1. Solve

\[
\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial \Theta}{\partial \rho} \right) = \frac{1}{\alpha^2} \frac{\partial \Theta}{\partial \tau}
\]

with boundary and initial conditions

\begin{align*}
\Theta &\text{ is finite } @ \rho = 0 \\
\Theta &\text{ = 0 } @ \rho = 1 \\
\Theta &\text{ = 1 } @ \tau = 0
\end{align*}

Assume separation of variables \( \Theta(\rho, \tau) = R(\rho) T(\tau) \), then

\[
\frac{1}{\rho^2 R} \left[ \rho^2 R' \right]' = \frac{T'}{T} = -\lambda^2
\]

or

\[
\rho^2 R'' + 2 \rho R' + \lambda \rho^2 R = 0,
\]

and

\[
T' + \lambda^2 T = 0.
\]

The solutions are

\[
R = A j_0(\lambda \rho) + B y_0(\lambda \rho),
\]

where \( j_0(\lambda \rho), y_0(\lambda \rho) \) are spherical Bessel functions, and

\[
T = Ce^{-\lambda^2 \tau}.
\]

At \( \rho = 0 \), \( R \) is finite, hence \( B = 0 \), and at \( \rho = 1 \), \( R = 0 \), hence \( j_0(\lambda) = 0 \). But

\[
J_0(\frac{\sin z}{Z}) \quad \text{(and } Y_0(\frac{\cos z}{z}) \text{). Then}
\]

\[
\lambda_n = n\pi.
\]

Note that 1) if \( n = 0 \) that \( j_0(z) = 1 \neq 0 \) and hence \( n \neq 0 \), and 2) if \( n < 0 \) that

\[
J_0(\lambda_{-n} \rho) = \frac{-\sin(n \pi \rho)}{-n \pi \rho} = \frac{\sin(n \pi \rho)}{n \pi \rho} = J_0(\lambda_n \rho)
\]

and we do not need to include \( n < 0 \). Hence
\[ \lambda_n = n\pi, \ n = 1, 2, 3, \ldots \]

Then the solution is

\[ \Theta = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2t} \frac{\sin(n\pi\rho)}{n\pi} \]

The initial condition yields

\[ 1 = \sum_{n=1}^{\infty} C_n \frac{\sin(n\pi\rho)}{n\pi}, \]

or

\[ \rho = \sum_{n=1}^{\infty} \frac{C_n}{n\pi} \sin(n\pi\rho). \]

But

\[ \int_{0}^{1} \sin(n\pi\rho) \sin(m\pi\rho) \, d\rho = 0 \quad \text{if} \quad n \neq m, \]

while

\[ \int_{0}^{1} \sin^2(n\pi\rho) \, d\rho = \frac{1}{2} \quad \text{if} \quad n = m. \]

Then

\[ \int_{0}^{1} \rho \sin(m\pi\rho) \, d\rho = \frac{C_m}{m\pi} \int_{0}^{1} \sin^2(n\pi\rho) \, d\rho = \frac{C_m}{2m\pi}. \]

Integrating the left hand side by parts let \( u = \rho \), then \( du = d\rho \) and \( dv = \sin(n\pi\rho) \, d\rho \) gives

\[ \frac{C_n}{2m\pi} = \left[ -\frac{\rho}{m\pi} \cos(m\pi\rho) \right]_{0}^{1} + \frac{1}{m\pi} \int_{0}^{1} \cos(m\pi\rho) \, d\rho, \]

or

\[ C_n = 2m\pi \left[ -\frac{1}{m\pi} \cos(m\pi) \right] = 2(-1)^{m-1}. \]

Then
An alternate method to solve the differential equation in the radial coordinate for a spherical solution of the Laplace equation

\[ \left[ \rho^2 R' \right]' + \rho^2 \lambda^2 R = 0, \]

can be developed as follows.

First expand the derivative

\[ \rho^2 R' + 2 \rho R'' + \rho^2 \lambda^2 R = 0. \]

Let

\[ z = \lambda \rho. \]

Then

\[ z^2 \frac{d^2 R}{dz^2} + 2z \frac{dR}{dz} + z^2 R = 0 \]

and let
\[ R = \frac{X}{z}. \]

Then

\[ \frac{dR}{dz} = \frac{1}{z} \frac{dX}{dz} - \frac{1}{z^2} X, \]

and

\[ \frac{d^2 R}{dz^2} = \frac{1}{z} \frac{d^2 X}{dz^2} - \frac{2}{z^2} \frac{dX}{dz} + \frac{2}{z^3} X. \]

The differential equation becomes

\[ z^2 \left[ \frac{1}{z} \frac{d^2 X}{dz^2} - \frac{2}{z^2} \frac{dX}{dz} + \frac{2}{z^3} X \right] + 2z \left[ \frac{1}{z} \frac{dX}{dz} - \frac{1}{z^2} X \right] + z^2 \left[ \frac{X}{z} \right] = 0. \]

Simplifying

\[ z \frac{d^2 X}{dz^2} + z X = 0. \]

Since \( z \neq 0 \) everywhere, then

\[ \frac{d^2 X}{dz^2} + X = 0 \]

The solution is

\[ X = C_1 \sin z + C_2 \cos z. \]

Substituting for \( X, z \)

\[ R = C_1 \frac{\sin (\lambda \rho)}{\lambda \rho} + C_2 \frac{\cos (\lambda \rho)}{\lambda \rho}. \]