Investigation of Local Stability Transitions in the Spectral Delay Space and Delay Space

The stability boundaries of LTI time-delayed systems with respect to the delays are studied in two different domains: (i) delay space (DS) and (ii) spectral delay space (SDS), which contains pointwise frequency information as well as the delay. SDS is the preferred domain due to its advantageous boundedness properties and simple construct of stability transition boundaries. These transitions at the mentioned boundaries, however, present some conceptual challenges in SDS. This transition property enables us to extract the corresponding local stability variation properties in the DS, while it does not have any implication in the preferred SDS. The novel aspect of the investigation is to introduce a comparison mechanism between these two domains, DS and SDS, from the stability transition perspective. Interestingly, we are able to prove their equivalency, which provides complementary insight to the parametric stability variations. [DO: 10.1115/1.4027171]

1 Introduction

This paper studies an interesting and practical feature of the stability outlook of linear time-invariant, retarded multiple time-delay systems (LTI-MTDS). For the ease of conveyance and formulation, a system with two independent delays is considered here

\[ \dot{x} = Ax + B_1x(t - \tau_1) + B_2x(t - \tau_2) \]

where \( A, B_1, B_2 \in \mathbb{R}^{n \times n}, x \in \mathbb{R}^{n 	imes 1}, \tau_1, \tau_2 \in \mathbb{R}^+, \text{rank}(B_1) = \text{rank}(B_2) = 1, \text{rank}(B_1 + B_2) = 2. \)

The proofs in the document are given for systems in this state space form; however, they can easily be extended to more complicated systems with more than two delays and additional commensurate delays. The class of characteristic equations considered here is

\[ CE_\tau(s, \tau_1, \tau_2) = a_0(s) + a_1(s)e^{-s\tau_1} + a_2(s)e^{-s\tau_2} + a_3(s)e^{-(\tau_1 + \tau_2)s} = 0, \quad \tau_1, \tau_2 > 0 \]

where the subscript \( \tau \) of \( CE \) denotes that the characteristic equation is in DS and \( a_1(s) \) are polynomials of \( s \) with real coefficients while the highest degree of \( s \), say \( n \), is only appearing in \( a_0(s) \), rendering Eq. (1) a “retarded system” [1–6]. Also, note that the delays \( \tau_1 \) and \( \tau_2 \) are considered to be constants in this study. The last term in Eq. (1) represents the crucial “cross-talk” between the two delays, which is also known as ‘interference’ in the literature [6]. Particularly the presence of this “cross-talk” term in the characteristic equation for LTI systems makes the stability analysis notoriously complex. Many dynamic systems have this class of characteristic equation and one common example to this kind of systems is the fully actuated cart-pendulum system in [7].

The stability robustness of single time-delay systems against delay uncertainties has been extensively studied in the last five decades [4,8–11]. The same pursuit has been extended to much more complicated MTDS with very limited literature [2,5,6,12–14] which is complemented later on by a key paradigm, called cluster treatment of characteristic roots (CTCR) [15,16]. This paradigm of the authors’ group provides a numerically efficient procedure to produce an exact and exhaustive stability posture for Eq. (1) and Refs. [10,15] within the domain of the delays. Several recent reports display the continued interest [17 and references therein] in this direction of research.

The asymptotic stability condition for Eq. (1) is that all its characteristic roots should be in the left half of the complex plane. However, due to the existence of the transcendental terms in Eq. (1), this pursuit is not trivial since infinitely many roots have to be examined. To overcome this difficulty, we apply a procedure known as the D-subdivision method (or the continuity argument) [18,19]. It simply states that the set of delay values at the stability boundaries should generate at least one pair of imaginary roots. The characteristic roots continuously vary with respect to the delays, and they may move from left to right half of the complex plane (or vice versa) crossing the imaginary axis only for some delay compositions. These compositions are located on some hypercurves in the 2D delay space. Once all these sets of hypercurves are determined, the problem reduces to checking the tendency of the imaginary roots for these sets of delays, i.e., their crossing directions to the right or the left-half of the complex plane. For the former case, the number of unstable roots (NU) increases by two and for the latter, NU decreases by two. Several degenerate cases are also possible to occur. For instance, the root might cross the imaginary axis at the origin; it might approach the imaginary axis tangentially or form an inflection point on the imaginary axis for some delay variations [6,20]. In some cases, the system can even have multiple imaginary roots [21] for a given delay composition. Such extreme degeneracies are kept out of the scope of this paper.

Various mathematical procedures can reveal the delay compositions for those potential stability transition locations (i.e., the crossing locations). This information serves as the preparatory step for the CTCR paradigm, which declares the stability picture entirely in the domain of the delays. The reader can refer to Refs. [16,22] for the details of two such procedures which feed into the CTCR paradigm. For the generation of the potential stability transition hypercurves in this paper, we deploy the “building block” (BB) concept from an interesting perspective [22,24]. BB construction is processed using a new domain called the SDS, where the earlier mentioned stability switching hypercurves exhibit a remarkably simple formation characteristic. The main contribution of this paper is to prove the equivalence of the local stability transition features between the two domains: DS and SDS.

The structure of the paper is as follows: Section 2 reviews the SDS concept and BB formation. The CTCR stability charts in DS and root tendency (RT) concept are revisited in Sec. 3. Section 4 contains the proof of equivalency of the local stability transition

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characteristics between the SDS and DS domains. Finally, an illustrative example is provided in Sec. 5, which validates these findings.

In the rest of the paper, bold face notation is used for vector quantities, bold capital letters for matrices and italic symbols for scalars. We refer to the right (and left) half open complex plane as $C^+(C^-)$, whereas $C^0$ is used to indicate the imaginary axis. Therefore, $C = C^+ \cup C^- \cup C^0$ represents the entire complex plane.

2 Spectral Delay Space

The concepts of SDS and BB are revisited here, from Ref. [24]. We present the definitions of kernel and offspring hypercurves first and then recite the properties of SDS concept and BB formation, which sets the stage for the main contribution of this paper.

Definition 1. Kernel hypercurves $\psi^{DS}_0$. The complete set of curves that contains all the points $(\tau_1, \tau_2) \in \mathbb{R}^2$ exhaustively, which cause an imaginary root $s = \omega i, \omega \in \mathbb{R}$ and satisfy the constraint $0 < \tau_1 \omega, \tau_2 \omega < 2\pi$. These curves are known as the kernel hypercurves.

Definition 2. Offspring hypercurves $\psi^{DS}_{p,j}$. The curves obtained from each pair $\tau(0, \tau_0)$ on the kernel curves by the following point-wise nonlinear transformation:

$$
\left\{ \begin{array}{l}
\tau_{00} = \tau_1 + \frac{2\pi}{\omega}, \\
\tau_{20} = \tau_2 + \frac{2\pi}{\omega}
\end{array} \right., \quad j_1, j_2 = 0, 1, 2, \ldots
$$

3 Stability Charts in DS and Root Tendency

We wish to stress an important point regarding the SDS analysis above. This domain yields a very convenient depiction of the potential stability switching hypercurves, but the determination of stability zones should be performed within the DS instead of the SDS. The reason for this is very simple: The SDS is a point-wise defined space, as it only exists at the points of delay composition $(\tau_1, \tau_2)$ where an imaginary characteristic root pair $(\tau_1, \tau_2, -\omega)$ exists. And the remaining infinitely many roots of the system have no representation in SDS unless they are also purely imaginary. Therefore, for an arbitrary delay composition $(\tau_1, \tau_2)$, there is no corresponding point in the SDS. Naturally, the study for determining the stability is handled strictly in the DS. At this point of nuance, however, this paper brings a very interesting additional conceptual development, which we present in Sec. 4.

In order to obtain the stability charts in DS, one needs to determine the sets of delay compositions which are defined in Eq. (2) exhaustively, with the corresponding imaginary roots, $s = \omega i$. To achieve this, we follow a mathematical procedure described in Ref. [24] which evaluates the building curves directly in SDS. Accordingly, in Eq. (1), the exponential terms are replaced by

$$
e^{-\pi k} = \frac{1 - T_k}{1 + T_k}, \quad k = 1, 2, \quad T_k \in \mathbb{R}$$

which is known as the Rekasius substitution and is exact for $s = \omega i, \omega \in \mathbb{R}$ with the following relation between $\tau_1$ and $\tau_2$:

$$
\tau_1 = \frac{2}{\omega} \tan^{-1}(T_k \omega) \pm j\pi, \quad k = 1, 2, \quad j = 1, 2, \ldots
$$

The above equation describes an asymmetric mapping, in which one $T$ is mapped into infinitely many $\tau$’s for a given $\omega$. Inversely for the same $\omega$, one particular $\tau$ corresponds to one $T$ only. Rekasius substitution in Eq. (1) converts the infinite dimensional characteristic equation to a finite dimensional polynomial of $\omega i$. In parallel, from Eq. (4b) one can create the following relations for the SDS domain parameters $(\nu, \nu_l)$$
\frac{\tau_1 \omega}{\tau_2} = \tan(\nu_k/2) = z_k, \quad \nu_k = \nu_2 \omega \in [0, 2\pi], \quad k = 1, 2
$$

These curves are known as the reflection hypercurves [24], corresponding to offspring hypercurves $\psi^{DS}_{p,j}(0, \tau_0)$ in DS, without an undesirable shape distortion. Obviously, $(j_1, j_2) = (3, 2)$ offspring is created by simply translating the building hypercurves by $(3 \times 2\pi, 2 \times 2\pi)$. This operation is referred to as “stacking” as it creates only translations of some integer multiples of $2\pi$ in each coordinate axes, i.e., $\tau_1 \omega$ and $\tau_2 \omega$. There are several other intriguing properties of the SDS and BB concepts which can be found in Ref. [24]. Here, we introduce a few of them leaving the proof to the cited paper.

Property 1. Kernel isolation property. The BB contains only the building hypercurves and has no trace of reflection curves in the SDS. In other words, the kernel requires only $(j_1, j_2) = (0, 0)$ in Eq. (3a) because the entire hypercurve is entrapped within $(2\pi, 2\pi)$ square due to the constraint of Eq. (2).

Property 2. Symmetry property of building curves. The point $(\pi, \pi)$ in SDS is the center of symmetry of the BB as well as the building curves. For each $(\tau_1, \tau_2, \omega)$ in the building block, there is also a $(\tau_1, \tau_2, -\omega)$ set, and these two points are symmetric with respect to the point $(\pi, \pi)$. This feature is very interesting in that, one needs to determine only one-half of the building hypercurves, and the other half can be created based on this symmetry property. Numerical advantage gained from this feature is considerable, as we will demonstrate in the example case studies.
With the substitution of Eqs. (4a) and (5) in Eq. (1), one obtains a polynomial of \( \omega \) with complex coefficients \( c_i \) which are parameterized in \( z_1 \) and \( z_2 \)

\[
q(\omega, z_1, z_2) = \sum_{k=0}^{n} c_k(z_1, z_2)(\omega)^k = 0
\]  

(6)

where \( n \) is the order of the non-delayed dynamics. Equation (6) implies

\[
\text{Re}[q(\omega, z_1, z_2)] = \sum_{k=0}^{n} f_k(z_1, z_2)\omega^k = 0 \quad (6a)
\]

\[
\text{Im}[q(\omega, z_1, z_2)] = \sum_{k=0}^{n} g_k(z_1, z_2)\omega^k = 0 \quad (6b)
\]

In order for Eqs. (6a) and (6b) to have a common root, the following Sylvester’s resultant matrix ought to be singular:

\[
M = \begin{bmatrix}
  f_n & f_{n-1} & f_{n-2} & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
  0 & f_n & f_{n-1} & f_{n-2} & \cdots & \cdots & \cdots & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
  g_n & g_{n-1} & g_{n-2} & g_{n-3} & \cdots & \cdots & \cdots & \cdots & 0 \\
  0 & g_n & g_{n-1} & g_{n-2} & g_{n-3} & \cdots & \cdots & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 
\end{bmatrix}_{2n \times 2n}
\]

(7)

That is,

\[
\det(M) = F(z_1, z_2) = F(\tan(\nu_1/2), \tan(\nu_2/2)) = 0
\]

(8)

which constitutes a closed form description of the kernel curves in the SDS \((\nu_1, \nu_2)\), i.e., the building hypercurves. To obtain these building hypercurves graphically, parameter \( \nu_2 \) is scanned in the range of \([0, 2\pi]\), and the corresponding \( \nu_1 \in \mathbb{R} \) values are calculated again within \([0, 2\pi]\). Notice that every point \((\nu_1, \nu_2)\) on these curves brings an imaginary characteristic root at \( \pm \omega i \) which can be evaluated from Eqs. (6a) or (6b), noting that they share the same \( \pm \omega i \) root. Due to Property 2, namely the symmetricity property of building hypercurves, we keep in mind that both points \((\tau_1, \tau_2, \omega)\) and \((\tau_1, \tau_2, -\omega)\) are involved in SDS. Backtransforming pointwise from the \((\nu_1, \nu_2)\) domain of SDS to the \((\tau_1, \tau_2)\) domain of DS, using relations in Eqs. (4b) and (5) with the appropriate positive \( \omega \) values, one generates the kernel and offspring hypercurves.

The kernel and offspring curves divide the \((\tau_1, \tau_2)\) domains into regions of possible stability and instability. Then, this information feeds into a recent stability paradigm CTCR [16]. To determine the stability nature of these regions we start from the non-delayed system (i.e., \( \tau_1 = \tau_2 = 0 \)). One should notice that delay compositions in these regions do not have representation in SDS, as the SDS map of a given system is conditional, and it only has validity at the possible stability switching boundaries. At those boundaries, however, some additional features could also be investigated, such as the “root tendencies” (defined next). A surprising feature from this angle appears as one looks at the variational characteristics of the root formations, which constitutes the key contribution of this paper.

**Definition 3. Root tendency.** The root tendency of an imaginary root \( \omega, i \), with respect to one of the delays \( (\tau_1 \text{ or } \tau_2) \), is defined as

\[
\text{RT} \bigg|_{\tau_1=\omega,i}^{\tau_2=\omega,i} = \text{sgn} \left[ \text{Re} \left( S'_{\tau_1} \big|_{\omega=\tau_1} \right) \right]
\]

(9)

where \( S'_{\tau_1} \big|_{\omega=\tau_1} = \frac{dS}{d\tau} \big|_{\omega=\tau_1} \) is the sensitivity of the root with one of the delays fixed.

This property indicates the tendency of transition of the imaginary root, as the other delay increases. A slightly different conceptualization of RT is defined within the SDS in the later segments of the paper. The highlight of the discussion comes at making sense of transition properties in a domain (SDS) which is only defined pointwise. The main contribution of the paper comes at this point.

**4 Main Contribution: Equivalency of Root Tendencies in SDS and DS Domains**

In this section, the unexpected equivalency of the local stability transitions along the boundaries in SDS and DS is proven. For this, we follow two corresponding radial lines passing through the origin with a common slope in both spaces. This treatment can eliminate the influence of \( \omega \) from DS to SDS. That is, if \((\tau_2/\tau_1) = g\) (where \( g \in \mathbb{R}^+ \)), the radial lines in both domains are identical (i.e., \((\tau_2/\tau_1) = g\)). The corresponding points on them, however, are still earmarked by \( \omega \) except the origin. Here, one may ask the following question: Departing from a certain delay composition that renders an imaginary characteristic root pair, \( \pm \omega i \), if we increase the delays slightly along the mentioned radial line, how does \( \text{NU} \) change? We look at this question within both SDS and DS. As the main contribution of the paper, we will prove that the local stability transition features within the SDS are identical to those within the DS. This discussion is obviously critical since it designates whether advancement along the radial line would have a stabilizing or destabilizing effect on the system.

Before we introduce the main lemma, a new root tendency is defined.

**Definition 4. Directional root tendency (DRT).** The directional root tendency for each imaginary root \( \omega, i \), with respect to time delays \( \tau_1 \) and \( \tau_2 \), is defined as

\[
\text{DRT} \bigg|_{\tau_1=\omega,i}^{\tau_2=\omega,i} = \text{sgn} \left[ \text{Re} \left( S'_{\tau_1} \big|_{\tau_1=\omega,i, \tau_2=\omega,i} \right) \right]
\]

(10)

where \( S'_{\tau_1} \big|_{\tau_1=\omega,i, \tau_2=\omega,i} = \frac{dS}{d\tau_1} \big|_{\tau_1=\omega,i, \tau_2=\omega,i} \) and \( g \in \mathbb{R}^+ \).

This definition indicates the tendency of transition of the imaginary root, as the delays change along a radial line passing through the origin.

**Main Lemma.** Along the two corresponding radial lines (i.e., lines passing through the origin) with a common slope \( g \in \mathbb{R}^+ \), one in the SDS and the other in the DS domain, the directional root tendencies of a purely imaginary roots remain identical.

**Proof.** A radial line is defined in SDS and DS as

\[
\tau_2 = g\tau_1
\]

(11)

where \( g \in \mathbb{R}^+ \) is the constant slope of the line. Substituting Eq. (11) into Eq. (1) yields

\[
\text{CE}_{\nu, \ell}(s, \tau_1) = a_0(s) + a_1(s)e^{-\ell \tau_1} + a_2(s)e^{-\ell \tau_2} + a_3(s)e^{-\ell \tau_1} + a_4(s)e^{-\ell \tau_2} = 0, \quad \tau_1 > 0
\]

(12)

which we name as directional characteristic equation. And it is, obviously, valid only on the line defined by Eq. (11) in both SDS and DS spaces. The DRT with respect to \( \tau_1 \) along the radial line can be found by using Eq. (10). Taking the total derivative of Eq. (12) with respect to \( \tau_1 \) yields

\[
\frac{\partial \text{CE}_{\nu, \ell}}{\partial \tau_1} \frac{ds}{d\tau_1} + \frac{\partial \text{CE}_{\nu, \ell}}{\partial \tau_2} \frac{ds}{d\tau_2} = 0
\]

(13)
which results in
\[
S_{s_1}^{\sigma_j} |_{s_\sigma_j,i} = - \frac{\partial CE_{g_T}}{\partial \tau_1} = s(H(s, \tau_1) - \tau_1)^{-1} |_{s_\sigma_j,i}
\]
(14)
where \( H(s, \tau_1) \) is given by the expression
\[
H(s, \tau_1) = \frac{(a_0 + a_1 e^{-s\tau_1} + a_2 e^{-2s\tau_1} + a_3 e^{-(1+g)s\tau_1} + \frac{a_4 e^{-s\tau_1} + g a_3 e^{-2s\tau_1} + (1+g) a_2 e^{-(1+g)s\tau_1)) \tau_1}{(a_1 e^{-s\tau_1} + g a_3 e^{-2s\tau_1} + (1+g) a_2 e^{-(1+g)s\tau_1)) \tau_1})\] and \( \alpha_i = da_i(s)/ds \).

The detailed derivation of Eq. (14) is provided in the Appendix. Simplifying \( P(s, T_1) \) using the properties of Eq. (4a), Eq. (19) can be reduced to a similar expression as Eq. (14).

\[
\frac{ds}{dT_i} |_{s_\sigma_j,i} = \frac{\partial H(s, \tau_1) - \tau_1}{m} |_{s_\sigma_j,i}
\]
(20)
where \( m = 2[(1-T_1)(1+T_1)]^{-1} \) for \( s = \alpha_i, j \) and \( H(s, \tau_1) \) is defined in Eq. (14). Using Eq. (20) in Eq. (10) for the DRT with respect to \( T_i \), one obtains
\[
DRT|_{s_\sigma_j,i}^{T_1} = \text{sgn} \left[ \text{Re} \left( S_{s_1}^{\sigma_j} |_{s_\sigma_j,i} \right) \right]
\]
(21)
which declares \( DRT|_{s_\sigma_j,i}^{T_1} = DRT|_{s_\sigma_j,i}^{T_1} \). Therefore, we conclude
\[
DRT|_{s_\sigma_j,i}^{T_1} = DRT|_{s_\sigma_j,i}^{T_1} = DRT|_{s_\sigma_j,i}^{T_1}
\]
(22)
with only one constraint as given in Eq. (11). That is the DRT’s are identical between SDS and DS so long as the variations are along the radial line segments.

Remark. According to the continuity argument [18,19], given any point \((r^0, \theta^0) \in \mathbb{R}^+$ in the domain of the delays, we can calculate NU of Eq. (1) on \( C^+ \) following the procedure below:

(a) Find the NU of Eq. (1) for the delay-free case when \((\tau_1, \tau_2, \tau_3) = (0, 0, 0)\);
(b) Generate the building and reflection hypercurves \( \psi_{SDS} \cup \chi_{DS} \) in SDS and the corresponding kernel and off-spring hypercurves \( \psi_{DS} \cup \chi_{DS} \) in DS;
(c) Draw two corresponding radial lines with a common slope \( g = (\tau_1^0/\tau_2^0) \), one in SDS and the other in DS; Then, find the intersection points \( \psi_{DS} \cup \chi_{DS} \) and the radial line, i.e., \( \{ (\tau_1^0, \tau_2^0) \in \text{DS} \}_0^{\infty} \); between origin and \( (\tau_0^0, \tau_2^0) \) in DS and locate the corresponding points \( \{ (\tau_0^0, \tau_2^0) \in \text{DS} \}_0^{\infty} \) in SDS;
(d) Calculate the DRT’s of each point in \( \{ (\tau_0^0, \tau_2^0) \in \text{DS} \}_0^{\infty} \) and use that information (as per main lemma) to find NU for the point \((\tau_0^0, \tau_2^0) \in \mathbb{R}^+$ in DS.

5 Illustrative Example

Next, an example case study is discussed to demonstrate this feature. For this, we take the system of Eq. (1) as
\[
\begin{align*}
CE_{g_T}(s, \tau_1, \tau_2) = s^2 + 7.1s + 21.1425 + (6s + 14.80)e^{-3s} \\
+ (2s + 7.3)e^{-7s} + 8e^{-2t_0+1}t_1 = 0
\end{align*}
\]
(23)
from Ref. [16]. By using the substitutions Eqs. (4a) and (5), the characteristic equation (23) can be rewritten as
\[
q(\alpha \omega, z_1, z_2) = (-z_1 - z_2 + 1 - z_1 z_2 - 1) \omega^2 + (-3.1z_1 - 11.1z_2 + 0.9z_2z_1 + 15.1) \omega + 5.6425z_1z_2 + 20.635z_1
\]
\[
- 7.0425z_2z_1 + 51.2425
\]
(24)
The real and imaginary parts of Eq. (24) must be zero, as
\[
\text{Re}[q(\alpha \omega, z_1, z_2)] = (-z_1 - z_2 - 1) \omega^2 + (-3.1z_1 - 11.1z_2) \omega - 7.0425z_2z_1 + 51.2425
\]
(24a)
\[
\text{Im}[q(\alpha \omega, z_1, z_2)] = (z_1 - z_2) \omega^2 + (0.9z_2z_1 + 15.1) \omega + 5.6425z_1 + 20.635z_2
\]
(24b)
In order for Eqs. (24a) and (24b) to have a common root, the following Sylvester resultant matrix should be singular:
This expression is the description of the kernel hypercurves in the SDS, namely the building curves. By sweeping \( v_k \) in the range of \([0, 2\pi]\), corresponding \( v_k \) values are found. By using the Property 2, the building block is obtained. Then, stacking the copies of the BB along the \((t_1, t_2)\) axes, the reflection curves are obtained in SDS (Fig. 1). Pointwise mapping from SDS \((t_1, t_2, \omega)\) to DS \((t_1, t_2)\) is performed using desirably dense frequency values, \( \omega \) (see Fig. 2).

Notice that in Fig. 1, the curves represent \((v_1, v_2, \omega)\) sets with both positive and negative \( \omega \) values. For back mapping to DS, we only consider positive \( \omega \) values. The lines in black in both figures represent \( g = 0.89 \) lines of Eq. (11). The points named as \( B_k \), \( k = 1, \ldots, 6 \) are the intersection points of \( g = 0.89 \) line and building and reflection curves in Fig. 1. They also correspond to the intersection points of \( g = 0.89 \) line in DS with the kernel and offspring hypercurves in the same space (see Fig. 2). Once we obtain all the \((t_1, t_2, \omega, v_1, v_2, T_1, T_2)\) values, the equivalency of the respective DRT \( DRT_{x=c_1}^{T_1} \) and DRT \( DRT_{x=c_2}^{T_1} \) are verified, as we display in Table 1. Please note again, the equivalency of identical directional root tendencies from Eq. (22) DRT \( DRT_{x=c_1}^{T_1} \) and DRT \( DRT_{x=c_2}^{T_1} \) is performed using desirably dense frequency values, \( \omega \) (see Fig. 2).

In conclusion, we can predict the stability transitions through these points while walking along the \( g = 0.89 \) radial line in SDS, without back-transforming to DS. One can easily determine that the NU = 0 at the origin (i.e., nondelayed case). Consequently the system remains stable for nonzero delay case, until a characteristic root pair appears on the imaginary axis and moves to the right half of the complex plane. CTCR declares the stable regions in DS which are gray-shaded ones in Fig. 2. Further advancing along the radial line, we recalculate the NU for each line segment using the DRT information (that is obtained from SDS) for each intersection point.

### Table 1 The corresponding parameters for each intersection point \( B_k \), \( k = 1, \ldots, 6 \)

<table>
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<tr>
<th></th>
<th>B1</th>
<th>B2</th>
<th>B3</th>
<th>B4</th>
<th>B5</th>
<th>B6</th>
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<td>1.59</td>
<td>5.26</td>
<td>3.63</td>
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<td>( t_2 )</td>
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<td>1.79</td>
<td>5.91</td>
<td>4.08</td>
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<td>( \omega )</td>
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<td>5.44</td>
<td>1.79</td>
<td>4.18</td>
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</tr>
<tr>
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<td>8.67</td>
<td>9.43</td>
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<tr>
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<td>-242</td>
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</tr>
<tr>
<td>( T_2 )</td>
<td>0.68</td>
<td>-2.0</td>
<td>-1.2</td>
<td>-0.84</td>
<td>-0.31</td>
<td>-0.34</td>
</tr>
<tr>
<td>DRT ( x=c_1 )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>DRT ( x=c_2 )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>DRT ( x=c_3 )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

![Fig. 1 SDS for example case study](image)

![Fig. 2 \((t_1, t_2)\) DS for the example case study](image)
6 Conclusions

For the LTI-MTDS, the equivalency of the local stability transition features along radial lines within the domains of SDS and DS is proven for the first time in literature. Although the proof is given here for the LTI systems with two delays, the treatment introduced in the paper could be extended to LTI systems with more than two delays. This property enables us to investigate the stability transition features within the SDS before going to the DS. Then, the number of unstable roots can be calculated in DS to reveal the stable regions. The main benefit of studying the stability transition features in SDS is that in SDS the potential stability changing curves are confined within the building blocks which are stacked identically along the axes. Therefore, the extraction of DRT information in SDS is more practical than in DS, which contains a shape distorting nonlinear mapping. Finally, an illustrative example is shown to verify this important property.

Appendix

In this section, some intermediate steps of the proof are provided. First, further details in the derivation of Eq. (14) are given. The terms \( \frac{\partial C_{E_{x,T}}}{\partial t_1} \) and \( \frac{\partial C_{E_{x,T}}}{\partial s} \) in Eq. (13) are calculated by partial differentiation of Eq. (12) with respect to \( t_1 \) and \( s \), respectively

\[
\frac{\partial C_{E_{x,T}}}{\partial t_1} = -sa_1e^{-\tau_2s} - gsa_2e^{-\tau_3s} - (1 + g)a_1e^{-(1 + g)\tau_1s}
\] (A1)

\[
\frac{\partial C_{E_{x,T}}}{\partial s} = a_0' + a_1'e^{-\tau_2s} - \tau_1(a_1' + a_2'e^{-\tau_3s} - \tau_1g_2e^{-\tau_4s})
+ a_3'e^{-(1 + g)\tau_1s} - \tau_1(1 + g)a_3e^{-(1 + g)\tau_1s}
\] (A2)

where \( a_j' = da_j(s)/ds \) for \( j = 0, 1, 2, \ldots \) Then, using Eqs. (A1) and (A2), \( ds/dt_1 \) is calculated as follows:

\[
\frac{ds}{dt_1} = \frac{\partial C_{E_{x,T}}}{\partial t_1} = s \left[ a_0' + a_1'e^{-\tau_2s} + a_2'e^{-\tau_3s} + a_3'e^{-(1 + g)\tau_1s} - \tau_1 \right]^{-1}
\] (A3)

Using Eq. (A3), one can obtain Eq. (14). Next, some additional steps for the derivation of Eq. (19) are provided. Similarly, the terms \( \frac{\partial C_{E_{x,T}}}{\partial t_1} \) and \( \frac{\partial C_{E_{x,T}}}{\partial s} \) in Eq. (18) are calculated using partial differentiation of Eq. (16) with respect to \( t_1 \) and \( s \), respectively.

\[
\frac{\partial C_{E_{x,T}}}{\partial t_1} = a_1 - s(1 + T_1s) - s(1 - T_1s) + a_2\frac{-sg(1 - T_1s)^{g-1}(1 + T_1s)^g - sg(1 + T_1s)^{g-1}(1 - T_1s)^g}{(1 + T_1s)^{g+2}}
+ a_3\frac{s(1 - T_1s)^{g+1} - (1 + T_1s)^{g+1} - s(1 + T_1s)^{g+1} - (1 - T_1s)^{g+1}}{(1 + T_1s)^{g+2}}
\] (A4)

\[
\frac{\partial C_{E_{x,T}}}{\partial s} = a_0' + a_1'\frac{(1 - T_1s)^{g}}{1 + T_1s} + a_2'\frac{(1 - T_1s)^{g}}{1 + T_1s} + a_3'\frac{(1 - T_1s)^{g+1}}{1 + T_1s}
+ a_4\frac{\frac{-T_1s(1 - T_1s)^{g} - T_1s(1 + T_1s)^{g}}{(1 + T_1s)^{g+2}}}{(1 + T_1s)^{g+2}}
\] (A5)

where \( a_j' = da_j(s)/ds \) for \( j = 0, 1, 2, \ldots \) Then, using Eqs. (A4) and (A5), \( ds/dt_1 \) is calculated as follows and can be used to obtain Eq. (19):

\[
\frac{ds}{dt_1} = \frac{\partial C_{E_{x,T}}}{\partial t_1} = \frac{\partial C_{E_{x,T}}}{\partial s} = s \left[ a_0' + a_1'\frac{1 - T_1s}{1 + T_1s} + a_2'\frac{1 - T_1s}{1 + T_1s} + a_3'\frac{1 - T_1s}{1 + T_1s}
+ a_4\frac{\frac{1 - T_1s^{g+1} - T_1s^{g+1}}{(1 + T_1s)^{g+2}}}{(1 + T_1s)^{g+2}} - T_1 \right]^{-1}
\] (A6)
References