u, T_d)\) compositions for which Eq. (B.3) has at least one pair of purely imaginary spectra, \(s = \omega i\). To perform this task in an efficient manner, we move to a new domain called the spectral delay space. The coordinates of SDS are defined as \(v_i = \tau_i \omega, i = u, d\). Then, using the inverse of Eq. (B.2), one can obtain the following relation:

\[
T_i = \tan(v_i/2)/\omega = z_i/\omega = u, d \tag{B.4}
\]

Note from Eq. (B.4), \(v_i \in [0, 2\pi]\) and by varying the coordinate in this interval, one should be able to recover all values of the corresponding \(T_i\). Finiteness of this coordinate provides a big advantage for SDS. With these definitions of \(v_i\) and \(z_i\), Eq. (B.3) becomes:

\[
P(z_u, z_d, \omega) = CE(s, z_u/\omega, z_d/\omega) \bigg|_{s = \omega i} = \sum_{k=0}^{N} C_k(z_u/\omega, z_d/\omega) (\omega i)^k \tag{B.5}
\]

For \(P(z_u, z_d, \omega)\) to have a solution \(\omega \in \mathbb{R}\), both its real and imaginary parts must be equal to zero:

\[
\text{Re}[P(z_u, z_d, \omega)] = \sum_{k=0}^{n} \alpha_k(z_u, z_d) \omega^k = 0 \tag{B.6}
\]
\[
\text{Im}[P(z_u, z_d, \omega)] = \sum_{k=0}^{n} \beta_k(z_u, z_d) \omega^k = 0 \tag{B.7}
\]

For the necessary and sufficient condition for Eqs. (B.6) and (B.7) to have a common root, their Sylvester matrix (Fazelinia et al., 2007):

\[
N(z_u, z_d) = \begin{bmatrix}
\alpha_m & \alpha_{m-1} & \alpha_{m-2} & \cdots & \cdots & \cdots & 0 \\
0 & \alpha_m & \alpha_{m-1} & \alpha_{m-2} & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \alpha_2 & \alpha_1 & \alpha_0 \\
\beta_m & \beta_{m-1} & \beta_{m-2} & \cdots & \cdots & \cdots & 0 \\
0 & \beta_m & \beta_{m-1} & \beta_{m-2} & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \beta_2 & \beta_1 & \beta_0 \\
\end{bmatrix}_{2m \times 2m} \tag{B.8}
\]

ought to be singular; hence, \(\text{det}(N) = 0\). This expression is known as the resultant of Eqs. (B.6) and (B.7). It is in the form of an implicit multinomial function in terms of \((z_u, z_d)\) only and this equation is exhaustive (i.e., there cannot be a common root if this condition does not hold). In accordance with Eq. (B.4), we scan \(v_u\) in the interval of \([0, 2\pi]\) and evaluate the corresponding \(z_u\) values. They are then substituted in the resultant to solve for \(z_d\). With \((z_u, z_d)\) compositions known, corresponding \(\omega \in \mathbb{R}\) values can be solved from either Eq. (B.6) or Eq. (B.7). Finally, using this information, we create the kernel hypersurface exhaustively and non-conservatively in the delay space \((\tau_u, \tau_d) \in \mathbb{R}^{2+}\) through Eq. (B.2).