Name:______________________________

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Problem 1:

a) Approximate the derivative of \( f(x) = e^{-x} \), at \( x = 0 \) using a centered difference approximation and let \( \Delta x = h = 1 \). What is the true relative error? (10pts)

\[
f'(0) \approx \frac{f(0+1) - f(0-1)}{2(1)} = \frac{e^{-1} - e^{(-1)}}{2(1)} = -1.175
\]

The true derivative is \( f'(x) = -e^{-x} \), which at \( x=0 \) is equal to 1. Therefore, the true relative error is:

\[
e_r = \frac{-1 - (-1.175)}{-1} = -17.5\
\]

b) Iterate by halving the interval size until the approximate error is less than 1% (10pts)

\( h=0.5: \)

\[
f'(0) \approx \frac{f(0+0.5) - f(0-0.5)}{2(0.5)} = \frac{e^{-0.5} - e^{(-0.5)}}{2(0.5)} = -1.042
\]

\[
\epsilon_a = \frac{-1.042 - (-1.175)}{-1.042} = 12.76\
\]

\( h=0.25: \)

\[
f'(0) \approx \frac{f(0+0.25) - f(0-0.25)}{2(0.25)} = \frac{e^{-0.25} - e^{(-0.25)}}{2(0.25)} = -1.010
\]

\[
\epsilon_a = \frac{-1.010 - (-1.042)}{-1.010} = 3.14\
\]

\( h=0.125: \)

\[
f'(0) \approx \frac{f(0+0.125) - f(0-0.125)}{2(0.125)} = \frac{e^{-0.125} - e^{(-0.125)}}{2(0.125)} = -1.003
\]

\[
\epsilon_a = \frac{-1.003 - (-1.010)}{-1.003} = 0.78\
\]
Problem 2

Solve the following ODE using Heun’s method. Solve at x=0.5 with a step size of 0.5. The initial conditions are y(0)=4 and y'(0)=0 (20 pts)

\[
d''(y) + 0.5 \frac{dy}{dx} + 7y = 0
\]

Convert into two first-order ODEs:

\[
\frac{dy}{dx} = z \quad y(0) = 4
\]

\[
\frac{dz}{dx} = -0.5z - 7y \quad z(0) = 0
\]

\[y'(0) = 0\]

\[z'(0) = -0.5(0) - 7(4) = -28\]

\[y''(0) = y(0) + h\frac{dy}{dx}(0) = 4 + 0.5(0) = 4\]

\[z''(0) = z(0) + h\frac{dz}{dx}(0) = 0 + 0.5(-28) = -14\]

\[y'(0.5) = y(0) + h\frac{f_y(0,0,4) + f_y(0.5,4,-14)}{2} = 4 + 0.5\frac{0 + (-14)}{2} = 0.5\]

\[z'(0.5) = z(0) + h\frac{f_z(0,0,4) + f_z(0.5,4,-14)}{2} = 0 + 0.5\frac{-28 + (-21)}{2} = -12.25\]

\[y''(0.5) = y(0) + h\frac{f_y(0,0,4) + f_y(0.5,0.5,-12.25)}{2} = 4 + 0.5\frac{0 + (-12.25)}{2} = 0.9375\]
Problem 3:

a) Evaluate the following integral using the trapezoid method with 5 intervals (10pts)

\[ \int_{0.5}^{2} \sqrt{\frac{e^x - e^{-x}}{e^x + e^{-x}}} \, dx \]

\[ h = \frac{2 - 0.5}{5} = 0.3 \]

\[ I = h \frac{f(0.5) + 2f(0.8) + 2f(1.1) + 2f(1.4) + 2f(1.7) + f(2.0)}{2} \]

\[ = h \frac{0.6798 + 2(0.8149) + 2(0.8947) + 2(0.9409) + 2(0.9672) + 0.9818}{2} \]

\[ = 1.334552 \]

b) Solve the integral using Simpson’s 1/3 rule with h=0.375 (10pts)

\[ I = h \frac{f(0.5) + 4f(0.875) + 2f(1.25) + 4f(1.625) + f(2.0)}{3} \]

\[ = h \frac{0.6798 + 4(0.8390) + 2(0.9210) + 4(0.9619) + 0.9818}{3} \]

\[ = 1.338431 \]
Problem 4:

Show how you could find the point on the curve \( y = x^2 \) that is closest to the point (3,1) by transforming the problem to a minimum optimization problem? You do not have to solve the optimization problem. (10pts)

The distance between the curve and the point (3,1) is \( \sqrt{(x - 3)^2 + (y - 1)^2} \).

If you substitute \( y = x^2 \), you get \( \sqrt{(x - 3)^2 + (x^2 - 1)^2} \), and that is the function you need to minimize. If you solve the optimization problem, you will get \( x = 1.672982, y = 0.493359 \) as the closest point.

Problem 5

Why can’t you just reduce the continuously interval size to improve the relative error of a Simpson’s rule based integration? (5 pts)

If you continue to reduce the interval size, you will start encountering round-off error and the relative error will start to increase instead of decreasing.

Problem 6

What is the difference between global error and local error when solving an ODE? (5 pts)

Local error is the error that you get at each step as you move away from the initial conditions. The global error is total error accumulated over all steps from the initial conditions.
Problem 7

Solve the following linear programming problem

Maximize \[ Z = 15x_2 + 6x_4 + 25x_5 \]
\[ x_1 = x_2 + x_3 \]
\[ x_3 = x_4 + x_5 \]
\[ x_1 \leq 5 \]
\[ x_2 \leq 4 \]
\[ x_3 \leq 3 \]
\[ x_4 \leq 1 \]
\[ x_5 \leq 2 \]
\[ x_1, x_2, x_3, x_4, x_5 \geq 0 \]

(20pts)

You can simplify the problem a little if you remove \( x_1 \) and \( x_2 \) from the constraints as follows:

\[ x_2 + x_4 + x_5 \leq 5 \]
\[ x_2 \leq 4 \]
\[ x_4 + x_5 \leq 3 \]
\[ x_4 \leq 1 \]
\[ x_5 \leq 2 \]
\[ x_1, x_2, x_3, x_4, x_5 \geq 0 \]

The initial tableau is shown below:

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<th>( x_5 )</th>
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<th>( S_2 )</th>
<th>( S_3 )</th>
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Pivot on \( x_5 \) since it has the largest coefficient: Replace \( S_1 \) and remove \( x_4 \) from other rows.

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Pivot on \( x_2 \) since it has the largest coefficient: Replace \( S_j \) and remove \( x_2 \) from other rows.

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No remaining positive \( Z \) coefficients, so we have arrived at final solution: \( x_2=3, x_4=0, x_5=2, Z=95 \). Using the equalities from above, \( x_1=5 \) and \( x_3=2 \),