Partial Differential Equations

Linear P.D.Es. contains no powers, or products, of the dependent variables and/or any of its derivatives, and can occasionally be solved. Consider for example $z = z(x, y)$ and

$$\frac{\partial z}{\partial x} = a_i$$

(sometimes written as $z_1 = a_i$)

then we can integrate to get

$$z = a_i x + f_i(y)$$

We need a boundary condition to complete the solution, e.g. if

$$z = z_0(y) \text{ at } x = 0$$

then

$$z_0(y) = f_i(y)$$

and

$$z = a_i x + z_0(y)$$

(of course if $z_0(y) = 0$ then $z = a_i x$)

What if

$$2z_1 - z_2 = 0$$

Let

$$t = \alpha x + \beta y, \ s = \gamma x + \delta y$$

then

$$z_{,1} = \frac{\partial z}{\partial t} \frac{\partial t}{\partial x} + \frac{\partial z}{\partial s} \frac{\partial s}{\partial x} = \alpha \frac{\partial z}{\partial t} + \gamma \frac{\partial z}{\partial s}$$

Similarly

$$z_{,2} = \frac{\partial z}{\partial t} \frac{\partial t}{\partial y} + \frac{\partial z}{\partial s} \frac{\partial s}{\partial y} = \beta \frac{\partial z}{\partial t} + \delta \frac{\partial z}{\partial s}$$

then

$$\left(2\alpha - \beta\right) \frac{\partial z}{\partial t} + (2\gamma - \delta) \frac{\partial z}{\partial s} = 0.$$
Take $\delta = 2\gamma$ but the Jacobian must not vanish

$$\frac{\partial (t,s)}{\partial (x,y)} = \begin{vmatrix} \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix} = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \alpha \delta - \gamma \beta$$

Within the constraint we are free to take $\alpha = 1, \beta = 0, \gamma = 1$ then $\delta = 2$ and

$$\frac{\partial (t,s)}{\partial (x,y)} = 2 \neq 0$$

Then

$$2z_{rs} = 0$$

or

$$z = f(s) = f(\gamma x + \delta y) = f(x + 2y)$$

As another example consider

$$z_{rs} - 2z_{rr} = 1$$

Let

$$z = z^c + z^p$$

where

$$z^c_{rr} - 2z^c_{rr} = 0$$

Assume

$$z^c = f(\alpha x + \beta y) + g(\gamma x + \delta y)$$

Recall from the Jacobian that $\alpha \delta - \beta \gamma \neq 0$. The derivatives are:

$$z^c_{,rs} = \alpha f^c + \gamma g^c,$$

$$z^c_{,rs} = \alpha^2 f^r + \gamma^2 g^r,$$

$$z^c_{,rs} = \alpha \beta f^c + \gamma \delta g^c,$$

$$z^c_{,rs} = \beta f^r + \delta g^r,$$

and

$$z^c_{,rs} = \beta^2 f^r + \delta^2 g^r.$$

Then

$$\alpha^2 f^r + \gamma^2 g^r - \alpha \beta f^r - \gamma \delta g^r - 2 \beta^2 f^r - 2 \delta^2 g^r = 0,$$

or

$$(\alpha^2 - \alpha \beta - 2 \beta^2) f^r + (\gamma^2 - \gamma \delta - 2 \delta^2) g^r = 0,$$

and
$$(\alpha - 2\beta)(\alpha + \beta)f^{''} + (\gamma - 2\delta)(\gamma + \delta)g^{''} = 0.$$  

Both coefficients are zero if  
$$\alpha = 2\beta, \quad \delta = -\gamma$$  
then  
$$\alpha\delta - \beta\gamma = -2\beta\gamma - \beta\gamma = -3\beta\gamma \neq 0$$  
We can take  
$$\beta = \gamma = 1$$  
then  
$$\alpha = 2, \quad \delta = -1$$  
and  
$$z^e = f(2x + y) + g(x - y)$$  
We can take $z^p$ to be any solution of  
$$z^p_{,xx} - z^p_{,xy} - 2z^p_{,yy} = 1$$  
For example  
$$z^p = -xy,$$  
then  
$$z^p_{,x} = -y$$  
and  
$$z^p_{,xx} = 0,$$  
and  
$$z^p_{,xy} = -1$$  
Also  
$$z^p_{,y} = -x$$  
and  
$$z^p_{,yy} = 0$$  
then $0 - (-1) - 2 \cdot 0 = 1$ which checks.  
Since $z = z^e + z^p$ then  
$$z = f(2x + y) + g(x - y) - xy$$  
is the general solution of the equation.
**Linear First Order P.D.E.**

A general 1st order linear P.D.E. can be written

\[ P(x, y)z_x + Q(x, y)z_y = zR_1(x, y) + R_0(x, y) = R(x, y, z) \]

A solution can be written as

\[ u(x, y, z) = c \]

where \( c \) = constant, and the solution is referred to as an integral surface. Then

\[ u_x + u_z z_x = 0 \quad \text{and} \quad u_y + u_z z_y = 0 \]

or \( z_x = -u_x / u_z \quad \text{and} \quad z_y = -u_y / u_z \)

and

\[ -P \frac{u_x}{u_z} - Q \frac{u_y}{u_z} = R \]

or

\[ Pu_x + Qu_y + Ru_z = 0 \]

Let

\[ \vec{V} = \vec{P}i + \vec{Q}j + \vec{R}k, \]

since

\[ \vec{V} \cdot \vec{u} = u_x \frac{dx}{ds} + u_y \frac{dy}{ds} + u_z \frac{dz}{ds} = 0 \]

Recall that \( \vec{V} \cdot \vec{u} \) is perpendicular to \( u(x, y, z) = c \) then \( \vec{V} \) is perpendicular to \( \vec{u} \) or tangent to \( u(x, y, z) = c \). Any curve traced out by a particle moving such that its direction is along \( \vec{V} \) on the surface is a characteristic curves. Let \( \vec{r}(s) \) be a characteristic curve, where \( s \) is the arc length, then

\[ \frac{d\vec{r}}{ds} = \frac{dx}{ds} \vec{i} + \frac{dy}{ds} \vec{j} + \frac{dz}{ds} \vec{k} \]

which must point along \( \vec{V} \) or

\[ \mu \frac{d\vec{r}}{ds} = \vec{V}, \]

then

\[ P = \mu \frac{dx}{ds}, \quad Q = \mu \frac{dy}{ds}, \quad R = \mu \frac{dz}{ds}, \quad \text{or} \quad \frac{ds}{P} = \frac{dx}{Q} = \frac{dz}{R} \]

There are three equations the last two can be integrated to yield,

\[ u_1(x, y, z) = C_1 \quad \text{and} \quad u_2(x, y, z) = C_2 \]

any \( C_1 \) and \( C_2 \) yields a point on the surface or

\[ F(C_1, C_2) = 0 \]

represents the surface.
Example 1: Consider \( 2 \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0 \) then \( P = 2, Q = -1, \) and \( R = 0. \)

Since
\[
\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}
\]

then
\[
\frac{dx}{2} = \frac{dy}{-1} = \frac{dz}{0}
\]

and we must have
\( dz = 0 \)
as one equation or

\( z = C_2 = u_2(x, y, z) \)

and for the second
\[-dx = 2dy\]
or
\( dx + 2dy = 0 \)

making
\( x + 2y = C_1 = u_1(x, y, z) \)

Finally
\( F(C_1, C_2) = 0 \)

becomes
\( F(x + 2y, z) = 0 \)

Equivalently
\( z = f(x + 2y). \)

Note that in general
\[
a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} = 0
\]

has for the solution
\( F(bx - ay, z) = 0 \)
or
\( z = f(bx - ay) \)
Example 2: Consider

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$$

then

$$P = x, \; Q = y, \; R = z$$

or

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\ln x + \ln C_1 = \ln y \quad \ln x + \ln C_2 = \ln z$$

or

$$C_1 = \frac{y}{x} = u_1$$

and

$$C_2 = \frac{z}{x} = u_2$$

then

$$F\left(\frac{y}{x}, \frac{z}{x}\right) = 0$$

or

$$z = xf\left(\frac{y}{x}\right)$$
Second Order Equations

The linear equations can be written as:

\[ az_{xx} + bz_{xy} + cz_{yy} + dz_{x} + ez_{y} + fz = g \]

We will be concerned with \( a,b,c,d,e,f = \) constant (not \( g \)). The characteristics are determined by \( a,b,\& c \). Consider

\[ d = e = f = g = 0 \]

then

\[ az_{xx} + bz_{xy} + cz_{yy} = 0 \]

Let

\[ z = f(y + mx) \]

then

\[ z_{xx} = m^2 f^*, \quad z_{xy} = mf^* \quad \text{and} \quad z_{yy} = f^* \]

If \( f^* \neq 0 \) then

\[ am^2 + bm + c = 0 \]

or

\[ m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

There are three cases:

1. \( b^2 - 4ac > 0 \) is hyperbolic
2. \( b^2 - 4ac = 0 \) is parabolic
3. \( b^2 - 4ac < 0 \) is elliptic

and

\[ z = f(y + m_1 x) + g(y + m_2 x) \]

If \( a = 0 \),

\[ bz_{xy} + cz_{yy} = 0 \]

or

\[ (bz_{x} + cz_{y})_y = 0 \]

and

\[ bz_{x} + cz_{y} = bf(x) \]

Note that \( bf'(x) = f_i(x) \)

Since

\[ bz_{x} + cz_{y} = 0 \]

has for a solution

\[ z = g(y + mx) \]

Then
\[ z_{,x} = mg' \]

and

\[ z_{,y} = g' \]

or

\[(mb + c)g' = 0, \]

and

\[ m = -c/b \]

The complete solution is then

\[ z = g \left( y - \frac{c}{b} x \right) \]

Expanding in a Taylor Series,

\[ g_1[y + (m + \Delta)x] = g_1(y + mx) + \Delta g_1'(y + mx) \]

or

\[ z = f_1(y + mx) + g_1(y + mx) + \frac{\Delta g_1'(y + mx)}{f(y + mx)} \]

and

\[ z = f(y + mx) + xg(y + mx) \]
Example: Consider
\[ z_{xx} - 2z_{xy} + z_{yy} = 0 \]
then
\[ m_{1,2} = \frac{2 \pm \sqrt{4 - 4}}{2} = \frac{2}{2} = 1 \]
Then
\[ z = f(y + x) + xg(y + x) \]

Note: Since \( b^2 - 4ac = 4 - 4 = 0 \), the equation is parabolic.

Laplace’s Equation Example:
\[ \nabla^2 z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0, \]
then
\[ a = 1, \]
\[ b = 0, \]
\[ c = 1, \]
\[ b^2 - 4ac < 0 \]
making
\[ m = \frac{\pm \sqrt{-4}}{2} = \pm \sqrt{-1} = \pm i. \]
Then
\[ z = f(x + iy) + g(x - iy), \]
Note that
\[ i(y + ix) = -x + iy = -(x - iy) \]
and
\[ (y - ix)k = x + iy \]
Wave Equation

The wave equation is

\[ \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0 \]

then

\[ a = 1, \quad b = 0, \quad c = -1, \quad b^2 - 4ac > 0 \]

Hence

\[ m = \pm \frac{\sqrt{4}}{2} = \pm 1 \]

and

\[ z = f(x + y) + g(x - y) \]

Usually the wave equation is written as

\[ \frac{\partial^2 \varphi}{\partial t^2} = c^2 \frac{\partial^2 \varphi}{\partial x^2} \]

and is subject to initial conditions

\[ \varphi(x,0) = F(x), \quad \varphi_t(x,0) = G(x) \]

Letting

\[ ct = y, \quad \varphi(x,t) = f(x + ct) + g(x - ct) \]

For the initial conditions this yields

\[ \varphi(x,0) = F(x) = f(x) + g(x) \]

and

\[ \varphi_t(x,t) = cf'(x + ct) - cg'(x - ct) \]

Hence

\[ \varphi_t(x,0) = G(x) = c[f'(x) - g'(x)] \]
Integrating

\[ f(x) - g(x) = \frac{1}{c} \int_0^x G(x_1) \, dx_1 + C_i \]

But

\[ f(x) + g(x) = F(x), \]

Hence

\[ f(x) = \frac{1}{2} \left[ F(x) + \frac{1}{c} \int_0^x G(x_1) \, dx_1 + C_i \right], \]

\[ g(x) = \frac{1}{2} \left[ F(x) - \frac{1}{c} \int_0^x G(x_1) \, dx_1 - C_i \right] \]

Then

\[ \phi(x, t) = \frac{1}{2} \left\{ F(x + ct) + \frac{1}{c} \int_0^{ct} G(x') \, dx' + C_1 + F(x - ct) - \frac{1}{c} \int_0^{ct} G(x') \, dx' - C_1 \right\} \]

or

\[ \phi(x, t) = \frac{1}{2} \left[ F(x + ct) + F(x - ct) \right] + \frac{1}{2c} \int_{ct}^{ct} G(x') \, dx' \]
**Constant Coefficient Case**

For the case of all the derivatives present (but the coefficients constant)

\[ a z_{,xx} + b z_{,xy} + c z_{,yy} + d z_{,x} + e z_{,y} + f z = 0 \]

Assume for now

\[ z = A e^{\alpha + \beta y}, \]

then

\[ a \alpha^2 + b \alpha \beta + c \beta^2 + d \alpha + e \beta + f = 0 \]

or

\[ c \beta^2 + (b \alpha + e) \beta + a \alpha^2 + d \alpha + f = 0 \]

Solving for \( \beta \)

\[ \beta = \varphi_1(\alpha), \varphi_2(\alpha), \]

and

\[ z = A e^{\alpha + \varphi_1(\alpha) y}, \]

or

\[ z = B e^{\alpha + \varphi_2(\alpha) y} \]

or

\[ z = \int_{\alpha_1}^{\alpha_2} A(\alpha) e^{\alpha + \varphi_1(\alpha) y} + B(\alpha) e^{\alpha + \varphi_2(\alpha) y} d\alpha \]

Note this is not a general solution!

Example: Consider

\[ z_{,xx} - z_{,xy} - z_{,x} + z_{,y} = 0 \]

then

\[ \alpha^2 - \beta^2 - \alpha + \beta = 0 = (\alpha - \beta)(\alpha + \beta - 1) \]

or

\[ \beta = \alpha, 1 - \alpha \]

and

\[ z = \sum_{\alpha} A(\alpha) e^{\alpha(x+y)} + \sum_{\alpha} B(\alpha) e^{\alpha(x-y)} \]

or

\[ z = f(x + y) + e^y g(x - y) \]

This does require some imagination!
Homework No. 12

1. Find the general solution of
   \[ x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = z \]

2. Find the general solution of
   \[ xz \frac{\partial z}{\partial x} + yz \frac{\partial z}{\partial y} = xy \]

3. Find the solution of
   \[ \frac{\partial^2 \varphi}{\partial t^2} = c^2 \frac{\partial^2 \varphi}{\partial x^2} \]
   that satisfies
   \[ \varphi(0,t) = 0, \quad \varphi(l,t) = 0 \]
   for all \( t \), and satisfies
   \[ \varphi(x,0) = F(x), \quad \varphi_t(x,0) = G(x) \]
   for \( 0 < x < l \).
   
   a. Show the boundary conditions at \( x = 0, l \) result in
      \[ f(u) + g(-u) = 0 \quad (\text{where } u = ct) \]
      and \( f(u + 2l) = f(u) \) (where \( u = ct - l \)),
      and show the initial conditions result in
      \[ f(u) + g(u) = F(u) \quad \text{and} \quad c \left[ f'(u) - g'(u) \right] = G(u) \quad \text{for } 0 < u < l. \]

   b. Show that if let \( H'(u) = G(u) \) that the initial conditions become
      \[ f(u) = \frac{1}{2} \left[ F(u) + \frac{1}{c} H(u) \right] \quad \text{and} \quad g(u) = \frac{1}{2} \left[ F(u) - \frac{1}{c} H(u) \right] \quad \text{for } 0 < u < l. \]
      
      Also show that for all \( u \), \( f(u) \) is periodic with period \( 2l \) and that
      \[ f(u) = -g'(-u). \]

   c. Show that all the conditions are satisfied if we set
      \[ f(u) = \frac{1}{2} \left[ \mathcal{F}(u) + \frac{1}{c} \mathcal{K}(u) \right] \quad \text{and} \quad g(u) = \frac{1}{2} \left[ \mathcal{F}(u) - \frac{1}{c} \mathcal{K}(u) \right] \]
      for all \( u \) where \( \mathcal{F}(u) \) and \( \mathcal{G}(u) \) are odd periodic functions with period \( 2l \).
      
      Note that \( \mathcal{F}(u) \) and \( \mathcal{G}(u) \) are equal to \( F(u) \) and \( G(u) \) in \( 0 < u < l \) and that \( \mathcal{K}'(u) = \mathcal{G}(u) \). The general solution then becomes
      \[ \varphi(x,t) = \frac{1}{2} \left[ \mathcal{F}(x + ct) + \mathcal{F}(x - ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \mathcal{G}(\xi) \, d\xi. \]