Steady Temperature in a Plate

Consider the case of two dimensional heat conduction in a plate as shown.

\[-\left( q_x + \frac{\partial q_x}{\partial x} dx \right)hdy + q_x hdy -
\left( q_y + \frac{\partial q_y}{\partial y} dy \right)hdx + q_y hdx = 0\]

Then

\[-\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} = 0,\]

But from the Fourier Heat Conduction law

\[q_x = -K \frac{\partial T}{\partial x} \quad \text{and} \quad q_y = -K \frac{\partial T}{\partial y},\]

and if \(K\) is constant

\[K \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) = 0.\]

Example: A Plate Hotter on One Edge

For the plate shown at the right, assume

\[T = X(x)Y(y)\]

then

\[X''Y + XY'' = 0,\]

\[\frac{X''}{X} + \frac{Y''}{Y} = 0.\]

Take

\[\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2.\]

Then

\[X = A' \sin \lambda x + B' \cos \lambda x\]

and

\[Y = C' e^{\lambda y} + D' e^{-\lambda y}.\]

But

\[X(0) = X(a) = 0\]

\[B' = 0 \quad A' \sin \lambda a = 0\]
then
\[ \lambda_n a = n\pi \text{ for } n = 1, 2, \ldots \]
Hence
\[ \lambda_n = \frac{n\pi}{a} \]
Since
\[ Y(0) = 0 \]
then
\[ C' + D' = 0 \]
or
\[ C' = -D'. \]
Which makes
\[ Y_n = C'_n \left[ e^{\lambda_n y} - e^{-\lambda_n y} \right] = C'_n \sinh \lambda_n y \]
Then
\[ T = \sum_{n=1}^{\infty} C_n \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{n\pi y}{a} \right) \]
but at
\[ y = b, \ T = T_o \]
or
\[ T = \sum_{n=1}^{\infty} C_n \sin \left[ \frac{n\pi b}{a} \right] \sin \left( \frac{n\pi x}{a} \right) = T_o \]
Then
\[ C_n \sin \left[ \frac{n\pi b}{a} \right] \int_0^{a} \sin \left[ \frac{n\pi x}{a} \right] dx = T_o \int_0^{a} \sin \left[ \frac{n\pi x}{a} \right] dx \]
The integrations yield
\[ \left\{ C_n \sin \left( \frac{n\pi b}{a} \right) \right\} \left[ \frac{a}{2} - \frac{T_o a}{n\pi} \left[ \cos(n\pi) - 1 \right] \right] = \frac{T_o a}{n\pi} \left[ 1 - \cos n\pi \right] \]
and
\[ C_n = \begin{cases} \frac{4T_o}{n\pi} \sinh \left( \frac{n\pi b}{a} \right) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \]
Then
\[ T = \frac{4T_o}{\pi} \sum_{n=1, 3, 5, \ldots \text{odd}} \frac{\sin \left( \frac{n\pi x}{a} \right) \sinh \left( \frac{n\pi y}{a} \right)}{n \sinh \left( \frac{n\pi b}{a} \right) \sinh \left( \frac{n\pi y}{a} \right)} \]
Transient Temperature Distribution in a Rod

\[- \left( q + \frac{\partial q}{\partial x} \right) A + qA = \left( \rho C_p \frac{\partial T}{\partial t} \right) A dx \]

then

\[ - \frac{\partial q}{\partial x} = \rho C_p \frac{\partial T}{\partial t} \]

Applying the Fourier Heat Conduction Law

\[ q = -K \frac{\partial T}{\partial x} \]

or

\[ K \frac{\partial^2 T}{\partial x^2} = \rho C_p \frac{\partial T}{\partial t} \]

Let

\[ \alpha^2 = \frac{K}{\rho C_p} \]

then

\[ \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial T}{\partial t} \]

Rod raised to temperature \( T_0 \) at one end and initially at \( T_i \), other end insulated, then the governing equations are:

\[ \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial T}{\partial t} \quad \text{for} \quad 0 \leq t < \infty, \quad 0 \leq x \leq \ell \]

\[ T = T_i \quad \text{for} \quad t = 0, \quad 0 \leq x \leq \ell \]

\[ T = T_0 \quad \text{for} \quad x = 0, \quad 0 \leq t \]

\[ \frac{\partial T}{\partial x} = 0 \quad \text{for} \quad x = \ell, \quad 0 \leq t \]

The equations can be nondimensionalized using

\[ \varepsilon = x/\ell, \]

\[ \tau = \alpha^2 t/\ell^2, \]

and

\[ \theta = (T - T_i)/(T_0 - T_i). \]

Then

\[ \frac{(T_0 - T_i)}{\ell^2} \frac{\partial^2 \theta}{\partial \varepsilon^2} = \frac{1}{\alpha^2} \frac{(T_0 - T_i)}{\ell^2} \left( \frac{\alpha^2}{\ell^2} \right) \frac{\partial \theta}{\partial \tau} \]

or
\[
\frac{\partial^2 \theta}{\partial \varepsilon^2} = \frac{\partial \theta}{\partial \tau} \\
\theta = 0 \quad \tau = 0 \\
\theta = 1 \quad \varepsilon = 0 \\
\frac{\partial \theta}{\partial \varepsilon} = 0 \quad \varepsilon = 1
\]

Assume

\[\theta = X(\varepsilon)T(\tau)\]

The partial differential equation yields

\[X'T = XT' \Rightarrow \frac{X'}{X} = \frac{T'}{T} = -\lambda^2,\]

Then we can set

\[X = A' \cos \lambda \varepsilon + B' \sin \lambda \varepsilon,\]

and integrate to get

\[T = Ce^{-\lambda \varepsilon}\]

then since \(T(0) = 0, \ C = 0\) and solution doesn’t work. We need \(\theta = 0 \quad \varepsilon = 0\) and a non-zero initial condition on \(\theta\) at \(\tau = 0\). Try

\[\theta = 1 + \varphi,\]

then

\[\frac{\partial^2 \varphi}{\partial \varepsilon^2} = \frac{\partial \varphi}{\partial \tau} \\
\varphi = -1 \quad \tau = 0, \\
\varphi = 0 \quad \varepsilon = 0, \\
\frac{\partial \varphi}{\partial \varepsilon} = 0 \quad \varepsilon = 1\]

Now try

\[\varphi = X(\varepsilon)T'(\tau)\]

then

\[X = \tilde{A} \cos \lambda \varepsilon + \tilde{B} \sin \lambda \varepsilon,\]

and

\[T = C'e^{-\lambda \varepsilon}\]

Now \(X(0) = 0\), yields \(\tilde{A} = 0\) and \(X' = \lambda \tilde{B} \cos(\lambda \varepsilon)\) making

\[\lambda \cos \lambda = 0\]

hence

\[\lambda_n = 0, \quad \lambda_n = \frac{(2n + 1)}{2} \pi, \quad (n = 0, \pm 1, \pm 2, \ldots)\]
Note that $X_{-1} = B_{-1} \sin \lambda_{-1} t = 0$ and negative values for $n$ are the same eigenfunctions as positive values.

$$\varphi = \sum_{n=0}^{\infty} C_n e^{\frac{(2n+1)^2}{4} \tau \tau} \sin \left[ \frac{(2n+1) \pi e}{2} \right]$$

But @ $t = 0$

$$\varphi = -1 = \sum_{n=0}^{\infty} C_n \sin \left[ \frac{(2n+1) \pi e}{2} \right]$$

Applying the orthogonal conditions

$$- \int_0^1 \sin \left[ \frac{(2n+1) \pi e}{2} \right] de = C_n \int_0^1 \sin^2 \left[ \frac{(2n+1) \pi e}{2} \right] de$$

Integrating the left hand side

$$\left( \frac{2}{(2n+1) \pi} \right) \cos \left[ \frac{(2n+1) \pi e}{2} \right]_0^1 = \frac{C_n}{2} \int_0^1 \left[ 1 - \cos \left( \frac{(2n+1) \pi e}{2} \right) \right] de$$

and the the right hand side

$$2 \left\{ \cos \left( \frac{2n+1 \pi}{2} \right) - 1 \right\} = \frac{C_n}{2}$$

and solving for $C_n$

$$C_n = \frac{4 \left\{ \cos \left( \frac{2n+1 \pi}{2} \right) - 1 \right\}}{(2n+1) \pi}$$

or

$$C_n = -\frac{4}{(2n+1) \pi}.$$ 

Then

$$\varphi = -4 \sum_{n=0}^{\infty} e^{-\frac{(2n+1)^2}{4} \tau \tau} \sin \left[ \frac{(2n+1) \pi e}{2} \right] \left( \frac{2n+1}{2n+1} \pi \right)$$

or

$$\theta = 1 - 4 \sum_{n=0}^{\infty} e^{-\frac{(2n+1)^2}{4} \tau \tau} \sin \left[ \frac{(2n+1) \pi e}{2} \right] \left( \frac{2n+1}{2n+1} \right)$$

Recall that $t = \ell^2 \tau / \alpha^2$, $x = \ell e$, and $T = T_0 + (T_0 - T_1) \theta$.
**Wave Equation for a Rod**

Consider a rod that is suddenly subjected to an external force at one end. The we can use equilibrium to find the motion.

First the force, stress and strain are
\[
F = \sigma A , \quad \sigma = E \varepsilon , \quad \varepsilon = \frac{\partial u}{\partial x} = \varepsilon
\]

The mass times the acceleration is equal to the unbalanced force. Then
\[
dm \frac{\partial^2 u}{\partial t^2} = \left( F + \frac{\partial F}{\partial x} dx - F \right) = \frac{\partial F}{\partial x} dx ,
\]

But
\[
dm = \rho Adx , \quad F = EA \frac{\partial u}{\partial x} .
\]

Note that the mass density \( \frac{\rho g}{g} \) (i.e., the weight density divided by the acceleration of gravity). Then
\[
\rho Adx \frac{\partial^2 u}{\partial t^2} = EA dx \frac{\partial^2 u}{\partial x^2}
\]

or
\[
\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2}
\]

Let
\[
C^2 = \frac{E}{\rho} = \text{Wave Speed}
\]

or
\[
\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}
\]

Now for the rod pushed by external force \( F \) at one end

At \( t = 0, u = 0, \frac{\partial u}{\partial t} = 0 \)

At \( x = 0, F = -EA \frac{\partial u}{\partial x} \) (- compressive)

At \( x = \ell, 0 = EA \frac{\partial u}{\partial x} \)

we need homogeneous boundary conditions. Consider the average motion \( \ddot{u} \) then
\[
m \ddot{u} = F \text{ where } m = \rho A \ell \text{ or } \ddot{u} = \frac{F}{\rho A \ell}
\]

and using the initial conditions
\[ \ddot{u} = \frac{Ft^2}{2 \rho A \ell} \]

may be used for the particular solution.

Try separation of variables by setting

\[ u = X(x)T(t) \]

then

\[ XT'' = C^2 X' T \]

or

\[ \frac{T''}{T} = C^2 \frac{X''}{X} = -\lambda^2 \]

then

\[ X = A' \sin \frac{\lambda x}{C} + B' \cos \frac{\lambda x}{C} \]

and

\[ T = C' \sin \lambda t + D' \cos \lambda t \]

But \( T(0) = T'(0) = 0 \) and hence \( D' = C' = 0 \) which is not a solution. Therefore move zero conditions to \( X \)

Since \( u_{xx} = -\frac{F}{EA} \) @ \( x = 0 \) & \( u_{xx} = 0 \) @ \( x = \ell \)

take

\[ u_{xx} = -\frac{F}{EA\ell} (\ell - x) + y_{xx} \]

or

\[ u = -\frac{F}{EA\ell} \left( \ell x - \frac{1}{2} x^2 \right) + y + G(t) \]

then

\[ u_{xx} = \frac{F}{EA\ell} + y_{xx} \]

but

\[ u_{tt} = y_{tt} + G''(t) \]

\[ C^2 u_{xx} = + \frac{C^2 F}{EA\ell} + C^2 y_{xx} = + \frac{F}{\rho A \ell} + C^2 y_{xx} \]

Note that

\[ \frac{C^2 F}{EA\ell} = \frac{EF}{\rho EA\ell} = \frac{F}{\rho A \ell} \]

and hence take

\[ G''(t) = + \frac{F}{\rho A \ell} \]
integrating

\[ G(t) = \frac{Ft^2}{2\rho A\ell} + G_i t + G_0 \]

Now

\[ u = y - \frac{F}{EA\ell} \left( \frac{x^2}{2} \right) + \frac{Ft^2}{2\rho A\ell} + G_i t + G_0 \]

\[ u_{xx} = y_{xx} - \frac{F}{EA\ell} (\ell - x) \]

\[ u_{xx} = y_{xx} + \frac{F}{\rho A\ell} \]

\[ u_{xx} = y_{xx} - \frac{F}{EA\ell} \]

\[ u = y + \frac{F}{\rho A\ell} \]

The governing partial differential equation and initial conditions can now be found

\[ u_{xx} = C^2 u_{xx} \Rightarrow y_{xx} + \frac{F}{\rho A\ell} = C^2 y_{xx} + \frac{F}{\rho A\ell} \]

or

\[ y_{xx} = C^2 y_{xx} \]

\[ @ t = 0, u = 0 = y - \frac{F}{EA\ell} \left( \frac{x^2}{2} \right) + G_0 \]

For symmetry take

\[ G_0 = + \frac{F\ell^2}{2EA\ell} \]

then

\[ G_0 = - \frac{F}{EA\ell} \left( \frac{x^2}{2} \right) = - \frac{F}{EA\ell} \left( \frac{\ell^2}{2} + x\ell - \frac{x^2}{2} \right) = \frac{F}{2EA\ell} \left[ \ell^2 - 2x\ell + x^2 \right] = - \frac{F}{2EA\ell} (\ell - x)^2 \]

and

\[ @ t = 0, y = - \frac{F}{2EA\ell} (\ell - x)^2 \]

\[ @ t = 0, u_{xx} = y_{xx} + G_i = 0 \]

Take

\[ G_i = 0 \]

then

\[ @ t = 0, y_{xx} = 0 \]
\( \forall \ x = 0 \quad u_x = -\frac{F}{EA} = y_x - \frac{F}{EA} \)

or

\( \forall \ x = 0, \ y_x = 0 \)

\( \forall \ x = \ell \quad u_x = 0 = y_{xx} \)

or

\( \forall \ x = \ell, \ y_{xx} = 0 \)

Summarizing:

\[
\begin{align*}
\frac{\partial^2 y}{\partial t^2} &= C^2 \frac{\partial^2 y}{\partial x^2} &: 0 \leq t < \infty \\
\frac{\partial y}{\partial x} &= 0 &: x = 0, \ell \\
y &= -\frac{F}{2EA\ell} (\ell - x)^2 &: t = 0 \\
\frac{\partial y}{\partial t} &= 0 &: t = 0
\end{align*}
\]

and recall that

\[
u = y + \frac{F}{3EA\ell} (\ell - x)^2 + \frac{Ft^2}{2\rho A\ell}
\]

We can now use separation of variables let \( y = X(x)T(t) \) then

\[
X = A' \frac{\lambda x}{C} + B' \cos \frac{\lambda x}{C},
\]

\[
T = C' \sin \lambda t + D' \cos \lambda t, \text{ and}
\]

\[
X' = \frac{\lambda}{C} \left( A' \cos \frac{\lambda x}{C} - B' \sin \frac{\lambda x}{C} \right)
\]

the boundary conditions can be applied

\( X'(0) = X'(\ell) = 0 \)

\( X'(0) \rightarrow A' = 0 \)

and

\( X'(\ell) \rightarrow \frac{\lambda}{C} B' \sin \frac{\lambda \ell}{C} = 0 \)

then

\( \frac{\lambda n \ell}{C} = n\pi \quad n = 0,1,2,... \)

or

\( \lambda_n = \frac{n\pi}{\ell} \)

Hence

\[
X_n = B_n' \cos \left( \frac{n\pi x}{\ell} \right)
\]
and  
\[ T'(0) = 0 \]
or
\[ T'(t) = \lambda C' \cos \lambda t - \lambda D' \sin \lambda t , \]
hence
\[ C' = 0 \]
unless
\[ \lambda_n = 0 , \]
which is true for \( n = 0 \). Then
\[ T_n = \begin{cases} 
C_0' \sin \lambda_0 t + D_0' \cos \lambda_0 t & (\text{but } \lambda_0 = 0) \\
D_n' \cos \lambda_n t & n > 0 
\end{cases} \]
or
\[ T_n = D_n' \cos \left( \frac{n \pi t}{\ell} \right) \]
or
\[ y_n = C_n \cos \left( \frac{n \pi}{\ell} \right) \cos \left( \frac{n \pi t}{\ell} \right) \]
or
\[ y = \sum_{n=0}^{\infty} C_n \cos \left( \frac{n \pi}{\ell} \right) \cos \left( \frac{n \pi t}{\ell} \right) = C_0 + \sum_{n=1}^{\infty} C_n \cos \left( \frac{n \pi}{\ell} \right) \cos \left( \frac{n \pi t}{\ell} \right) \]
The last condition we need to apply @ \( t = 0 \) where
\[ y(x,0) = -\frac{F}{2EA\ell} (\ell - x)^2 = \sum_{n=0}^{\infty} C_n \cos \left( \frac{n \pi}{\ell} \right) \]
For \( n = 0 \):
\[ -\int_{0}^{\ell} \frac{F}{2EA\ell} (\ell - x)^2 \, dx = \int_{0}^{\ell} C_0 \, dx = C_0 \ell \]
or
\[ -\frac{F \ell^2}{2EA} \int_{0}^{\ell} \left[ 1 - \frac{x}{\ell} \right]^2 d \left( \frac{x}{\ell} \right) = C_0 \ell \]
Let \( \epsilon = 1 - \frac{x}{\ell} \) then
\[ d \left( \frac{x}{\ell} \right) = -d\epsilon \ @ \ x = 0, \epsilon = 1 @ \ x = \ell, \epsilon = 0 \]
and the integral becomes
\[
C_0 = -\frac{F \ell}{2EA} \int_0^1 e^2 \, de = -\frac{F \ell}{2EA} \int_0^1 e^2 \, de = -\frac{F \ell e^3}{6EA} \bigg|_0^1 = -\frac{F \ell}{6EA}.
\]

For \( n > 0 \)

\[
-\int_0^1 \frac{F \ell^2}{2EA} \left(1 - \frac{x}{\ell}\right)^2 \cos\left(\frac{n \pi x}{\ell}\right) \, dx = \int_0^1 C_n \cos^2\left(\frac{n \pi x}{\ell}\right) \, dx
\]

but

\[
\cos^2 x = \frac{1 + \cos 2x}{2}
\]
can be used.

Let \( \eta = x/\ell \) then

\[
-\frac{F \ell^2}{2EA} \int_0^1 (1 - \eta)^2 \cos(n \pi \eta) \, d\eta = \frac{C_n \ell}{2} \int_0^1 (1 + \cos 2n \pi \eta) = \frac{C_n \ell}{2}
\]
or

\[
C_n = -\frac{F \ell}{EA} \int_0^1 (1 - \eta)^2 \cos(n \pi \eta) \, d\eta
\]
and use \( \varepsilon = 1 - \eta \) then

\[
C_n = -\frac{F \ell}{EA} \int_0^1 \varepsilon^2 \cos[\pi \varepsilon (1 - \varepsilon)] \, d\varepsilon
\]
But

\[
\cos[\pi \varepsilon (1 - \varepsilon)] = \cos \pi \varepsilon \cos \pi \varepsilon - \sin \pi \varepsilon \sin \pi \varepsilon
\]
Then

\[
C_n = (-1)^{n+1} \frac{2F \ell}{EA} \int_0^1 \varepsilon^2 \cos \pi n \varepsilon \, d\varepsilon
\]

Integrating by parts let \( u = \varepsilon^2 \) then \( du = 2\varepsilon \, d\varepsilon \), and \( v = \frac{1}{\pi n} \sin \pi n \varepsilon \), so that \( dv = \cos \pi n \varepsilon \, d\varepsilon \). The integral becomes

\[
C_n = (-1)^{n+1} \left\{ \frac{\varepsilon^2 \sin \pi n \varepsilon}{\pi n} \bigg|_0^1 - \frac{2}{\pi n} \int_0^1 \varepsilon \sin \pi n \varepsilon \, d\varepsilon \right\}
\]
or

\[
C_n = (-1)^n \frac{2F \ell}{2n \pi EA} \int_0^1 \varepsilon \sin \pi n \varepsilon \, d\varepsilon
\]
Integrating by parts \( u = \varepsilon \) then \( du = d\varepsilon \), and \( v = -\frac{1}{n \pi} \cos n \pi \varepsilon \), so that \( dv = \sin n \pi \varepsilon \), makes
\[ C_n = (-1)^n \frac{2F\ell}{2n\pi EA} \left\{ \frac{\varepsilon}{n\pi} \cos n\pi \bigg|_0^1 + \frac{1}{n\pi} \int_0^1 \cos (n\pi\varepsilon) \right\} \]

or

\[ C_n = (-1)^{n+1} \left( \frac{2F\ell}{\pi^2 n^2 EA} \right)^{\frac{(-1)^n}{\cos n\pi}} = -\frac{2F\ell}{\pi^2 n^2 EA} \]

Hence

\[ y = -\frac{F\ell}{6EA} - \frac{2F\ell}{\pi^2 EA} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \left( \frac{n\pi x}{\ell} \right) \cos \left( \frac{n\pi t}{\ell} \right) \]

and then

\[ u = -\frac{F\ell}{6EA} + \frac{F\ell}{2EA} \left( 1 - \frac{x}{\ell} \right)^2 + \frac{Ft^2}{2\rho A\ell} - \frac{2F\ell}{\pi^2 EA} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \left( \frac{n\pi x}{\ell} \right) \cos \left( \frac{n\pi t}{\ell} \right). \]

Let \( \eta = \frac{x}{\ell}, \tau = \frac{ct}{\ell}, V = \frac{EAu}{F\ell} \)

the solution can now be written as

\[ V = -\frac{1}{6} + \frac{1}{2} (1-\eta)^2 + \frac{1}{2} \tau^2 + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cos(n\pi\eta)\cos(n\pi\tau)}{n^2} \]
We will evaluate $V$ at $\tau = 0.0, 0.5, 1.0, 1.5, 2.0$ for $\eta = 0.0, 0.1, ..., 1.0$
note that we can use $\cos[(n + 1)\pi \eta] = \cos n \pi \eta \cos \pi \eta - \sin n \pi \eta \sin \pi \eta$ to help with the calculations.
Homework No. 13

Find the transient temperature distribution in a rod held at zero at ends with a linear initial temperature distribution \( T = x \) or

\[
\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}
\]

\( T = x \) @ \( t = 0 \)

\( T = 0 \) @ \( x = 0 \) & 1

Use separation of variables directly.

My answer: \( T = \frac{2}{\pi} \sum_{n=1}^{\infty} \left(-1\right)^{n+1} \frac{e^{-x^2n^2}}{n} \sin(n\pi x) \)