Transient Temperature Distribution in a Sphere

Transient temperatures are described by

$$\nabla^2 T = \frac{1}{\alpha^2} \frac{\partial T}{\partial t}; \quad T = 0 \quad @ \quad r = a; \quad T = T_0(r, \theta, \phi)$$

In spherical coordinates this becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial T}{\partial \phi} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 T}{\partial \theta^2} = \frac{1}{\alpha^2} \frac{\partial T}{\partial t}$$

Applying separation of variables

$$T = R(r) \Phi(\phi) \Theta(\theta) \xi(t)$$

then

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 R' \right) + \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \Phi' \right) + \frac{1}{r^2 \sin^2 \phi} \Theta' \frac{\Theta'}{\Theta} = \frac{1}{\alpha^2} \frac{T'}{T}$$

Take

$$\frac{\Theta'}{\Theta} = -\beta^2$$

Then

$$\Theta' + \beta^2 \Theta = 0$$

The solution is

$$\Theta = A \cos \beta \theta + B \sin \beta \theta,$$

Continuity @ $\theta = 0, 2\pi$ requires, $\Theta(0) = \Theta(2\pi), \Theta'(0) = \Theta'(2\pi), \Theta''(0) = \Theta''(2\pi), $

Then $A = A \cos 2\pi \beta, B = B \cos 2\pi \beta, \beta = m = 0, 1, 2, \ldots$ and

$$\Theta = A_m \cos m \theta + B_m \sin m \theta$$

Note that for $m = 0, \Theta = A_0$

Now take

$$\frac{1}{r^2 \sin \phi} \frac{(\sin \phi \Phi')'}{\Phi} - \frac{m^2}{r^2 \sin^2 \phi} = \frac{-\gamma^2}{r^2}$$

then

$$\Phi' + \cot \phi \Phi + \left( \frac{\gamma^2}{\sin^2 \phi} - \frac{m^2}{\sin^2 \phi} \right) \Phi = 0$$

or
\[ \Phi = BP_n^m(\cos \phi) + \overline{B}Q_n^m(\cos \phi) \]

where \( P_n^m(\cos \phi), Q_n^m(\cos \phi) \) are associated Legendre polynomials and \( \gamma^2 = n(n+1) \)

and \( n \) is an integer.

Since the solution is continuous \( @ \phi = 0, \pi \) \( \overline{B} = 0 \)

Note that \( P_n^m(\cos \phi) = \left(1 - \cos^2 \phi\right)^{m/2} P_n(\cos \phi) \)

For the time dependence

\[ \frac{1}{\alpha^2} T' = -\lambda^2 \Rightarrow T' + \alpha^2 \lambda^2 T = 0 \]

or

\[ T = Ce^{-\alpha^2 \lambda^2 t} \]

Finally for \( R(r) \),

\[ \frac{1}{r^2 R} \left(r^2 R'\right) - \frac{\gamma^2}{r^2} = -\lambda^2 \]

or

\[ R'' + \frac{2}{r} R' + \left[ \lambda^2 - \frac{n(n+1)}{r^2} \right] R = 0 \]

This is almost Bessel’s equation. Hence try \( R = r^p S \), then pick \( p \) to make \( \frac{2}{r} R' = 1 S' \). Then

\[ R' = r^p S' + pr^{p+1} S, \]

\[ R'' = r^p S'' + 2 pr^{p+1} S' + p(p-1)r^{p-2} S \]

and

\[ r^p S'' + 2pr^{p-1} S' + p(p-1)r^{p-2} S + 2pr^{p-2} S + \lambda^2 r^p S - n(n+1)r^{p-2} S = 0 \]

or

\[ r^{-1/2} S'' + r^{-3/2} S' + \lambda^2 r^{-1/2} S + \left\{ -\frac{1}{2} \left( -\frac{3}{2} \right) - 1 - n^2 - n \right\} r^{-5/2} S = 0 \]

Then
\[ S^\sigma + \frac{1}{r} S^\prime + \left( \lambda^2 - \left( \frac{n^2 + n + \frac{1}{4}}{r^2} \right) \right) S^\prime = 0 \]

or

\[ S^\sigma + \frac{1}{r} S^\prime + \left( \lambda^2 - \left( \frac{n + \frac{1}{2}}{r^2} \right) \right) S = 0 \]

then

\[ S = D' J_{n + \frac{1}{2}}(\lambda r) + D' Y_{n + \frac{1}{2}}(\lambda r) \]

or

\[ R = D' J_{n + \frac{1}{2}}^{\frac{1}{2}}(\lambda r) + D' Y_{n + \frac{1}{2}}^{\frac{1}{2}}(\lambda r) \]

These are the spherical Bessel functions defined by

\[ j_n(z) = \frac{\pi}{2z} J_{n + \frac{1}{2}}(z), \quad y_n(z) = \frac{\pi}{2z} Y_{n + \frac{1}{2}}(z) \]

Note that

\[ j_0(z) = \frac{\sin z}{z}, \quad y_0(z) = -\frac{\cos z}{z} \]

\[ j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}, \quad y_1(z) = -\frac{\cos z}{z^2} - \sin z \]

etc. (use a recurrence relation for the rest)

Then

\[ R = D j_n(\lambda r) + D y_n(\lambda r) \]

and \( R \) finite @ \( r = 0 \) hence \( D = 0 \) while \( R(0) = 0 \), hence \( j_n(\lambda a) = 0 \) which gives \( \lambda_p \).

Then

\[ T = \sum_{n, m = 0}^{\infty} C_{mnp} e^{-\alpha^2 \tilde{z}_i} j_n(\tilde{\lambda}_p t) P_m^\nu(\cos(\phi)) \cos(m\theta) + \sum_{n = 0}^{\infty} C_{mnp} e^{-\alpha^2 \tilde{z}_i} j_n(\tilde{\lambda}_p t) P_m^\nu(\cos(\phi)) \sin(m\theta) \]

We can get \( C \) and \( \widetilde{C} \) by an orthogonal series expansion @ \( t = 0 \), when \( T = T_0(r, \theta, z) \).
**Potential Flow About A Sphere**

Let \( \vec{v}(x, y, z) \) be the fluid velocity and \( \rho(x, y, z) \) be the mass density. Then \( \vec{Q} = \rho \vec{v} \) is the flow/unit area perpendicular to \( \vec{v} \). Then the flow into volume \( V \) with surface \( S \) and outward normal \( \vec{n} \) is

\[
\int_S \vec{Q} \cdot \vec{n} dS = -\int_V \frac{\partial \rho}{\partial t} dV
\]

i.e., flow out is rate of loss of \( m \) in \( V \), or

\[
\int_V \left( \nabla \cdot \vec{Q} + \frac{\partial \rho}{\partial t} \right) dV = 0
\]

for all \( V \) or

\[
\nabla \cdot (\rho \vec{v}) + \frac{\partial \rho}{\partial t} = 0
\]

For incompressible flow \( \rho = \text{constant} \) and the governing equation becomes

\[
\nabla \cdot \vec{v} = 0
\]

Now consider a rotating fluid where \( \vec{v} = \vec{\omega} \times \vec{r} \), then

\[
\nabla \cdot \vec{v} = \nabla \cdot (\vec{\omega} \times \vec{r}) = (\nabla \times \vec{\omega}) \cdot \vec{r} + \vec{\omega} \cdot (\nabla \times \vec{r})
\]

but \( \nabla \times \vec{r} = 0 \) and since \( \vec{\omega} = \text{constant} \) and for this case \( \nabla \cdot \vec{v} = 0 \) (i.e., the flow is incompressible)

Finally since \( \nabla \times \vec{v} = \nabla \times (\vec{\omega} \times \vec{r}) \) becomes \( \nabla \times \vec{v} = 2\vec{\omega} \). Then if \( \vec{\omega} \) is zero the flow is irrotational and incompressible. This is referred to as potential flow and satisfies

\[
\nabla \cdot \vec{v} = 0
\]

and

\[
\nabla \times \vec{v} = 0
\]

Take \( \vec{v} = \nabla \varphi \) then \( \nabla \times \vec{v} = \nabla \times (\nabla \varphi) = 0 \) is identically satisfied and

\[
\nabla \cdot \vec{v} = \nabla^2 \varphi = 0
\]

governs the flow.
For the axisymmetric flow about a sphere all the \( \theta \) positions look the same (The flow enters along \( \phi = 0 \) and leaves along \( \phi = \pi \)) so that
\[
\frac{\partial}{\partial \theta} = 0 \quad \text{and}
\]
\[
\nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial \phi}{\partial \phi} \right) = 0.
\]
At \( r = \infty \), \( \vec{v} = -v_0 \hat{k} \) or
\[
\nu_r = -v_0 \cos \phi
\]
@ \( r = a \), \( \vec{v} \) is parallel to the sphere or
\[
\nu_r = 0.
\]
Assume \( \phi = R(r)\Phi(\phi) \) then
\[
\frac{1}{r^2} \left( r^2 R' \right) + \frac{1}{r^2 \Phi \sin \phi} \left( \Phi' \sin \phi \right) = 0
\]
Assume \( \frac{1}{\Phi \sin \phi} \frac{d}{d\phi} \left( \sin \phi \frac{\partial \Phi}{\partial \phi} \right) = \lambda^2 \) but from previous discussions for continuity @ \( \phi = 0, \pi \), \( \lambda^2 = n(n+1) \) or
\[
\Phi = A P_n(\cos \phi) + B Q_n(\cos \phi)
\]
Finiteness at \( \phi = 0, \pi \) gives \( B = 0 \)
We require that
\[
\left( r^2 R' \right) - n(n+1)R = 0, \text{ or } r^2 R'' + 2rR' - n(n+1)R = 0
\]
This is an Euler or equidimensional equation. The same thing happens for other axisymmetric problems in polar coordinates. Hence take
\[
R = Cr^\alpha
\]
then
\[
\alpha(\alpha - 1)Cr^\alpha + 2\alpha Cr^\alpha - n(n+1)Cr^\alpha = 0
\]
Since \( C, r^\alpha \neq 0 \) in general then
\[
\alpha^2 - \alpha + 2\alpha - n(n+1) = 0, \text{ or } \alpha^2 + \alpha - n(n+1) = 0,
\]
and
\[
\alpha = -1 \pm \sqrt{1 + 4n(n+1)}
\]
But \( 4n(n+1) + 1 = 4n^2 + 4n + 1 = (2n+1)^2 \) hence
\[ \alpha = \frac{-1 \pm (2n + 1)}{2} = n, \quad - (n + 1) \]

Note that \( \alpha^2 + \alpha - n \left( n + 1 \right) = (\alpha - n)(\alpha + n + 1) \). Then

\[ R = C_n r^n + D_n / r^{(n+1)} \]

Since flow is finite at \( r = \infty \) and \( v \sim R' \) we must set \( C_n = 0 \) for \( n = 2, 3, \ldots, \infty \) then

\[ \varphi = C_0 P_0 (\cos \phi) + C_1 r P_1 (\cos \phi) + \frac{D_0}{r} P_0 (\cos \phi) + \frac{D_1}{r^2} P_1 (\cos \phi) + \sum_{n=2}^{\infty} \frac{D_n}{r^{n+1}} P_n (\cos \phi) \]

but \( @ r = a, \ v_r = 0 \Rightarrow \frac{\partial \varphi}{\partial r} = 0, @ r = \infty \),

and

\[ \ddot{v} = -v_0 \ddot{k} \quad (v_r = -v_0 \cos \phi) \]

or

\[ \frac{\partial \varphi}{\partial r} = -v_0 \cos \phi \quad @ r = \infty \]

and from \( \frac{\partial \varphi}{\partial r} = 0 \quad @ r = a \)

\[ C_1 P_1 (\cos \phi) - \frac{D_0}{a^2} P_1 (\cos \phi) - \frac{2D_1}{a^3} P_1 (\cos \phi) + \sum_{n=2}^{\infty} \frac{-(n + 1)D_n}{a^{n+2}} P_n (\cos \phi) = 0 \]

Then \( D_0 = 0, \ D_n = 0, \ n \geq 2, \ D_1 = \frac{a^3}{2} C_1 \) and hence

\[ \varphi = C_0 P_0 (\cos \phi) + C_1 \left[ r + \frac{a^3}{2r^2} \right] P_1 (\cos \phi) \]

But \( P_0 (\cos \phi) = 1, \ P_1 (\cos \phi) = \cos \phi \) so that

\[ \varphi = C_0 + C_1 \left( r + \frac{a^3}{2r^2} \right) \cos \phi \]

Finally at \( r = \infty, \ \frac{\partial \varphi}{\partial r} = -v_0 \cos \phi = C_1 \cos \phi \) or \( C_1 = -v_0 \) then
\[
\phi = C_0 - \nu_0 \left( r + \frac{a^3}{2r^2} \right) \cos \phi
\]

and using \( \vec{v} = \vec{\nabla} \phi = \vec{u}_r \frac{\partial \phi}{\partial r} + \vec{u}_\phi \frac{\partial \phi}{\partial \phi} \) we find that

\[
\vec{v} = -\nu_0 \left( 1 - \frac{a^3}{r^3} \right) \cos \phi \vec{u}_r + \nu_0 \left( 1 + \frac{a^3}{2r^4} \right) \sin \phi \vec{u}_\phi
\]

Recall that \( \phi = \text{const} \) is perpendicular to the flow lines. Take along the flow line \( \psi = \text{const} \) which are easier to interpret then

\[\vec{\nabla} \phi \cdot \vec{\nabla} \psi = 0\]

or

\[\vec{v} \cdot \vec{\nabla} \psi = 0\]

and since

\[\vec{\nabla} \psi = \vec{u}_r \frac{\partial \psi}{\partial r} + \vec{u}_\phi \frac{\partial \psi}{\partial \phi}\]

then

\[
\frac{\partial \psi}{\partial r} \left[ \nu_0 \left( 1 - \frac{a^3}{r^3} \right) \cos \phi \right] + \frac{1}{r} \frac{\partial \psi}{\partial \phi} \left[ \nu_0 \left( 1 + \frac{a^3}{2r^4} \right) \sin \phi \right] = 0
\]

or

\[
\left( 1 - \frac{a^3}{r^3} \right) \cos \phi \frac{\partial \psi}{\partial r} + \left( 1 + \frac{1}{r} \frac{a^3}{2r^4} \right) \sin \phi \frac{\partial \phi}{\partial \phi} = 0
\]

which is satisfied by

\[
\psi = C_1 - \frac{\nu_0}{2} \left( r^2 - \frac{a^3}{r} \right) \sin^2 \phi.
\]

Since \( z = r \cos \phi \), \( x = r \sin \phi \) then at \( r = \infty \), \( \psi = C_1 - \frac{\nu_0}{2} r^2 \sin^2 \phi = C_1 - \frac{\nu_0 x_\infty^2}{2} \)

Pick \( x = x_\infty \) as point at \( \infty \) to label a “streamline” then

\[
C_1 - \frac{\nu_0 x_\infty^2}{2} = C_1 - \frac{\nu_0}{2} \left( r^2 - \frac{a^3}{r} \right) \sin^2 \phi
\]

or
\[ x^2 = \left( r^2 - \frac{a^3}{r} \right) \sin^2 \phi = x^2 - \frac{a^3 \sin^2 \phi}{r} = x^2 - \frac{a^3 x}{r^3} \]

and

\[ x^2 = x^2 \left( 1 - \frac{a^3}{r^3} \right) \text{ then } \frac{x}{x_{\infty}} = \frac{\left( 1 - \frac{a^3}{r^3} \right)}{r}^{1/2} \]
One Electron Atom (Hydrogen)

Quantum Mechanics requires that measurement quantities be represented as operators. For example:
1) Position is multiplication \( \hat{x} \cdot (\ ) \)
2) Momentum is \(-i\hbar \hat{\nabla}(\ )\)
3) Time is multiplication, and
4) Energy is \(i\hbar \frac{\partial}{\partial t}(\ )\).

These quantities do not commute: e.g. consider \( x \) direction
\[ [x, p_x]\psi = (xp_x - p_x x)\psi = -i\hbar \frac{\partial \psi}{\partial x} + i\hbar \frac{\partial}{\partial x} (x\psi) \]
\[ [x, p_x]\psi = i\hbar \frac{\partial \psi}{\partial x} + i\hbar \frac{\partial \psi}{\partial x} + i\hbar \psi = i\hbar \psi \text{ or } [x, p_x] = i\hbar \]
The energy is
\[ E = \frac{1}{2} mv^2 + V = \frac{p^2}{2m} + V \]
then
\[ E\psi = \frac{1}{2m} p^2 \psi + V\psi \]
But \( p^2 = -\hbar^2 \nabla^2 \), and \( E = i\hbar \frac{\partial}{\partial t} \) or
\[ i\hbar \frac{\partial \psi}{\partial t} = -\hbar^2 \frac{\nabla^2 \psi}{2m} + V\psi \]

From quantum mechanics postulates the probability for the electron to be at \( \vec{r} \) is proportional to \( \psi^* \vec{r} \psi(\vec{r}) \).

If the system is stationary \( E \) is a constant (i.e., an eigenvalue) or
\[ -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi = E\psi \]
For the electron of mass \( m = m_e \) and charge \( e \) for the proton at the origin
\[ V = \frac{e^2}{r} \]
Then if \( \psi = R(r) \Phi(\phi) \Theta(\theta) \)
\[
\frac{\hbar^2}{2m_e} \left\{ \frac{1}{r^2} R \left( r^2 R' \right) + \frac{1}{r^2 \sin \phi} \left( \sin \phi \Phi \right)' + \frac{\Theta^*}{r^2 \sin^2 \theta} \right\} + \frac{e^2}{r} + E = 0
\]
then take
\[
\frac{\Theta^*}{\Theta} = -m^2,
\]
and
\[
\Theta = Ae^{im\theta} + Be^{-im\theta}.
\]
Also
\[
\frac{1}{r^2 \sin \phi} \left( \sin \phi \Phi \right)' - \frac{m^2}{r^2 \sin^2 \phi} = 0,
\]
or
\[
\Phi = CP_{\ell}^m(\cos \phi) + DQ_{\ell}^m(\cos \phi),
\]
For finite \( \Phi, D = 0 \).
We can normalize so that
\[
\int_0^{2\pi} \int_0^{\pi} \left| KP_{\ell}^m(\cos \phi) e^{im\theta} \right|^2 \sin \phi d\phi d\theta = 1,
\]
and let
\[
KP_{\ell}^m(\cos \phi) e^{im\theta} = Y_{\ell}^m(\phi, \theta)
\]
See table of hydrogen functions to find \( Y_{\ell}^m \).
Finally for the radial dependence
\[
\frac{1}{r^2} \left( r^2 R \right)' - \frac{\ell (\ell + 1)}{r^2} + \frac{2m_e}{\hbar^2} \left[ E + \frac{e^2}{r} \right] = 0
\]
or
\[
R^* + \frac{2}{r} R' + \frac{2m_e}{\hbar^2} \left[ E - \frac{e^2}{r} - \frac{\hbar^2 \ell (\ell + 1)}{2m_e r^2} \right] R = 0
\]
Use a Frobenius series to solve but first the following substitutions help.

Let \( n = \left\{ \frac{e^2}{2(-E)a_0} \right\}^{1/2} \)
where \( a_0 = \frac{\hbar^2}{m_e e^2} \),
\[
\rho = \sqrt{\frac{8m_e (-E)}{\hbar^2}} r
\]
Results in
\[
\frac{d^2 R}{d\rho^2} + 2 \frac{dR}{\rho \, d\rho} + \left[ \frac{1}{4} \frac{n}{\rho} - \frac{\ell(\ell + 1)}{\rho^2} \right] R = 0
\]

and let \( R = e^{-\rho^2/2} F(\rho) \) then we will get

\[
\frac{d^2 F}{d\rho^2} + \left( \frac{2}{\rho} - 1 \right) \frac{dF}{d\rho} + \left[ \frac{n-1}{\rho} - \frac{\ell(\ell + 1)}{\rho^2} \right] F = 0
\]

Now set \( F = \sum a_j \rho^j \) this will give gives \( a_{j+1} = \left[ \frac{j-(n-1)}{j(j+1) + 2(j+1) - \ell(\ell + 1)} \right] a_j \)

Note denominator vanishes when \((j+2)(j+1) = \ell(\ell + 1) \) or \( j = \ell - 1 \) hence \( a_j = 0 \) for \( j \leq \ell - 1 \) 1st term is \( a_\ell \rho \) but the answer must also be finite \( \rho = \infty \) which results in \( n = \text{integer} \) and \( -E_n = \frac{e^2}{2a_0 \hbar^2} \) (Note that \( \ell < n \).)

The resulting polynomials are associated Laguerre functions \( L_{\ell n} \) and are normalized by \[ \int_0^\infty \rho^2|R_{n\ell}|^2 \, d\rho = 1 \] where \( R_{n\ell} = e^{-\rho^2/2} \rho^\ell L_{\ell n}(\rho) \).

**Normalized Hydrogen Wave Functions**

\[
\sigma = r / a_0 = \frac{n\rho}{\hbar}, \quad \Psi_{n\ell m} = a_0^{3/2} \Psi_{n\ell m}, \quad a_0 = \frac{\hbar^2}{me^2}
\]

\[
\Psi_{100} = e^{-\sigma / 2} \sqrt{\frac{8}{\pi}}
\]

\[
\Psi_{200} = \frac{e^{-\sigma / 2}}{4\sqrt{2\pi}} (2 - \sigma)
\]

\[
\Psi_{210} = \frac{e^{-\sigma / 2}}{4\sqrt{2\pi}} \sigma \cos \phi
\]

\[
\Psi_{21\pm 1} = \frac{e^{-\sigma / 2}}{8\sqrt{2\pi}} \sigma \sin \phi e^{\pm i\theta}
\]

\[
\Psi_{300} = \frac{e^{-\sigma / 3}}{81\sqrt{3\pi}} (27 - 18\sigma + 2\sigma^2)
\]

\[
\Psi_{310} = \frac{\sqrt{2} e^{-\sigma / 3}}{81\sqrt{\pi}} (6 - \sigma) \sigma \cos \phi
\]

\[
\Psi_{31\pm 1} = \frac{e^{-\sigma / 3}}{81\sqrt{\pi}} (6 - \sigma) \sigma \sin \phi e^{\pm i\theta}
\]

\[
\Psi_{320} = \frac{e^{-\sigma / 3}}{81\sqrt{6\pi}} \sigma^2 (3\cos^2 \phi - 1)
\]

\[
\Psi_{32\pm 1} = \frac{e^{-\sigma / 3}}{81\sqrt{\pi}} \sigma^2 \sin \phi \cos \phi e^{\pm i\theta}
\]

\[
\Psi_{32\pm 2} = \frac{e^{-\sigma / 3}}{162\sqrt{\pi}} \sin^2 \phi e^{\pm 2i\theta}
\]
Electron Probability Density In The Hydrogen Atom

Note: The density distribution is rotationally symmetric about the z-axis

<table>
<thead>
<tr>
<th>State</th>
<th>( n, \ell )</th>
</tr>
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<td>1s</td>
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<tr>
<td>2s</td>
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<tr>
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<td>2, 1</td>
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<td>3s</td>
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<tr>
<td>3d</td>
<td>3, 2</td>
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</tbody>
</table>
Homework #15

In a solid sphere find the transient temperature distribution for a uniform initial temperature, \( T_o \), and a constant surface temperature, where \( T_a < T_o \).

The problem is axisymmetric hence

\[
\frac{\partial T}{\partial \theta} = \frac{\partial T}{\partial \phi} = 0 \text{ (i.e. } T = T(r,t))
\]

The governing equation is

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) = \frac{1}{\alpha^2} \frac{\partial T}{\partial t}
\]

& \quad T \text{ is finite} \at r = 0

\[
T = T_a \at r = a
\]

\[
T = T_0 \at t = 0
\]

Let \( \Theta = \frac{T - T_a}{T_0 - T_a}, \ \rho = r/a, \ \tau = \alpha^2 t / a^2 \)

Then

\[
\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial \Theta}{\partial \rho} \right) = \frac{\partial \Theta}{\partial \tau}
\]

& \quad \Theta \text{ is finite} \at \rho = 0

\[
\Theta = 0 \at \rho = 1
\]

\[
\Theta = 1 \at t = 0
\]