ECE257 Numerical Methods and Scientific Computing

Ordinary Differential Equations
Today’s class:

• Stiffness
• Multistep Methods
Stiff Equations

• Stiffness occurs in a problem where two or more independent variables change at very different scales

• Example: \( \frac{dy}{dt} = -1000y + 3000 - 2000e^{-t} \) \( y(0) = 0 \)

• Analytical Solution:
  \[ y = 3 - 0.998e^{-1000t} - 2.002e^{-t} \]
Stiff Equations

\[ y = 3 - 0.998e^{-1000t} - 2.002e^{-t} \]

- As \( t \) moves away from zero, the solution settles to \( y = 3 - 2.002e^{-t} \)
- You still need to account for the \( e^{-1000t} \) term even after you move away from zero
- If you don’t do so, the system will be unstable
- Adaptive methods will not work
Stiff Equations

\[ \frac{dy}{dt} = -ay \]

\[ y_{i+1} = y_i + \frac{dy_i}{dt} h \]

\[ y_{i+1} = y_i - ay_i h \]

\[ y_{i+1} = y_i (1 - ah) \]

For the system to be stable \(|1-ah|\) must be less than 1, or \(h < \frac{2}{a}\)

\[ \frac{dy}{dt} = -1000y + 3000 - 2000e^{-t} \]

\[ \Rightarrow h < \frac{2}{1000} \]
Stability

• A numerical method is **stable** if errors occurring at one stage of the process do not tend to be **magnified** at later stages.

• A numerical method is **unstable** if errors occurring at one stage of the process tend to be **magnified** at later stages.

• Numerical methods which may be unstable should be carefully dealt with or avoided.

• In general, the stability of a numerical scheme depends on the step size. Usually, **large** step sizes lead to unstable solutions.

• Implicit methods are in general more stable than explicit methods.
Stiff Equations

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## Stiff Equations

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<thead>
<tr>
<th>h</th>
<th>y(0.01)</th>
<th>y(0.1)</th>
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<tr>
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<td>4.346x10^{24}</td>
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<tr>
<td>0.001</td>
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Stiffness

- Stiffness can also cause problems even when the system is stable

- Example: \[ \frac{d^2 y}{dx^2} = 100y \quad y(0) = 1, y'(0) = -10 \]

\[ \frac{dy_1}{dx} = y_2 \]
\[ \frac{dy_2}{dx} = 100y_1 \]

Analytical Solution \[ y_1 = e^{-10x}, y_2 = -10e^{-10x} \]
\[ y(0.5) = 0.00674 \]
## Stiffness

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## Stiffness

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</table>
Stiffness

• Roundoff or truncation error can cause inaccuracies in initial values and lead us to follow an inaccurate solution

\[ \frac{d^2y}{dx^2} = 100y \quad y(0) = 1, y'(0) = -10 \]

\[ y = Ae^{-10x} + Be^{10x} \]

• As x gets large, the \( e^{10x} \) dominates and because of accumulated errors as we integrate from the initial values, we will introduce large errors
Stiffness

- Implicit (backward) differencing (Euler’s)

\[ y_{i+1} = y_i + \frac{dy_{i+1}}{dt} h \]
\[ y_{i+1} = y_i - ay_{i+1}h \]
\[ y_{i+1} = \frac{y_i}{(1 + ah)} \]

- This method is unconditionally stable

- True for linear ODEs
Stiffness

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Stiffness

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Multi-step Methods

- R- K methods use only one previous approximation \( (y_i) \), known as one-step method.
- Multi-step methods use several previous points \( (y_i, y_{i-1}, \ldots) \) - explicit \( (b_0 = 0) \) & implicit methods.

\[
y_{i+1} = a_1 y_i + a_2 y_{i-1} + \ldots + h \cdot [ b_0 f(x_{i+1}, y_{i+1}) + b_1 f(x_i, y_i) + b_2 f(x_{i-1}, y_{i-1}) + \ldots ]
\]
- Open & closed formula, non-self start Huen method
- Adams-Bashforth Methods (explicit methods \( b_0 = 0 \))
- Adams-Moulton Methods (implicit methods)
- Predictor-Corrector Methods
Multi-step methods

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Non-Self-Starting Heun Method

- **Heun Method**
  \[ y_{i+1}^0 = y_i + f(x_i, y_i)h \]
  \[ y_{i+1} = y_i + h \left( \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)}{2} \right) \]

- **Instead of using a** \( O(h^2) \) **forward Euler method as the predictor, use an** \( O(h^3) \) **predictor**
  \[ y_{i+1}^0 = y_{i-1} + f(x_i, y_i)2h \]
  \[ y_{i+1} = y_i + h \left( \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)}{2} \right) \]
Non-Self-Starting Heun Method

- Multiple iterations
  \[
  y_{i+1}^0 = y_{i-1} + f(x_i, y_i)2h
  \]
  \[
  y_{i+1}^j = y_i + h \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{j-1})}{2}
  \]
  \[
  |\epsilon_a| = \left| \frac{y_{i+1}^j - y_{i+1}^{j-1}}{y_{i+1}^j} \right|
  \]
Non-Self-Starting Heun Method

- Derivation

\[
\frac{dy}{dx} = f(x, y)
\]

\[
\int_{y_{i-1}}^{y_{i+1}} dy = \int_{x_{i-1}}^{x_{i+1}} f(x, y)dx
\]

\[
y_{i+1} = y_{i-1} + \int_{x_{i-1}}^{x_{i+1}} f(x, y)dx
\]

\[
y_{i+1} = y_{i-1} + f(x_i, y_i)2h
\]
Non-Self-Starting Heun Method

- **Predictor Error**
  \[ E_p = \frac{4}{5} \left( y_i^m - y_i^0 \right) \]

- **Refine value using predictor modifier**
  \[ y_{i+1}^0 \leftarrow y_{i+1}^0 + \frac{4}{5} \left( y_i^m - y_i^0 \right) \]

- **Corrector Error**
  \[ E_c = -\frac{y_{i+1}^0 - y_{i+1}^m}{5} \]

- **Refine value using corrector modifier**
  \[ y_{i+1}^m \leftarrow y_{i+1}^m + \frac{y_{i+1}^0 - y_{i+1}^m}{5} \]
Non-Self-Starting Heun Method

• Integrate \( y' = 4e^{0.8x} - 0.5y \) and \( y(0) = 2, \ y(-1) = -0.393 \)
  – Analytical solution: \( y = \frac{4}{1.3} \left( e^{0.8x} - e^{-0.5x} \right) + 2e^{-0.5x} \)
  – Heun’s Method (step size of 1):

\[
y_1^0 = y_{-1} + f(x_0, y_0)2h = -0.393 + (3)(2) = 5.607 \quad \varepsilon_t = 10.54\%
\]
\[
y_1^1 = y_0 + h \frac{f(x_0, y_0) + f(x_1, y_1^0)}{2} = 2 + 1 \left( \frac{3 + 6.099}{2} \right) = 6.549 \quad \varepsilon_t = -5.73\%
\]
\[
y_1^2 = y_0 + h \frac{f(x_0, y_0) + f(x_1, y_1^1)}{2} = 2 + 1 \left( \frac{3 + 5.627}{2} \right) = 6.314 \quad \varepsilon_t = -1.92\%
\]

ECE 257 Numerical Methods and Scientific Computing
Fall 2004
Lecture 18

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Non-Self-Starting Heun Method

\[ y_1^1 \leftarrow y_1^1 - \frac{y_1^1 - y_1^0}{5} = 6.549 - \frac{6.549 - 5.607}{5} = 6.361 \]
\[ \epsilon_t = -2.68\% \]

\[ y_1^2 \leftarrow y_1^2 - \frac{y_1^2 - y_1^0}{5} = 6.314 - \frac{6.314 - 5.607}{5} = 6.173 \]
\[ \epsilon_t = 0.356\% \]
Predictor-Corrector Methods

• Instead of using midpoint method and trapezoidal method, use higher order integration methods as the predictor and corrector
  – Newton-Cotes
  – Adams
Predictor-Corrector Methods

\[ y_{i+1} = y_{i-2} \int_{x_{i-2}}^{x_{i+1}} f(x, y) \, dx \]

(a)

\[ y_{i+1} = y_i \int_{x_i}^{x_{i+1}} f(x, y) \, dx \]

(b)
Predictor-Corrector Methods

- Newton Cotes based
  - Integrate by fitting an nth degree polynomial to n+1 points

\[
\frac{dy}{dx} = f(x, y)
\]

\[
\int_{y_{i-n}}^{y_{i+1}} dy = \int_{x_{i-n}}^{x_{i+1}} f(x, y) dx
\]

\[
y_{i+1} = y_{i-n} + \int_{x_{i-n}}^{x_{i+1}} f(x, y) dx
\]
Predictor-Corrector Methods

- Open Formulas

  - n=1
  \[ y_{i+1} = y_{i-1} + \int_{x_{i-1}}^{x_{i+1}} f(x, y)dx \]
  \[ = y_{i-1} + 2hf_i \]

  - n=2
  \[ y_{i+1} = y_{i-2} + \int_{x_{i-2}}^{x_{i+1}} f(x, y)dx \]
  \[ = y_{i-2} + \frac{3h}{2}(f_i + f_{i-1}) \]

  - n=3
  \[ y_{i+1} = y_{i-3} + \int_{x_{i-3}}^{x_{i+1}} f(x, y)dx \]
  \[ = y_{i-3} + \frac{4h}{3}(2f_i - f_{i-1} + 2f_{i-2}) \]
Predictor-Corrector Methods

- Closed Formulas
  - n=1
  \[ y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} f(x, y) dx \]
  \[ = y_{i-1} + \frac{h}{2} (f_i + f_{i+1}) \]
  - n=2
  \[ y_{i+1} = y_{i-1} + \int_{x_{i-1}}^{x_{i+1}} f(x, y) dx \]
  \[ = y_{i-1} + \frac{h}{3} (f_{i-1} + 4f_i + f_{i+1}) \]
Adams-Bashforth Methods

- Open Formulas
  - Taylor series expansion
    \[ y_{i+1} = y_i + f_i h + f'_i \frac{h^2}{2!} + f''_i \frac{h^3}{3!} + \cdots \]
    \[ y_{i+1} = y_i + h \left( f_i + f'_i \frac{h}{2!} + f''_i \frac{h^2}{3!} + \cdots \right) \]
  - Backward difference substitution for \( f' \)
    \[ y_{i+1} = y_i + h \left( f_i + \frac{f_i - f_{i-1}}{h} + f''_i \frac{h}{2!} + O(h^2) \right) \left( \frac{h}{2!} + f''_i \frac{h^2}{3!} + \cdots \right) \]
    \[ y_{i+1} = y_i + h \left( \frac{3}{2} f_i - \frac{1}{2} f_{i-1} \right) + \frac{5h^3}{12} f''_i + O(h^4) \]
Adams-Bashforth Formulas

• 2nd order

\[ y_{i+1} = y_i + h \left( \frac{3}{2} f_i - \frac{1}{2} f_{i-1} \right) + \frac{5h^3}{12} f_i'' + O(h^4) \]

• n-th order

\[ y_{i+1} = y_i + h \sum_{k=0}^{n-1} \beta_k f_{i-k} + O(h^{n+1}) \]
Adams-Moulton Methods

• Closed Formulas
  – Taylor series expansion

\[ y_i = y_{i+1} - f_{i+1} h + f'_{i+1} \frac{h^2}{2!} - f''_{i+1} \frac{h^3}{3!} + \cdots \]

\[ y_{i+1} = y_i + h \left( f_{i+1} - f'_{i+1} \frac{h}{2!} + f''_{i+1} \frac{h^2}{3!} + \cdots \right) \]

– Backward difference substitution for \( f' \)

\[ y_{i+1} = y_i + h \left( f_{i+1} - \left[ f_{i+1} - f_i + f''_{i+1} \frac{h}{2!} + O(h^2) \right] \frac{h}{2!} + f''_{i+1} \frac{h^2}{3!} + \cdots \right) \]

\[ y_{i+1} = y_i + h \left( \frac{1}{2} f_{i+1} + \frac{1}{2} f_i \right) - \frac{h^3}{12} f_i'' - O(h^4) \]
Adams-Moulton Formulas

- 2nd order

\[ y_{i+1} = y_i + h \left( \frac{1}{2} f_{i+1} + \frac{1}{2} f_i \right) - \frac{h^3}{12} f_i'' - O(h^4) \]

- n-th order

\[ y_{i+1} = y_i + h \sum_{k=0}^{n-1} \beta_k f_{i+1-k} + O(h^{n+1}) \]
Milne’s Method

• Predictor-Corrector
  – Predictor: 3-point Newton-Cotes open
    \[
    y_{i+1} = y_{i-3} + \frac{4h}{3} (2f_i - f_{i-1} + 2f_{i-2})
    \]
    \[
    E_p = \frac{28}{29} (y^m_i - y_i^0)
    \]
  – Corrector: 3-point Newton-Cotes closed
    \[
    y_{i+1}^j = y_{i-1} + \frac{h}{3} \left( f_{i-1} + 4f_i + f_{i+1} \right)
    \]
    \[
    E_c = -\frac{1}{29} (y^m_{i+1} - y^0_{i+1})
    \]
Milne’s Method

- Integrate \( y' = 4e^{0.8x} - 0.5y \) and \( y(0) = 2 \)
  - Use an RK method to compute “previous” points
    - \( y(-3) = -4.547, y(-2) = -2.306, y(-1) = -0.393 \)

\[
y_1^0 = y_{-3} + \frac{4h}{3} \left( 2f_0 - f_{-1} + 2f_{-2} \right)
\]
\[
= -4.547 + \frac{4(1)}{3} \left( 2(3) - 1.994 + 2(1.961) \right)
\]
\[
= 6.023 \quad \epsilon_t = 2.8\%
\]

\[
y_1^1 = y_{-1} + \frac{h}{3} \left( f_{-1} + 4f_0 + 2f_{1}^0 \right)
\]
\[
= -0.393 + \frac{1}{3} \left( 1.994 + 4(3) + 5.891 \right)
\]
\[
= 6.235 \quad \epsilon_t = -0.66\%
\]
Fourth-Order Adams

- Use fourth-order Adams-Bashforth as predictor
  \[ y_{i+1} = y_i + h \left( \frac{55}{24} f_i - \frac{59}{24} f_{i-1} + \frac{37}{24} f_{i-2} - \frac{9}{24} f_{i-3} \right) \]
  \[ E_c = \frac{251}{270} (y^m_i - y_i^0) \]

- Use fourth-order Adams-Moulton as corrector
  \[ y^j_{i+1} = y_i + h \left( \frac{9}{24} f_{i+1}^j - \frac{5}{24} f_{i-1} + \frac{19}{24} f_{i-2} \right) \]
  \[ E_c = -\frac{19}{270} (y^m_{i+1} - y^0_{i+1}) \]
Fourth-Order Adams

- Integrate \( y' = 4e^{0.8x} - 0.5y \) and \( y(0) = 2 \)
  - Use an RK method to compute “previous” points

- \( y(-3) = -4.547, y(-2) = -2.306, y(-1) = -0.393 \),

\[
y_1^0 = y_0 + h\left(\frac{55}{24}f_0 - \frac{59}{24}f_{-1} + \frac{37}{24}f_{-2} - \frac{9}{24}f_{-3}\right)
\]

\[
= 2 + 1\left(\frac{55}{24}3 - \frac{59}{24}1.994 + \frac{37}{24}1.961 - \frac{9}{24}2.637\right)
\]

\[
= 6.008 \quad \varepsilon_t = 3.1\%
\]

\[
y_1^1 = y_0 + h\left(\frac{55}{24}f_0 - \frac{59}{24}f_{-1} + \frac{37}{24}f_{-2} - \frac{9}{24}f_{-3}\right)
\]

\[
= 2 + 1\left(\frac{9}{24}5.898 + \frac{19}{24}3 - \frac{5}{24}1.994 + \frac{1}{24}1.961\right)
\]

\[
= 6.253 \quad \varepsilon_t = -0.96\%
\]
Multistep Methods

• Because of the smaller error coefficients, Milne’s Method is slightly more accurate than Fourth-Order Adams methods

• However, Milne’s can often be unstable

• Example: \( y' = -y, \ y(0) = 1, \ h = 0.5 \)
Multistep Methods

Milne’s method

True solution

ECE 257 Numerical Methods and Scientific Computing
Fall 2004
Lecture 18
Next Lecture

• Ordinary Differential Equations
  – Boundary-Value Problems
  – Eigenvalue Problems

• Read Chapter 27

• HW6 due 11/16

• Exam 2 11/9