ECE257 Numerical Methods and Scientific Computing

Optimization Methods
Today’s class:

• Optimization

• One-dimensional unconstrained
Optimization

• Given a function $f(x_1, x_2, x_3, \ldots, x_n)$ find the set of values that minimize or maximize the function

• Examples:
  – Lowering power usage in a circuit while maximizing speed
  – Least-cost management of supply chain
Optimization

- From calculus, optimization means setting the derivative \( f'(x_1, x_2, x_3, \ldots, x_n) \) to zero and finding solutions to that equation.

- If \( f''(x_1, x_2, x_3, \ldots, x_n) > 0 \), it is a minimum and if \( f''(x_1, x_2, x_3, \ldots, x_n) < 0 \), it is a maximum.

- To find where the derivative is zero, we can use the root-finding techniques we discussed before.
Optimization

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- Critical point: $f(x) = 0$
- Maximum: $f''(x) < 0$
- Minimum: $f''(x) > 0$
- Root: $f'(x) = 0$

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Lecture 11

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Optimization

- In VLSI circuits, you typically use cascaded inverters to reduce transistor sizes when driving large capacitances

\[ t_d = (n-1)aRC + \frac{R}{a^{n-1}}C_L \]

Take the derivative and get

\[ \frac{dt_d}{da} = (n-1)RC - (n-1)\frac{R}{a^n}C_L \]

\[ = (n-1)\left(1 - \frac{C_L}{C}a^{-n}\right) \]

Set the derivative to 0 and get

\[ a^n = \frac{C_L}{C} \]
Transistor Sizing

\[ t_d = (n-1)aRC + \frac{R}{a^{n-1}}C_L \]

\[ = (n-1)aRC + \frac{R}{a}aC_L \]

\[ = (n-1)aRC + aRC \]

\[ = naRC \]

\[ a^n = \frac{C_L}{C} \]

\[ n = \log_a \left( \frac{C_L}{C} \right) \]

\[ t_d = \log_a \left( \frac{C_L}{C} \right)aRC \Rightarrow \frac{dt_d}{da} = RC \ln \left( \frac{C_L}{C} \right) \left[ \frac{\ln a - 1}{(\ln a)^2} \right] \Rightarrow a = e \]

- Minimum delay is reached when \( a \) is equal to \( e \)(~2.72)

- At optimum, \[ n = \ln \left( \frac{C_L}{C} \right) \]
Optimization

- Finding the roots of the derivative requires knowing what the derivative is
- Often, the function is not well-defined and you can not analytically derive the derivative
- Requires finite-difference approximation of the derivative
Optimization

• General optimization problem:

Find \( x \), which minimizes or maximizes \( f(x) \) subject to

\[
d_i(x) \leq a_i \quad i = 1,2,\ldots,m
\]

\[
e_i(x) = b_i \quad i = 1,2,\ldots,m
\]

where \( x \) is a \( n \)-dimensional design vector, \( f(x) \) is the objective function, \( d_i(x) \) are inequality constraints and \( e_i(x) \) are equality constraints.
Optimization

- Constrained optimization
  - Degrees of freedom
    - $n-p-m$
  - Under-constrained
    - $p+m\leq n$
    - Solution possible
  - Over-constrained
    - $p+m>n$
    - Solution unlikely
Optimization

- Linear Programming
  - Objective function and the constraints are all linear

- Quadratic Programming
  - Objective function is quadratic and the constraints are linear

- Nonlinear Programming
  - Objective function is not linear or quadratic and/or the constraints are nonlinear
Optimization

• Dimensionality
  – One-dimensional
    • Single function variable
  – Multi-dimensional
    • Function is dependent on more than one variable
Optimization

• How do you know that the minimum or maximum that you found is the global minimum or maximum?
  – Multimodal systems

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Optimization

- How do you find the global extremum?
  - Use graphical methods to gain insight
  - Use multiple guesses to find multiple local extrema and then pick the largest as global
  - Slight perturbations of initial guesses to see if the extrema changes

- For large multi-dimensional problems it is sometimes too difficult to guarantee that you have found the global extremum
One-dimensional Unconstrained Optimization

- Similar to root-finding
- Bracketing methods
  - Golden-section search
  - Quadratic interpolation
- Open methods
  - Newton’s method
Golden-section search

• Also called Golden Ratio search

• Start with a interval bracket around the maximum

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Golden-section search

- Assume that the function is unimodal over interval

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Golden-section Search

\[
\ell_0 = \ell_1 + \ell_2
\]

\[
\frac{\ell_1}{\ell_0} = \frac{\ell_2}{\ell_1} = R
\]

\[
\frac{1}{R} = 1 + R
\]

\[
R^2 + R - 1 = 0
\]

\[
R = \frac{\sqrt{5} - 1}{2}
\]
Golden-section search

- Algorithm
  - Pick two interior points in the interval using the golden ratio

\[
x_1 = x_l + \frac{\sqrt{5} - 1}{2} (x_u - x_l) = x_l + d
\]

\[
x_2 = x_u - \frac{\sqrt{5} - 1}{2} (x_u - x_l) = x_u - d
\]
Golden-section search

- Two possibilities
  - $f(x_1) > f(x_2)$
    - The maximum is between $x_2$ and $x_u$
      - $x_{l,new} = x_2$, $x_{1,new} = x_{l,new} + R(x_u - x_{l,new})$
  - $f(x_1) < f(x_2)$
    - The maximum is between $x_l$ and $x_l$
      - $x_{u,new} = x_1$, $x_{1,new} = x_2$, $x_{2,new} = x_{u,new} - R(x_{u,new} - x_l)$
Golden section search

• Example
  – Find minimum of $f(x) = \sin x - x^2$
  – Start with $x_l=0$ and $x_u=1$

$x_1 = 0 + \frac{\sqrt{5} - 1}{2} (1 - 0) = 0.618 \quad \Rightarrow f(x_1) = 0.197$

$x_2 = 1 - \frac{\sqrt{5} - 1}{2} (1 - 0) = 0.382 \quad \Rightarrow f(x_2) = 0.226$
Golden section search

\[ f(x_2) > f(x_1) \]

\[ x_{u,new} = x_1 = 0.618 \]

\[ x_{1,new} = x_2 = 0.382 \]

\[ x_{2,new} = x_{u,new} - \frac{\sqrt{5} - 1}{2} (x_{u,new} - x_1) \]

\[ = 0.618 - 0.618(0.618 - 0) \]

\[ = 0.236 \]
### Golden section search

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<tr>
<th>$i$</th>
<th>$x_l$</th>
<th>$x_2$</th>
<th>$x_1$</th>
<th>$x_u$</th>
<th>$f(x_2)$</th>
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<td>0.2323</td>
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Golden section search

• Error analysis
  – Worst-case error is when true value is at the far-end of the sub-interval
  • Take case when optimum value is in upper interval
    – True value is at the left
      \[
      \Delta x_a = x_1 - x_2 \\
      = x_l + R(x_u - x_l) - \left[ x_u - R(x_u - x_l) \right] \\
      = (2R - 1)(x_u - x_l) \\
      = 0.236(x_u - x_l)
      \]
Golden section search

- True value is at the right

\[
\Delta x_a = x_u - x_l \\
= x_u - [x_l + R(x_u - x_l)] = (1 - R)(x_u - x_l) \\
= 0.382(x_u - x_l)
\]

- Relative error is

\[
\epsilon_a = (1 - R) \left| \frac{x_u - x_l}{x_{opt}} \right|
\]
Quadratic interpolation

- Use a second order polynomial as an approximation of the function near the optimum
Quadratic interpolation

\[ x_3 = \frac{f(x_0)(x_1^2 - x_2^2) + f(x_1)(x_2^2 - x_0^2) + f(x_2)(x_0^2 - x_1^2)}{2f(x_0)(x_1 - x_2) + 2f(x_1)(x_2 - x_0) + 2f(x_2)(x_0 - x_1)} \]

- Use the above formula to get a new guess and then use same method as with Golden-search to get rid of one of the guesses
Quadratic interpolation

- Example
  - Find minimum of $f(x) = \sin x - x^2$
  - Start with $x_0=0$, $x_1=1$ and $x_2=4$

$$
x_3 = \frac{f(0)(1^2 - 4^2) + f(1)(4^2 - 0^2) + f(4)(0^2 - 1^2)}{2f(0)(1-4) + 2f(1)(4-0) + 2f(4)(0-1)}
$$

$$
= \frac{0(-15) + 1.583(16) + (-3.114)(-1)}{2(0)(-3) + 2(1.583)(4) + 2f(-3.114)(-1)}
$$

$$
= 1.5055
$$
Quadratic interpolation

\[ f(x_3) = f(1.5055) = 1.7691 \]
\[ f(x_1) = f(1) = 1.5829 \]

\[ x_3 > x_1 \text{ and } f(x_3) > f(x_1) \]

\[ x_{0,new} = x_1 = 1 \]
\[ x_{1,new} = x_3 = 1.5055 \]
\[ x_{2,new} = x_2 \]
Newton’s Method

• Newton-Raphson could be used to find the root of an function

\[ x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \]

• When finding a function optimum, use the fact that we want to find the root of the derivative and apply Newton-Raphson

\[ x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)} \]
Newton’s Method

• Example
  – Find minimum of $f(x) = 2\sin x - \frac{x^2}{10}$
  – Start with $x_0 = 2.5$

$$f'(x) = 2\cos x - \frac{x}{5}$$
$$f''(x) = -2\sin x - \frac{1}{5}$$

$$x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} = 2.5 - \frac{2\cos(2.5) - \frac{2.5}{5}}{-2\sin(2.5) - \frac{1}{5}} = 0.99508$$
Newton’s Method

• Example

\[ x_2 = x_1 - \frac{f'(x_1)}{f''(x_1)} = 0.995 - \frac{2\cos(0.995) - \frac{0.995}{5}}{-2\sin(0.995) - \frac{1}{5}} = 1.46901 \]

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<th>( f(x) )</th>
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<td>1.77573</td>
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</table>
Newton’s Method

• Like Newton-Raphson for roots, Newton’s method for finding optima may also diverge
• Can use secant-like methods using finite-difference approximations if the derivative is not available
• Usually used only near the optima
• Use Hybrid methods
  – Bracketing methods to get near the optimum
  – Open methods to quickly converge to the optimum
Next Lecture

- Multi-dimensional Optimization
- Read Chapter 13 and 14