Introduction

Knowledge of the apparent radiative properties of the semitransparent hemispherical shell placed on an opaque surface is of fundamental interest and is also important to a number of applications in materials processing and manufacturing. One particular example is glass manufacturing. During glass melting, the raw materials (also called batch) are fed into the glass melting furnace and float on the free surface of the melt as islands until they are completely melted. The chemical reactions and moisture evaporation take place in the batch during its heating, fusion, and melting, and it results in the large number of bubbles, many of which are trapped on the surface of the batch and produce batch foam. These foams scatter the incident thermal radiation from the combustion space and provide significant resistance to radiant heating and melting of the batch [1], thereby diminishing the energy efficiency of the glass melting process. Unlike the foams formed on the free surface of the molten glass that possess a structure of multiple layers of spherical or polyhedral bubbles separated by the liquid lamella, the batch foam is a collection of often non-overlapping individual gas bubbles that only partially emerge from the batch Fig. 1(a). Hence, a thin hemispherical shell placed on top of the opaque surface as shown in Fig. 1(b) can be considered as a fairly good simplified representation of the batch foam. Due to significant structural differences, the radiative transfer models developed for conventional glass foams [2,3] cannot be used for the batch foams, and thus prediction of the apparent radiative characteristics of batch foams is the subject of this paper.

In addition to glass manufacturing, understanding of the local radiant heating of a semitransparent hemispherical shell is needed for optimization of the novel process for fabrication of hollow perforated beads for acoustic and thermal insulation [4]. In this process, the gas is injected from a small orifice into the molten droplet placed on the substrate to form a bubble surrounded by the thin lamella of the liquid material, which is then solidified by rapid cooling through the substrate to form a solid hemispherical bead. The parameters (e.g., the shell thickness and structure) and quality of the final product are defined by the relative magnitude of the local heating rates due to collimated thermal radiation incident from the top and the local cooling rates needed for solidification.

The objective of this paper is to develop a complete fundamental understanding and analytical tools for predicting radiative characteristics and heat transfer in semitransparent hemispherical shell suspended in the radiatively non-participating environment. In our previous work [7], we developed approximate, closed-form analytical expressions for the total apparent reflectance, transmittance, and absorptance of such a shell and validated our analysis by using numerical ray tracing solution of the problem. The focus of this work is on the local, spatially resolved radiation characteristics of the semitransparent hemispherical shell, such as the scattering phase function and the local volumetric rate of radiant heating. These properties are critically important to optical, spectroscopic studies of the foam morphology (i.e., phase function) and mechanisms of the foam rapture due to the local thermally induced instabilities and stress (i.e., volumetric heating rate). To further elucidate the results, the comparison is made of the total apparent transmittance of the hemispherical shell to that for a plane parallel layer of the same semitransparent material [5,6].

Analysis

The schematic of the idealized physical arrangement and the coordinate system are shown in Fig. 1(b). Several simplifying assumptions are made to make the problem amenable to analytical solution.

Assumptions.

1. Incident radiation is normal to the substrate, collimated and its intensity is constant.
2. The surrounding medium is radiatively nonparticipating with the indices of refraction equal to unity.
3. The shell is made of a homogeneous, gray, and weakly absorbing material characterized by the constant complex index of refraction \( n_i = n_2 - ik_2 \).
4. The shell is treated as a “cold” medium, i.e., self-emission of radiation is neglected.
5. The diameter and the thickness of the shell are much greater than the wavelength of the incident radiation, i.e., diffraction and interference effects are not considered.

6. Polarization effects are not considered.

7. All interfaces are optically smooth and obey geometric optic principles.

**Closed-Form Solution and Ray Tracing Algorithm.** An analytical ray tracing procedure is used and results in the analytical expressions for the apparent radiative properties of the thin hemispherical shell, owing to the existence of a recursive pattern in the physical domain. The ray tracing method follows the optical path of a single energy bundle of incident radiation as it undergoes a repeating pattern of multiple reflections within the shell (Fig. 2). Here we only present the summary of formulas needed for performing the calculations, and an interested reader is referred to the paper by Fan and Fedorov [7] for the detailed discussion of the model development. Specifically, the total apparent reflectance (R), transmittance (T), and absorbance (A) for any energy bundle are given by the following expressions:

\[
R = \frac{\rho_{12} + (1 - \rho_{12}) \rho_{23} \tau^2}{1 - \rho_{12} \rho_{23} \tau^2} \left(1 - \rho_{12} \rho_{23} \tau^2\right) \left(1 - \tau\right) (1 - \rho_{12}^2)(1 + \rho_{23}^2 \tau^2)
\]

(1)

\[
T = \frac{\rho_{12} (1 - \rho_{12}) \rho_{23} \tau^2}{1 - \rho_{12} \rho_{23} \tau^2}
\]

(2)

\[
A = \frac{(1 - \tau)(1 - \rho_{12}^2)(1 + \rho_{23}^2 \tau^2)}{1 - \rho_{12} \rho_{23} \tau^2}
\]

(3)

respectively, where

In Eqs. (1–6), \(\rho_{12}\) is the reflectivity of the surrounding/shell interface at the point where a given energy bundle is incident on the top surface of the shell, \(\rho_{12} = \rho_{21}\) is the reflectivity of shell/surrounding interface, \(\rho_{23}\) is the reflectivity of shell/inside gas interface, and \(\tau\) is the transmissivity associated with a single pass of the energy bundle across the shell thickness and equal to exp \([-\kappa (r_2 - r_1) \cos \theta_2 \cos \theta_3]\). In the last expression, \(\kappa\) is an absorption coefficient of the shell material, which relates to its absorptive index \(k_2\) through the relationship \(\kappa = 4\pi k_2/\lambda_0\).

In the case of the radially uniform irradiation, the discrete energy bundles possess the same energy and located equidistantly in the radial direction. Other kinds of energy distribution of incident radiation can also be investigated within the deterministic ray tracing algorithm developed by simply adjusting the distance and/or energy content of the ray bundles. Application of the Snell’s law and a simple geometric analysis [7] result in the following recursive relations for the angle of initial incidence (\(\theta^{(1)}_i\)) and for the angles of consecutive reflections within the shell (\(\theta^{(i)}_1, \theta^{(i)}_2, \theta^{(i)}_3, \theta^{(i)}_4\)) for the \(i\)th energy bundle (Fig. 2):

\[
\theta^{(1)}_i = \sin^{-1}\left(\frac{1 - \rho_{12}^i (1 - \rho_{23}^i) \tau^2}{1 - \rho_{12}^i \rho_{23}^i \tau^2}\right)
\]

(4)

\[
\theta^{(i)}_2 = \sin^{-1}\left(\frac{r_2}{r_1} \sin \theta^{(i)}_1\right),
\]

(5)

\[
\theta^{(i)}_3 = \sin^{-1}\left(\frac{r_1}{r_2} \sin \theta^{(i)}_2\right),
\]

(6)

\[
\theta^{(i)}_4 = \sin^{-1}\left(\frac{n_2}{n_1} \sin \theta^{(i)}_3\right),
\]

where \(N\) is the total number of bundles considered, and \(i\) is the bundle index starting from 0 at the centerline of the hemisphere. Finally, in the case of radially uniform incidence, the total apparent absorbance, reflectance and transmittance are computed by summing up the contributions from all energy bundles, i.e.,

\[
A_{total} = \frac{\sum_{i=1}^{N} A^{(i)}}{N}, \quad R_{total} = \frac{\sum_{i=1}^{N} R^{(i)}}{N}, \quad T_{total} = \frac{\sum_{i=1}^{N} T^{(i)}}{N}
\]

(8)

respectively. To validate the analytical expressions for radiative properties given by Eq. (8), a numerical ray tracing algorithm [7] was developed and applied to the problem in hand. The comparison of the results [7] indicates that the approximate analytical results agree within 3.7 percent with the results of numerical simulations as long as the thickness of shell does not exceed 5 percent of the shell radius.

A general three-dimensional ray tracing technique [8] uses analytical representation of the geometry in a three-dimensional vector space to find the intersection points of the ray bundle with the discrete elements it crosses. This approach requires some additional modifications if the solution of the intersection equation is not unique. Our simplified approach considers two concentric hemispherical interfaces, defined by the inner and the outer radii \(r_1\) and \(r_2\), respectively. Because of the azimuthal independence, instead of solving the bundle-interface intersection equation in the vector form, the intersection points on the two-dimensional plane can be found analytically (using Eq. (13) from [7]), at least in the case of normal incidence of collimated radiation, thereby reducing the computational complexity associated with implementation of the ray-tracing algorithm. Also, if the angle of incidence for a given ray bundle is larger than the critical angle, the reflectivity is
assigned to a unit value. Furthermore, the substrate is assumed to be “cold” and black so that no energy is reflected back or emitted towards the shell by the substrate, thereby significantly reducing the complexity of the shell-substrate interactions and the computational procedure.

In the presented ray tracing algorithm, any discrete element inside the computational domain can have infinitely many possibilities of incoming bundle directions that result in crossing its boundaries, and the intensity of each such energy bundle decays along the traveled path according to the Beer’s Law, \( I(r,s) = I_0 \exp(-s f_{ij} \kappa ds) \). In order to compute the local volumetric heating rate at every point within the shell, an average intensity of radiation needs to be computed by using the local radiant energy conservation and calculating the energy fraction deposited on each element from every direction. That is [7]:

\[
I_{avg}(r,s) = \sum_{ij} \frac{I(r,s)[1 - \exp(-s f_{ij} \kappa ds)]dx}{\kappa dA d\Omega}
\]

(9)

where \( r \) denotes the element location, \( s \) is the vector in the direction of propagation of the energy bundle, \( s' \) is the distance traveled by the energy bundle through a given discrete element (it is different for different bundle directions \( s \)), \( dx \) is a spacing between the bundles of incident collimated radiation (it is inversely proportional to a total number of energy bundles \( N \) used in calculations), \( dA \) is the cross-sectional projection area of a given element in respect to the direction of propagation of a given energy bundle, and \( d\Omega \) is the incremental solid angle for a given energy bundle. Using an averaged local intensity, the local volumetric heating rate due to thermal radiation absorbed by an element of the computational domain is given by

\[
\bar{Q}_{\text{rad}}(r) = -\nabla \cdot q_{\text{rad}}(r,s) = \kappa \int d\Omega = \kappa \sum_{k=1}^{m} I_{avg}(k) \Delta \varphi(k)
\]

(10)

where \( \Delta \varphi \) is the increment angle in polar direction from a spherical symmetry plane. The total apparent absorbance of the hemispherical shell is expressed as a ratio of the energy absorbed by all elements in the computational domain and the incident energy [7], i.e.,

\[
A_{\text{total}} = \frac{\sum_{ij} \kappa (\sum_{i} I_{avg}(k) \Delta \varphi(k)) \Delta A_{ij}}{\kappa_{\text{rad}}}
\]

(11)

The three summations in Eq. (11) are needed to account for the fact that each element in the two-dimensional computational domain is defined by two independent indices \( (i,j) \) and an additional index \( (k) \) is used to trace directional dependence of the local average intensity field for each \( ij \)th element.

**Scattering Phase Function.** When the incident energy bundle hits the hemispherical shell, part of its energy is reflected immediately by the outer interface and the remaining part penetrates into the semitransparent shell. After that, a given energy bundle undergoes multiple internal reflections and during this process some of its energy is absorbed and some is leaked out (redirected to different scattering directions) through both inner and outer interfaces. In order to construct the scattering phase function, we consider the first reflection from the outer interface and the consecutive internal reflections accompanied by the radiation energy leakage through the shell interfaces as separate events. These events are characterized by the radiant energy redirection or scattering at an angle \( \beta \) (see Fig. 2) and by the fraction of the incident radiation energy leaving the shell due to the first reflection (\( \Lambda^1_\beta \)) or the radiant energy leakage after \( i \)th internal reflection through the outer (\( \Lambda^o_\beta \)) and inner (\( \Lambda^i_\beta \)) interfaces. In other words, we decompose the entire trajectory of any given energy bundle into the elements with well defined scattering characteristics (\( \beta, \Lambda \)), which can be readily computed analytically or numerically by tracing the bundle propagation. Eventually, we combine all contributions (\( \Lambda^1_\beta, \Lambda^o_\beta, \text{ and } \Lambda^i_\beta \)) for any given scattering direction (\( \beta \)) and rescale them in order to obtain a normalized scattering phase function. The analytical developments and the results shown next decisively indicate that the scattering phase function of the semitransparent hemispherical shell is well approximated by the popular Henyey–Greenstein phase function [5,6].

As shown by Fan and Fedorov [7], the recursive pattern traced by the energy bundle as it travels within the shell (see Fig. 2) is defined by the angles \( \theta_1 \) through \( \theta_i \) as well as by the reflectivities \( \rho_{ij}, \rho_{12}, \rho_{23} \) and transmissivity \( \tau \) that all depend on the angle of incidence \( \theta_1 \) only. Hence, by using Eqs. (4) and (7), the fraction of the incident radiation energy scattered by the shell due to the first reflection (\( \Lambda^1_\beta \)) can be expressed by a function of the scattering angle \( \beta \) as follows:

\[
\Lambda^1_\beta = \frac{1}{2} \left\{ \frac{\tan^2 \left( \frac{\pi - \beta}{2} \right) - \sin^2 \left( \frac{n_1}{n_2} \sin \left( \frac{\pi}{2} - \beta \right) \right)} {\tan^2 \left( \frac{\pi - \beta}{2} \right) + \sin^2 \left( \frac{n_1}{n_2} \sin \left( \frac{\pi}{2} - \beta \right) \right)} \right\}
\]

(12)

by noting that the angle of incidence of a given energy bundle \( \theta_1 \) and the scattering angle \( \beta \) are related through

\[
\theta_1 = \left( \frac{\pi}{2} - \beta \right)
\]

(13)

Analogously, due to repeating pattern of internal reflections created within the shell (Fig. 2), the fraction of the incident radiation energy leaving the shell by leakage after \( \text{th} \) internal reflection through the outer interface is given by

\[
\Lambda^o_i(\theta_1) = (1 - \rho_{12}^i(\theta_1))(1 - \rho_{12}^i(\theta_1)) \times (\rho_{23}^i(\theta_1))^{i-2} (\rho_{23}^i(\theta_1))^{i-1} (\tau(\theta_1))^{2i-2}, \quad i = 2 \sim \infty
\]

(14)

where the parameter \( \theta_1 \), in Eq. (14) is a function of the scattering angle \( \beta \),

\[
\theta_1 = \frac{1}{2} \left\{ \pi - \beta - (\pi/2 - 1) \left[ \sin^{-1} \left( \frac{r_2 n_1}{r_1 n_2} \sin \theta_1 \right) \right. \right. \\
- \left. \sin^{-1} \left( \frac{n_2}{n_1} \sin \theta_1 \right) \right] \right\}, \quad i = 2 \sim \infty
\]

(15)

Equation (15) is a nonlinear and implicit equation with respect to \( \theta_1 \) and, thus, requires an iterative numerical solution with the careful selection of an initial guess to achieve fast convergence. In general, Eq. (15) may have infinitely many solutions, however, the physically plausible solution is unique: that is the parameter \( \theta_1 \) must result in the scattering angle \( \beta \) that is between 0 and \( \pi \). By combining the contribution from the radiation reflected upon the initial incidence on the shell and leaked through the outer interface, the scattering distribution function resulting from the radiation/outer interface interactions can be determined,

\[
\Lambda^r_\beta = \Lambda^1_\beta + \sum_{i=2}^{\infty} \Lambda^o_i(\theta_1(\beta)), \quad 0 \leq \beta \leq \pi
\]

(16)

The same procedure can be also applied to the radiant energy escaping the shell through the inner interface, which results in the forward scattering only. Specifically, the fraction of the incident radiation energy leaving the shell due to leakage after \( 2^{\text{th}} \) internal reflection through the inner interface is given by (Fig. 2).
where the parameter $\theta_i$ depends on the scattering angle $\beta$ and satisfies the following implicit nonlinear equation:

$$
\theta_i = -\beta + \sin^{-1}\left(\frac{n_2}{n_1} \sin \theta_i\right) - (2i-1) \left[ \sin^{-1}\left(\frac{n_2}{n_1} \sin \theta_i\right) - \sin^{-1}\left(\frac{n_1}{n_2} \sin \theta_i\right) \right], \quad i = 1, \ldots, \infty
$$

The scattering distribution function resulting from the radiation/inner interface interactions is for the forward direction only, and it accounts for the radiant energy from all incident energy bundles leaving the shell through the inner interface at a given scattering angle $\beta$:

$$
\Lambda^L(\beta) = \sum_{i=1}^{\infty} \Lambda_i^L(\theta_i(\beta)) + \sum_{i=1}^{\infty} \Lambda_i^T(\theta_i(-\beta)), \quad 0 \leq \beta \leq \pi
$$

Note that the negative scattering angles $\beta$ are in principle possible, albeit hardly realized under practical conditions (see Fig. 2 for the coordinate system). However, for the sake of generality, this fact is elegantly incorporated into the scattering distribution function, Eq. (19), by invoking the symmetry of the problem in hand.

We now can combine both components of the total scattering distribution function, given by Eqs. (16) and (19), and, after normalization (i.e., $\int \Phi \ d\Omega = 4\pi$), the scattering phase function for the semitransparent hemispherical shell is given by

$$
\Phi(\beta) = \frac{4\pi (\Lambda^R(\beta) + \Lambda^T(\beta))}{\int_{4\pi} \Lambda^R(\beta) + \Lambda^T(\beta) \, d\Omega}, \quad 0 \leq \beta \leq \pi
$$

It is interesting to note that for an extremely thin hemispherical shell (i.e., relative to its radius), $\theta_k = \theta_1$, $\theta_3 = \theta_1$, and $r_1 = r_2$. Thus, Eqs. (16) and (19) can be reduced to Eqs. (1) and (2) with the following dependence on the scattering angle,

$$
\Lambda^R(\beta) = \frac{p_{12} + (1 - p_{12})p_{23} \tau}{1 - p_{12}p_{23} \tau}, \quad 0 \leq \beta = \pi - 2 \theta_k \leq \pi
$$

$$
\Lambda^T(\beta) = \frac{(1 - p_{12})(1 - p_{23}) \tau}{1 - p_{12}p_{23} \tau}, \quad \beta = 0
$$

Results and Discussion

The numerical simulations have been performed for different element sizes and number of energy bundles. The validity of the numerical ray-tracing algorithm has been carefully checked by ensuring that the results are independent of the number and the size of discrete elements in the computational domain and of the total number of energy bundles used in simulations. As it follows from Table 1(a), use of 1000 energy bundles produces fully converged results with the uncertainty in the total apparent reflectance and transmittance less than 0.1 percent as compared to much greater number (5000) of energy bundles. The sensitivity of the simulation results in respect to the number of discrete elements used is reported in Table 1(b). The dimensionless maximum local volumetric heating rate is chosen for the comparison of results because it has the greatest sensitivity to the number of elements used in simulations. It is clear that full convergence (with the uncertainty less than 0.5 percent) is achieved when 1000 energy bundles and $8 \times 36$ elements are employed. Further increase in the number of elements and energy bundles results in a very slightly improved accuracy and thus is not warranted. Thus, all simulation results reported hereafter have been obtained by using 1000 energy bundles and $8 \times 36$ elements. In addition, the overall energy conservation has always been achieved by ensuring that the sum of the total apparent absorptance, transmittance, and reflectance is equal to unity.

The focus of this work is on the local volumetric radiative heating and resolving angular dependence of radiation scattering by the hemispherical semitransparent shell. However, to provide a suitable introduction to the problem and facilitate the result interpretation, we first briefly summarize the results on the total apparent radiative properties of the shell and validation of the analytical model, which are discussed in detail in [7]. Figure 3 shows the comparison of the results of the total apparent absorptance ($A_{total}$) and reflectance ($R_{total}$) obtained using a closed-form, but approximate analytical solution of the problem and those representing an "exact" solution of the problem using the numerical ray-tracing technique. Clearly, the analytical and numerical models produce essentially the same results for the total apparent absorptance, whereas their predictions of the total apparent reflectance agree very well (i.e., within 1 percent) in the limit of the very thin shell (i.e., when $r_1/r_2 \rightarrow 1$). The main reasons for such a discrepancy are an approximation involved in using an infinite rather than a finite algebraic series representation for the total apparent reflectance as well as the trigonometric relationships [Eq. (7)] that are used for tracking the reflection angles, and this is discussed in greater detail in [7]. In short, the closed-form analytical solution of the problem presented here produces the results accurate within 3 percent only if the ratio of the shell inner and outer radii is

![Fig. 3 Total apparent absorptance and reflectance as functions of the normalized shell thickness ($n_2=1.5, \kappa=0.1$)](image-url)
The ratio \( r_1/r_2 \) should be 0.95. Otherwise, one should resort to the numerical ray tracing method in order to obtain an accurate prediction of the reflectance by the hemispherical shell.

To evaluate the resistance to radiation propagation introduced by the hemispherical shell, we compare the total apparent transmittance of the hemispherical shell and that of a plane parallel layer of the same thickness and made of the same semitransparent material. The apparent radiative properties for the planar configuration are readily available from the radiation textbooks [5,6]. The results presented in Fig. 4 clearly indicate that the hemispherical shell impedes much more significantly the propagation of the radiation relative to the planar layer, and this difference increases sharply as the thickness of the shell/layer increases (i.e., \( r_1/r_2 \) decreases). This fundamental result has very important practical implications. For example, in glass melting it permits to quantify the implications. For example, in glass melting it permits to quantify the resistance to radiant heating provided by the batch foams, thereby helping optimize the process and operating conditions.

In addition to the exact calculation of the total apparent radiative properties of the shell, the numerical ray tracing allows one to predict the local radiation intensity field at any point in the shell [Eq. (9)] and, thus, the local volumetric radiant heating rate [Eq. (10)]. The latter is illustrated in Fig. 5 by showing the isolines (contours) of the local heating rate normalized by the total irradiation \( r_1 I_0 \). As expected, the strongest heating spot \( Q_{\max} \) is predicted in the area within the shell, which is located near the point at the inner shell interface when the onset of the total internal reflection occurs. This is because the intensity of radiation concentrated in this region of the shell is the largest as it is not reduced anymore by the radiation leakage through the inner interface. It is also observed that the very top portion of the shell, where the incident radiation impinges essentially normal to the outer shell interface, is heated much more uniformly than the region near the shell circumference, where the local heating is defined by rather intricate superposition of radiation intensities resulting from multiple reflection within the shell. Once again, this fundamental result is of the great interest to a number of materials processing applications, where one is trying to heat the material properties are sensitive to the local temperature and, thus, the local volumetric radiant heating rate is readily available from the radiation textbooks [5,6]. Otherwise, one should resort to the numerical ray tracing method in order to obtain an accurate prediction of the reflectance by the hemispherical shell.

In Fig. 5, it is observed that the very top portion of the shell, where the incident radiation impinges essentially normal to the outer shell interface, is heated much more uniformly than the region near the shell circumference, where the local heating is defined by rather intricate superposition of radiation intensities resulting from multiple reflection within the shell. Once again, this fundamental result is of the great interest to a number of materials processing applications, where one is trying to heat the material properties are sensitive to the local temperature and have to be carefully controlled [4], one could use the fundamental understanding developed and the model as a tool to design the radiant heating system capable of producing uniform heating of all parts of the shell.

Figure 6 shows the normalized scattering phase function of the semitransparent hemispherical shell (solid line), and its components due to direct reflection of incident radiation from the outer interface (dashed line) and interactions of the radiation trapped within the shell with the outer (dash-dotted line) and inner (dash-double-dotted line) shell interfaces as given by Eqs. (20), (12), (14), and (17), respectively. Note that the component of the scattering phase function due to direct reflection of radiation (dashed line) represents a typical scattering phase function for a large spherical particle, whose scattering characteristics depend on the specular reflectivity of interface only [6]. The second contribution shown as the dash-dotted line is due to radiation leakage through the outer interface, and it exhibits a strong increase beginning at the scattering angle that corresponds to an onset of the total internal reflection regime at the inner shell interface. After that, this component of the phase functions remains relatively constant, providing the major contribution to the forward scattering energy with the directional \( \cos \beta \) larger than 0.5. Finally, as expected, the component of the scattering phase function due to radiation/inner interface interactions (dash-double-dotted line) has a single strong spike in forward direction characterized by the scattering angle

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**Fig. 4** Comparison of the total apparent transmittance of the planar layer and hemispherical shell \((n_2 = 1.5, \kappa = 0.1)\)

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**Fig. 6** Normalized scattering phase function and its three components \((\kappa = 0.1, r_1/r_2 = 0.8, n_2 = 1.5)\)
essentially equal to zero. As the shell thickness becomes thinner (i.e., $r_1/r_2$ decreases), the last two components of the scattering phase function shift more toward the forward direction. The total scattering phase function combines all three components and is depicted by the dashed line in Fig. 7.

Our calculations shown in Figs. 6 and 7 indicate that the scattering of normally incident collimated radiation by the hemispherical shell favors strongly the forward direction. Thus, in order to achieve some generality in presenting results and to permit comparison with other previously studied physical situations, we tried to express the scattering phase function in terms of the popular Henyey-Greenstein phase function [5,6].

$$\Phi(\beta) = \frac{1 - g^2}{(1 + g^2 - 2g \cos \beta)^{1/2}}$$  \hspace{1cm} (23)

with the dimensionless asymmetry factor $g$ given by

$$g = \frac{1}{4\pi} \int_4^\infty \Phi(\beta) \cos \beta \, d\Omega$$  \hspace{1cm} (24)

In Fig. 7, the Henyey-Greenstein (HG) phase function is used to approximate the predicted scattering phase function for the hemispherical shell, and the asymmetry factor $g$ is calculated for three different shell thickness. In the case of intermediate shell thickness ($r_1/r_2 = 0.9$), the HG approximation is quite accurate across the entire range of the scattering angle, except for the strong peak at $\cos \beta = 0.75$, corresponding to an onset of the total internal reflection at the inner shell interface. In the case of a thicker shell ($r_1/r_2 = 0.8$), the HG function provides satisfactory approximation for $\cos \beta = 0.50$, but overpredicts the probability of scattering in the forward direction. On the contrary, for the very thin shell with $r_1/r_2 = 0.99$, the forward scattering is underpredicted by the HG approximation.

Conclusions

This paper presents a closed-form analytical solution of the problem involving thermal radiation interactions with a hemispherical, semitransparent shell placed on the cold black surface. It permits calculation of the total apparent radiative properties of the shell, its scattering phase function, and the local volumetric radiant heating of the shell under conditions of the normal incidence of collimated radiation. The predictions of the analytical model have been carefully validated by comparing with the "exact" solution for the benchmark case obtained by using the numerical ray tracing technique. The comparison of the results indicate that the approximate analytical results agree within 3.7 percent with the results of numerical simulations as long as the thickness of shell does not exceed 5 percent of the shell radius.

To elucidate the relative magnitude of the resistance to radiation propagation introduced by the shell, the comparison is made of the total apparent transmittance of the hemispherical shell to that of the plane parallel layer of semitransparent material. The results indicate that the hemispherical shell impedes significantly the propagation of the radiation as compared to the plane parallel layer of the same thickness, and this difference increases sharply with an increase in thickness of the shell/layer.

Analysis of the local volumetric radiant heating indicates a relatively uniform heating of the shell near its top, where radiation is incident almost normal to the shell surface. At the same time, a significant non-uniformity in local heating is observed near the circumference of the shell with the maximum in the part of shell located near the point of an onset of the total internal reflection at the inner shell interface.

Scattering of radiation by the shell has also been analyzed in great details, and the key components of the scattering phase function has been identified and quantified. The results of calculations indicate a strongly forward character of the radiation scattering by the shell, and the scattering phase function can be sufficiently well represented by the Henyey-Greenstein approximation.

The results of this study not only provide an insight into the fundamentals of radiation/shell interactions but also contribute to improving a number of important applications ranging from metallurgical slag foaming to batch foams in glass melting to hollow bead fabrication.

Nomenclature

$A =$ absorptance  
$dA =$ cross-sectional projection area  
$I_0 =$ intensity of incident collimated radiation  
$k =$ absorptive index  
$N =$ total number of energy bundles  
$n =$ refractive index  
$\dot{Q}_{\text{rad}} =$ volumetric heating rate  
$q_{\text{rad}} =$ radiation heat flux  
$R =$ reflectance  
$r =$ radius  
$r_1, r_2 =$ radii of the inner and outer hemispheres, respectively  
$r =$ spatial position vector  
$s =$ distance traveled by an energy bundle  
$s =$ direction of ray propagation  
$T =$ transmittance

Greek Symbols

$\beta =$ scattering angle  
$\Omega =$ solid angle  
$\Lambda^s =$ fraction of incident radiation scattered by first reflection of the energy bundle

Fig. 7 Normalized scattering phase function and Henyey-Greenstein approximation ($\kappa = 0.1, n_2 = 1.5$)

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\( \Lambda_i^R \) = fraction of incident radiation scattered (i.e., leaving the shell) through the outer shell surface after \( i \)th refractive pass of the energy bundle

\( \Lambda_i^T \) = fraction of incident radiation scattered (i.e., leaving the shell) through the inner shell surface after \( i \)th refractive pass of the energy bundle

\( \Phi \) = scattering phase function

\( \varphi \) = polar angle

\( \bar{\kappa} \) = dimensionless absorption coefficient \( (\bar{\kappa} = \kappa r_2) \)

\( \rho \) = reflectivity of the interface

\( \tau \) = transmissivity

**Superscript**

\( (i) \) = refers to the bundle index

**Subscripts**

\( \text{total} \) = refers to total apparent properties

1 = refers to surrounding medium

2 = refers to shell material

3 = refers to medium inside the shell

**References**


