ECE257 Numerical Methods and Scientific Computing

Roots of Equations
Today’s class:

• Roots of Equations
• Polynomials
Polynomials

• A polynomial is of the form:
  \[ f_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \]

• The roots of a polynomial follow these rules:
  – There will be \( n \) roots to an \( n \)-th order polynomial.
    The roots may be real or complex and need not be distinct
  – If complex roots exist, they exist in conjugate pairs
    \((\lambda + \mu i \text{ and } \lambda - \mu i)\)
  – If \( n \) is odd, there is at least one real root
Polynomials

• Uses in electrical engineering
  – Solving for poles and zeros in transfer functions
  – Solving characteristic equations in linear ordinary differential equations
Polynomials

\[ f_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \]

\[ F = A[0] \]
\[ \text{FOR } I=1 \text{ to } N \]
\[ K = A[I] \]
\[ \text{FOR } J=1 \text{ to } I \]
\[ K = K * x \]
\[ \text{ENDFOR} \]
\[ F = F + K \]
\[ \text{ENDFOR} \]

\[ \frac{n(n+1)}{2} \text{ multiplications} \]
\[ n \text{ additions} \]
Polynomials

\[ f_n(x) = a_0 + x(a_1 + x(a_2 + \cdots + xa_n)) \]

F=A[N]
FOR I=N-1 TO 0
    F = F*X + A[I]
ENDFOR

\( n \) multiplications
\( n \) additions
Polynomial Deflation

- Given a polynomial and a single known root, deflating the polynomial can reduce the order of the polynomial and remove possible redundant roots

- Example:

\[ x^4 - 8x^3 - 20x^2 + 288x - 576 \]

- Root \( x=4 \) is known
Polynomial Deflation

\[
x^3 \quad -4x^2 \quad -36x \quad +144 \\
 x - 4 \overline{x^4 \quad -8x^3 \quad -20x^2 \quad +288x \quad -576}
\]

\[
x^4 \quad -4x^3 \\
 x^4 \quad -4x^3 \\
 \overline{-4x^3 \quad -20x^2}
\]

\[
\overline{-4x^3 \quad +16x^2}
\]

\[
\overline{-36x^2 \quad +288x}
\]

\[
\overline{-36x^2 \quad +144x}
\]

\[
144x \quad -576 \\
144x \quad -576 \\
\overline{0}
\]
Polynomial Deflation

• Synthetic Division

\[ x^4 - 8x^3 - 20x^2 + 288x - 576 \]

\[
\begin{array}{cccccc|c}
1 & -8 & -20 & 288 & -576 & 4 \\
4 & -16 & -144 & 576 & & \\
1 & -4 & -36 & 144 & 0 & \\
\end{array}
\]
Polynomial Deflation

- Synthetic Division

\[ x^3 - 4x^2 - 36x + 144 \]

\[
\begin{array}{cccc|c}
1 & -4 & -36 & +144 & 4 \\
\hline
\end{array}
\]

\[
\begin{array}{cccc}
4 & 0 & -144 \\
1 & 0 & -36 & 0 \\
\end{array}
\]
Polynomial Deflation

- Synthetic Division

\[ a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \left( b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1} \right) (x - t) + r \]

\[
\begin{align*}
r &= a[n] \\
b[n] &= 0 \\
\text{for } i &\text{ from } n-1 \text{ to } 0 \\
\quad &\quad b[i] = r \\
\quad &\quad r = r \times t + a[i] \\
\text{endfor}
\]
Polynomial Deflation

• If the known root is a calculated root, approximation error in that root may be compounded during the deflation process.

• With forward deflation, i.e. new polynomial coefficients are calculated in descending order, it is better to divide by roots of smallest absolute value first.

• Round-off errors can also be reduced by using root polishing.
Roots of polynomials

• Previous methods (Newton-Raphson, bracketing, etc.) have a few problems when applied to polynomials
  – Must determine an initial guess of the root
  – Can not find complex roots
  – May not converge

• Polynomial-specific methods
  – Müller’s method
  – Bairstow’s method
Müller’s method

- Similar idea to secant method

- Instead of projecting a straight line through two points to estimate the root, project a parabola through three points to estimate the root
Müller’s method

From Numerical Methods for Engineers, Chapra and Canale, Copyright © The McGraw-Hill Companies, Inc.
Müller’s method

• Find a parabola \( f_2(x) = a(x - x_2)^2 + b(x - x_2) + c \) such that it intersects the function at three points \( x_0, x_1, \) and \( x_2 \)

\[
\begin{align*}
  f(x_0) &= f_2(x_0) = a(x_0 - x_2)^2 + b(x_0 - x_2) + c \\
  f(x_1) &= f_2(x_1) = a(x_1 - x_2)^2 + b(x_1 - x_2) + c \\
  f(x_2) &= f_2(x_2) = a(x_2 - x_2)^2 + b(x_2 - x_2) + c
\end{align*}
\]
Müller’s method

• Solve for c

\[ c = f(x_2) \]

• Substitute back into set of equations

\[ f(x_0) - f(x_2) = a(x_0 - x_2)^2 + b(x_0 - x_2) \]
\[ f(x_1) - f(x_2) = a(x_1 - x_2)^2 + b(x_1 - x_2) \]

• Define new set of variables

\[ h_0 = (x_1 - x_0) \quad \quad h_1 = (x_2 - x_1) \]
\[ \delta_0 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad \quad \delta_1 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \]
Müller’s method

- Substitute new variables back into equations

\[
\begin{align*}
    h_0\delta_0 + h_1\delta_1 &= b(h_0 + h_1) - a(h_0 + h_1)^2 \\
    h_1\delta &= bh_1 - ah_1^2
\end{align*}
\]

- Solve for a and b

\[
\begin{align*}
    a &= \frac{\delta_1 - \delta_0}{h_0 + h_1} \\
    b &= ah_1 + \delta_1
\end{align*}
\]
Müller’s method

• Now that we have found the coefficients of the parabola, find where it intersects the x-axis to get the next root estimate

• The intersection point with the x-axis is simply the root of the parabola which we can find using the quadratic formula

\[ f_2(x_r) = a(x_r - x_2)^2 + b(x_r - x_2) + c = 0 \]

\[ x_r - x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2c}{b \pm \sqrt{b^2 - 4ac}} \]
Müller’s method

\[ x_r = x_2 + \frac{-2c}{b \pm \sqrt{b^2 - 4ac}} \]

- Which of the two quadratic roots do you choose?
  - Choose the sign that makes the denominator the largest
  - That will bring \( x_r \) closer to \( x_2 \)
Müller’s method

• How do you pick the three points for the next iteration?
  – If there are only real roots, choose $x_r$ and the closer two of the original three roots to $x_r$.
  – If there are complex roots, use a sequential approach: use $x_r$, $x_2$, and $x_1$. 
Example

- Find roots of \( f(x) = x^3 - 13x - 12 \)
- Use \( x_0 = 4.5 \), \( x_1 = 5.5 \), and \( x_2 = 5 \) as the initial guesses

\[
\begin{align*}
f(x_0) &= f(4.5) = 20.625 \\
f(x_1) &= f(5.5) = 82.875 \\
f(x_2) &= f(5) = 48
\end{align*}
\]

\[
\begin{align*}
h_0 &= (x_1 - x_0) = 5.5 - 4.5 = 1.0 \\
\delta_0 &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{82.875 - 20.625}{5.5 - 4.5} = 62.25
\end{align*}
\]

\[
\begin{align*}
h_1 &= (x_2 - x_1) = 5 - 5.5 = -0.5 \\
\delta_1 &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{48 - 82.875}{5 - 5.5} = 69.75
\end{align*}
\]
Example

\[ a = \frac{\delta_1 - \delta_0}{h_0 + h_1} = \frac{69.75 - 62.25}{1 - 0.5} = 15 \]

\[ b = ah_1 + \delta_1 = 15 \cdot (-0.5) + 69.75 = 62.25 \]

\[ c = f(x_2) = 48 \]

\[ x_r = x_2 + \frac{-2c}{b \pm \sqrt{b^2 - 4ac}} = 5 + \frac{-2 \cdot 48}{62.25 + \sqrt{(62.25)^2 - 4 \cdot 15 \cdot 48}} = 3.976487 \]

\[ \epsilon_a = \left| \frac{x_r - x_2}{x_r} \right| = \left| \frac{3.976487 - 5}{3.976487} \right| = 25.74\% \]
## Example

<table>
<thead>
<tr>
<th>i</th>
<th>$x_r$</th>
<th>$\varepsilon_a(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3.976487</td>
<td>25.74</td>
</tr>
<tr>
<td>2</td>
<td>4.00105</td>
<td>0.6139</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>0.0262</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0.0000119</td>
</tr>
</tbody>
</table>
Bairstow’s method

• Based on dividing the polynomial by \((x-t)\) where \(t\) is the root estimate

• If there is no remainder, we found the root

• If not, we adjust the guess
Bairstow’s method

\[ f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \]

- Factor by \( x-t \)

\[ f(x) = (x - t)(b_1 + b_2 x + b_3 x^2 + \cdots + b_n x^{n-1}) + b_0 \]

\[ b_n = a_n \]

\[ b_i = a_i + b_{i+1}t \quad \text{for } i = n-1 \text{ to } 0 \]

- Factor by \( x^2-rx-s \)

\[ f(x) = (x^2 - rx - s)(b_2 + b_3 x + b_4 x^2 + \cdots + b_n x^{n-2}) + b_1(x - r) + b_0 \]
Bairstow’s method

\[ b_n = a_n \]

\[ b_{n-1} = a_{n-1} + rb_n \]

\[ b_i = a_i + rb_{i+1} + sb_{i+2} \quad \text{for } i = n - 2 \text{ to } 0 \]

- We have to find values of \( r \) and \( s \) such that the remainder is 0
- The remainder is 0 if \( b_1 = b_0 = 0 \)
- \( b_1 \) and \( b_0 \) are both functions of \( r \) and \( s \), so we can give a Taylor series expansion of both

\[
 b_1(r + \Delta r, s + \Delta s) \approx b_1(r, s) + \frac{\partial b_1(r, s)}{\partial r} \Delta r + \frac{\partial b_1(r, s)}{\partial s} \Delta s
\]

\[
 b_0(r + \Delta r, s + \Delta s) \approx b_0(r, s) + \frac{\partial b_0(r, s)}{\partial r} \Delta r + \frac{\partial b_0(r, s)}{\partial s} \Delta s
\]
Bairstow’s method

\[ 0 = b_1(r,s) + \frac{\partial b_1(r,s)}{\partial r} \Delta r + \frac{\partial b_1(r,s)}{\partial s} \Delta s \]

\[ 0 = b_0(r,s) + \frac{\partial b_0(r,s)}{\partial r} \Delta r + \frac{\partial b_0(r,s)}{\partial s} \Delta s \]

\[ c_2 \Delta r + c_3 \Delta s = -b_1(r,s) \]

\[ c_1 \Delta r + c_2 \Delta s = -b_0(r,s) \]

\[ c_n = b_n \]

\[ c_{n-1} = b_{n-1} + rc_n \]

\[ c_i = b_i + rc_{i+1} + sc_{i+2} \quad \text{for } i = n - 2 \text{ to } 1 \]
Bairstow’s method

\[ c_2 \Delta r + c_3 \Delta s = -b_1(r, s) \]
\[ c_1 \Delta r + c_2 \Delta s = -b_0(r, s) \]

• Solve for \( \Delta r \) and \( \Delta s \)
• Use the results to adjust \( r \) and \( s \) and iterate
• Stop when the approximate error in \( r \) and \( s \) is sufficient

\[ |\varepsilon_{a,r}| = \left| \frac{\Delta r}{r} \right| \quad \text{and} \quad |\varepsilon_{a,s}| = \left| \frac{\Delta s}{s} \right| \]
Example

- Find roots of \( f(x) = x^5 - 3.5x^4 + 2.75x^3 + 2.125x^5 - 3.875x + 1.25 \)

- Use \( s = -1, r = -1 \) as the initial guesses

\[
\begin{align*}
  b_5 &= a_5 = 1 \\
  b_4 &= a_4 + rb_5 = -4.5 \\
  b_3 &= a_3 + rb_4 + sb_5 = 6.25 \\
  b_2 &= a_2 + rb_3 + sb_4 = 0.375 \\
  b_1 &= a_1 + rb_2 + sb_3 = -10.5 \\
  b_0 &= a_0 + rb_1 + sb_2 = 11.375 \\
  c_5 &= b_5 = 1 \\
  c_4 &= b_4 + rc_5 = -5.5 \\
  c_3 &= b_3 + rc_4 + sc_5 = 10.75 \\
  c_2 &= b_2 + rc_3 + sc_4 = -4.875 \\
  c_1 &= b_1 + rc_2 + sc_3 = -16.375
\end{align*}
\]
Example

• Solve linear system

\[ c_2 \Delta r + c_3 \Delta s = -b_1(r,s) \Rightarrow -4.875\Delta r + 10.75\Delta s = 10.5 \]
\[ c_1 \Delta r + c_2 \Delta s = -b_0(r,s) \Rightarrow -16.375\Delta r - 4.875\Delta s = -11.375 \]

\[ \Delta r = 0.3558 \]
\[ \Delta s = 1.1381 \]

• New s and r

\[ r = -1 + \Delta r = -0.6442 \]
\[ s = -1 + \Delta s = 0.1381 \]
Example
• Use $s=0.1381$, $r=-0.6442$

\[ b_5 = a_5 = 1 \]
\[ b_4 = a_4 + rb_5 = -4.1442 \]
\[ b_3 = a_3 + rb_4 + sb_5 = 5.5578 \]
\[ b_2 = a_2 + rb_3 + sb_4 = -2.0276 \]
\[ b_1 = a_1 + rb_2 + sb_3 = -1.8013 \]
\[ b_0 = a_0 + rb_1 + sb_2 = 2.1304 \]

\[ c_5 = b_5 = 1 \]
\[ c_4 = b_4 + rc_5 = -4.7884 \]
\[ c_3 = b_3 + rc_4 + sc_5 = 8.7806 \]
\[ c_2 = b_2 + rc_3 + sc_4 = -8.3454 \]
\[ c_1 = b_1 + rc_2 + sc_3 = -4.7874 \]
Example

- Solve linear system

\[ c_2 \Delta r + c_3 \Delta s = -b_1(r, s) \Rightarrow -8.3454 \Delta r + 8.7806 \Delta s = 1.8013 \]
\[ c_1 \Delta r + c_2 \Delta s = -b_0(r, s) \Rightarrow 4.7874 \Delta r - 8.3454 \Delta s = -2.1304 \]
\[ \Delta r = 0.1331 \]
\[ \Delta s = 0.3316 \]

- New s and r

\[ r = -0.6442 + \Delta r = -0.5111 \]
\[ s = 0.1381 + \Delta s = 0.4697 \]
Example

• Keep going until $s=0.5$ and $r=-0.5$

• Then solve the following:

\[
x^2 - rx - s = 0 \Rightarrow x^2 + \frac{x}{2} - \frac{1}{2} = 0
\]

\[
x = \frac{-0.5 \pm \sqrt{(0.5)^2 - 4 \cdot 1 \cdot (-0.5)}}{2} = 0.5, 1.0
\]
Example

- The quotient coefficients are as follows

  \[ b_5 = a_5 = 1 \]
  \[ b_4 = a_4 + rb_5 = -4 \]
  \[ b_3 = a_3 + rb_4 + sb_5 = 5.25 \]
  \[ b_2 = a_2 + rb_3 + sb_4 = -2.5 \]
  \[ b_1 = a_1 + rb_2 + sb_3 = 0 \]
  \[ b_0 = a_0 + rb_1 + sb_2 = 0 \]

- Now we have to solve:

  \[ f(x) = x^3 - 4x^2 + 5.25x - 2.5 = 0 \]
Bairstow’s method

• If no $s$ or $r$ are known, start with $s=0$, $r=0$
• Converges fast
• May diverge
Roots of equations

• Bisection
  – 2 initial guesses, slow convergence, will not diverge

• False-position
  – 2 initial guesses, slow-medium convergence, will not diverge
Roots of equations

- Newton-Raphson
  - 1 initial guess, fast convergence, may diverge

- Secant
  - 2 initial guesses, medium-fast convergence, may diverge
Roots of equations

- **Muller**
  - 3 initial guesses, medium-fast convergence, may diverge, only polynomials, complex roots

- **Bairstow**
  - 2 initial guesses, medium-fast convergence, may diverge, only polynomials, complex roots
Next Lecture

• Linear algebraic equations
• Read Chapter PT3, 9
• HW3 due 9/30
• Exam 1 on Tuesday 10/04