Homework No. 11

1. Minimize the distance between the origin \((x, y) = (0, 0)\) and the ellipse

\[
\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1
\]

using Lagrange multipliers.

The square of the distance, \(f(x, y)\), is to be minimized is given by

\[f(x, y) = x^2 + y^2\]

and subject to the constraint

\[g(x, y) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1 = 0,\]

We can minimize

\[F(x, y) = f(x, y) - \lambda g(x, y) = x^2 + y^2 - \lambda \left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1\right].\]

Then the equations to find the minimum distance are:

1) \(\frac{\partial F}{\partial x} = 0\) or \(2x - \frac{2\lambda x}{a^2} = 0\) and \(\left(1 - \frac{\lambda}{a^2}\right)x = 0\),

2) \(\frac{\partial F}{\partial y} = 0\) or \(2y - \frac{2\lambda y}{b^2} = 0\) and \(\left(1 - \frac{\lambda}{b^2}\right)y = 0\), and

3) \(g(x, y) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1 = 0\),

for the unknowns \((x, y, \lambda)\).
The first two equations have four possible solutions

1. \( x = 0, y = 0 \) but this is not on the ellipse \( g(x, y) = 0 \).
2. \( \lambda = a^2, y = 0 \) is a good solution.
3. \( x = 0, \lambda = b^2 \) is a good solution.
4. \( \lambda = a^2, \lambda = b^2 \) is a circle since \( a^2 = b^2 \) and all distances from the origin are equal.

For case 2 above \( x = \pm a \) and the square of the distance is \( f(x, y) = a^2 \).
For case 3 above \( y = \pm b \) and the square of the distance is \( f(x, y) = b^2 \).

2. Find the ordinary differential equation and initial condition(s) for

\[
y(x) = \frac{1}{2!} t^2 h(t) dt.
\]

Then

\[
y'(x) = \frac{x}{a} (x-t) h(t) dt + \frac{1}{2!} (x-t)^2 h(t) \left( \frac{dx}{dt} \right) \bigg|_{t=x} = \frac{x}{a} (x-t) h(t) dt
\]

or

\[
y''(x) = \frac{x}{a} h(t) dt + (x-t) h(t) \left( \frac{dx}{dt} \right) \bigg|_{t=x} = \frac{x}{a} h(t) dt
\]

and

\[
y'''(x) = h(t) \left( \frac{dx}{dt} \right) \bigg|_{t=x}.
\]

or

\[
y'''(x) = h(x)
\]

The initial conditions are:

\[
y(a) = 0, \quad y'(a) = 0, \quad \text{and} \quad y''(a) = 0.
\]