Vector and Tensor Algebra

Vector Analysis

Addition: \( \vec{c} = \vec{a} + \vec{b} \)

Subtraction: \( \vec{c} = \vec{a} - \vec{b} = \vec{a} + (\vec{-b}) \), where \( \vec{-b} \) points in the opposite direction of \( \vec{b} \)

Equality: Two vectors are equal if the magnitudes (lengths) and directions are the same.

Basis Vectors: In general vectors can be written using orthogonal vectors along the coordinate axes, \( \vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k} \). The components of \( \vec{v} \) are \( (v_x, v_y, v_z) \).

Scalar Product: The scalar product is defined as \( s = \vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z \) where the components of \( \vec{a} \) and \( \vec{b} \) are \( (a_x, a_y, a_z) \) and \( (b_x, b_y, b_z) \) respectively.

Magnitude: Using the scalar product the magnitude can be written as \( |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{\vec{a} \cdot \vec{a}} = a \).

We can also write \( \vec{a} = a \left( \frac{\vec{a}}{a} \right) \) where \( \left( \frac{\vec{a}}{a} \right) \) is a unit vector along \( \vec{a} \). Note that it can be shown that \( s = ab \cos \theta \) where \( \theta \) is the angle between \( \vec{a} \) and \( \vec{b} \).

Vector Product: the vector product is defined as \( \vec{v} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \). Note that \( \vec{v} \) is perpendicular to the plane of \( \vec{a} \) and \( \vec{b} \) with the positive sense determined by the right-hand rule, and \( |\vec{a} \times \vec{b}| = ab \sin \theta \), where \( \theta \) is the minimum angle between \( \vec{a} \) and \( \vec{b} \). The determinant (or the right-hand) rule means that \( \vec{a} \times \vec{b} \) is not commutative since \( \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \). Finally note that \( \vec{a} \times \vec{b} \) is the area of the parallelogram shown with a direction perpendicular to the plane in a right-hand sense.
Triple Scalar Product:

\[
(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{c} \cdot \begin{vmatrix}
  \hat{i} & \hat{j} & \hat{k} \\
  a_x & a_y & a_z \\
  b_x & b_y & b_z \\
\end{vmatrix} = \begin{vmatrix}
  c_x & c_y & c_z \\
  a_x & a_y & a_z \\
  b_x & b_y & b_z \\
\end{vmatrix},
\]

and is the volume of the parallelepiped shown below.

![Parallelepiped Diagram]

Triple Vector Product: The triple vector product \((\vec{a} \times \vec{b}) \times \vec{c} = \alpha_i \vec{a} + \alpha_2 \vec{b}\) must be perpendicular to \(\vec{c}\) and lie in the plane of \(\vec{a}\) and \(\vec{b}\). It can be shown that

\[
(\vec{a} \times \vec{b}) \times \vec{c} = -(\vec{b} \cdot \vec{c})\vec{a} + (\vec{a} \cdot \vec{c})\vec{b}.
\]

Let \(\vec{a} \rightarrow \vec{b}, \vec{b} \rightarrow \vec{c}\), and \(\vec{c} \rightarrow \vec{a}\) then

\[
(\vec{b} \times \vec{c}) \times \vec{a} = -(\vec{c} \cdot \vec{a})\vec{b} + (\vec{b} \cdot \vec{a})\vec{c}\) and \((\vec{b} \times \vec{c}) = (\vec{c} \cdot \vec{a})\vec{b} - (\vec{b} \cdot \vec{a})\vec{c} \).

Gradient: Let \(\phi\) be a scalar function of \(x, y,\) and \(z\), then the gradient of \(\phi\) is defined as

\[
\nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} = \nabla \phi.
\]

This certainly represents the direction of changes. Consider the position vector

\[
\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}
\]

then

\[
d\vec{r} = \hat{i} \, dx + \hat{j} \, dy + \hat{k} \, dz.
\]

Let

\[
ds = |d\vec{r}| = \sqrt{dx^2 + dy^2 + dz^2}
\]

then

\[
\frac{d\vec{r}}{ds} = \hat{i} \frac{dx}{ds} + \hat{j} \frac{dy}{ds} + \hat{k} \frac{dz}{ds},
\]

is a unit vector and

\[
\nabla \phi \cdot \frac{d\vec{r}}{ds} = \frac{d\phi}{ds}
\]

or

\[
d\phi = \nabla \phi \cdot d\vec{r}.
\]
The operator $\mathbf{\nabla} = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$ is the gradient operator.

Some properties include:

1) If $\phi(x, y, z) = 0$ (i.e., $\phi$ is a surface) then $d\phi = 0 = \mathbf{\nabla}\phi \cdot d\mathbf{r}$ and $\mathbf{\nabla}\phi$ is perpendicular to $d\mathbf{r}$ and hence is perpendicular to $\phi = 0$ since $d\mathbf{r}$ is tangent to $\phi = 0$.

2) If $\phi(x, y, z) \neq 0$ then each component of $\mathbf{\nabla}\phi$ is the derivative of $\phi$ in the direction of the maximum derivative of $\phi$ with a magnitude equal to the maximum derivative.

Divergence: If $\mathbf{\nabla} \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \text{div}\mathbf{F}$, then the divergence is defined by

\[
\mathbf{\nabla} \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \text{div}\mathbf{F}.
\]

Curl: If $\mathbf{\nabla} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$, then the curl is defined by

\[
\mathbf{\nabla} \times \mathbf{F} = \mathbf{i} \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \mathbf{j} \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \mathbf{k} \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) = \text{curl}\mathbf{F}.
\]

Also note that

1) $\mathbf{\nabla} \cdot \mathbf{\nabla}\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$,

2) $\mathbf{\nabla} \cdot \mathbf{\nabla}\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$ is the Laplacian,

3) $\mathbf{\nabla} \cdot \mathbf{\nabla}\phi = 0$, and

4) $\mathbf{\nabla} \cdot (\mathbf{\nabla} \times \mathbf{F}) = 0$. 


Transformation of Vectors

In two dimensions

$$\vec{V} = V_1 \hat{e}_1 + V_2 \hat{e}_2 + V_3 \hat{e}_3$$

From the figures at the right

$$V'_1 = V_1 \cos \theta + V_2 \sin \theta$$
$$V'_2 = -V_1 \sin \theta + V_2 \cos \theta$$

Let $\alpha_{ij}$ be the angle from $x_j$ to $x'_i$. Then

$$V'_1 = V_i \cos \alpha_{i1} + V_j \cos \alpha_{i2}$$
$$V'_2 = V_i \cos \alpha_{21} + V_j \cos \alpha_{22}$$

If $a_{ij} = \cos \alpha_{ij}$ are the direction cosines, then

$$V'_1 = V_i a_{i1} + V_j a_{i2}$$
$$V'_2 = V_i a_{21} + V_j a_{22}$$

In a similar manner

$$V_i = V'_i \cos \alpha_{i1} + V'_j \cos \alpha_{i2}$$
$$V_j = V'_i \cos \alpha_{21} + V'_j \cos \alpha_{22}$$

or

$$\begin{bmatrix} V'_1 \\ V'_2 \end{bmatrix} = \begin{bmatrix} a_{i1} & a_{i2} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

and

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} a_{i1} & a_{i2} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} V'_1 \\ V'_2 \end{bmatrix}.$$

Then $V' = AV$ and $V = A^T V'$ and, hence, $A^{-1} = A^T$. 
Indicial Notation

Note that

\[ V'_j = \sum_{i=1}^{3} a_{ji} V_i \]

and that

\[ V_k = \sum_{j=1}^{3} a_{jk} V'_j \]

In both cases repeated indices are summed. Assume in all cases (unless otherwise stated) that a repeated index is summed. Then

\[ V'_j = a_{ji} V_i \]

and

\[ V_k = a_{jk} V'_j. \]

Hence

\[ V_k = a_{jk} a_{ji} V_i. \]

In matrix form

\[ V = A^T A V \]

or

\[ A^T A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \]

Identity Matrix and the Kronecker Delta

Let

\[ \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \]

then

\[ a_{jk} a_{ji} = \delta_{ki} = \delta_{ik} \]

from

\[ A^T A = I. \]

Three Dimensional Vector Transformation

A three dimensional vector satisfies

\[ V'_i = a_{ij} V_j \]

where \( a_{ij} \) are the direction cosines and \( i, j = 1,2,3 \).
**Cartesian Tensors**

A scalar is a rank zero tensor (no indices). Examples include: temperature, pressure, density,... Under coordinate system rotations a scalar, $S$, transforms according to

$$S' = S.$$ 

A vector is a rank one tensor and contains one index. Examples include: position, velocity, force, Under coordinate system rotations a scalar, $V$, transforms according to

$$V'_i = a_{ij}V_j$$

A rank two tensor has two indices. Example include stress and strain, Under coordinate system rotations a rank two, $T_{ij}$, transforms according to

$$T'_{ij} = a_{ik}a_{jl}T_{kl}.$$ 

Similarly a rank three tensor, $T$, transforms under coordinate system rotation according to

$$T'_{ijk} = a_{ij}a_{jm}a_{kn}T_{lmn}.$$ 

Rank N tensors, $T$, transforms under coordinate system rotation according to

$$T'_{ijk...N} = a_{ij}a_{jm}a_{kn}...a_{NM}T_{lmn...M}.$$ 

The above transformations define Cartesian tensors where $a_{ij}$ are the direction cosines from axes $x'_i$ to axes $x_j$. 
**Transformation of the Kronecker Delta**

If the Kronecker delta is a tensor it must satisfy

\[
\delta_{ij} = a_{ik} a_{jl} \delta_{kl}
\]

Expanding the sums over \( k \) and \( l \),

\[
\delta'_{ij} = a_{i1} a_{j1} \delta_{11} + a_{i1} a_{j2} \delta_{12} + a_{i1} a_{j3} \delta_{13} + a_{i2} a_{j1} \delta_{21} + a_{i2} a_{j2} \delta_{22} + a_{i2} a_{j3} \delta_{23} + a_{i3} a_{j1} \delta_{31} + a_{i3} a_{j2} \delta_{32} + a_{i3} a_{j3} \delta_{33}
\]

But

\[
\delta_{11} = \delta_{22} = \delta_{33} = 1
\]

and

\[
\delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = \delta_{23} = \delta_{32} = 0
\]

then

\[
\delta'_{ij} = a_{i1} a_{j1} + a_{i2} a_{j2} + a_{i3} a_{j3} = a_{ik} a_{jk}
\]

(i.e., \( a_{ik} a_{jl} \delta_{kl} = a_{ik} a_{jk} \) and is referred to as contraction)

Since

\[
a_{ik} a_{jk} \equiv AA^T = I \equiv \delta'_{ij}
\]

then

\[
\delta'_{ij} = \delta_{ij}.
\]

This means that \( \delta_{ij} \) is a rank 2 isotropic tensor (i.e., the Kronecker delta looks the same in all directions). It can be shown that Kronecker delta is the ‘only’ rank 2 isotropic tensor.
\textbf{Permutation Symbol and Determinants}

Let

\[
|A| = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}
\]

If we switch any two rows the sign changes. We can write

\[
\epsilon_{ijk} |A| = \begin{vmatrix} A_{i1} & A_{i2} & A_{i3} \\ A_{j1} & A_{j2} & A_{j3} \\ A_{k1} & A_{k2} & A_{k3} \end{vmatrix}
\]

where recall \( \epsilon_{ijk} = \begin{cases} 1 & \text{even permutation} \\ -1 & \text{odd permutation} \\ 0 & \text{not a permutation} \end{cases} \).

Switching the columns

\[
\epsilon_{lmn} |A| = \begin{vmatrix} A_{l1} & A_{lm} & A_{ln} \\ A_{2l} & A_{2m} & A_{2n} \\ A_{3l} & A_{3m} & A_{3n} \end{vmatrix}.
\]

Combining

\[
\epsilon_{ijk} \epsilon_{lmn} |A| = \begin{vmatrix} A_{i1} & A_{im} & A_{in} \\ A_{jl} & A_{jm} & A_{jn} \\ A_{kl} & A_{km} & A_{kn} \end{vmatrix}.
\]

Set \( A_{ij} = \delta_{ij} \) (i.e., \( A = I \)) then

\[
|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1
\]

then

\[
\epsilon_{ijk} \epsilon_{lmn} = \begin{vmatrix} \delta_{ij} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}.
\]
Transformation of the Permutation Symbol

For a rank three tensor

\[ \varepsilon'_{ijk} = a_{il} a_{jm} a_{kn} \varepsilon_{lmn} \]

Since there are only 6 non-zero \( \varepsilon_{ijk} \), that is

\[ \varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1 \] and \( \varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1 \)

then

\[ \varepsilon'_{ijk} = a_{i1} a_{j2} a_{k3} + a_{i2} a_{j3} a_{k1} + a_{i3} a_{j1} a_{k2} - a_{i1} a_{j3} a_{k2} - a_{i2} a_{j1} a_{k3} - a_{i3} a_{j2} a_{k1} \]

Note that if any of the two indices are equal that \( \varepsilon'_{ijk} \) vanishes, and that

\[ \varepsilon'_{ijk} = \begin{vmatrix} a_{i1} & a_{i2} & a_{i3} \\ a_{j1} & a_{j2} & a_{j3} \\ a_{k1} & a_{k2} & a_{k3} \end{vmatrix} = \begin{cases} |A| & i,j,k \text{ are an even permutation} \\ -|A| & i,j,k \text{ are an odd permutation} \\ 0 & i,j,k \text{ are not a permutation} \end{cases} \]

But recall that the determinant of the direction cosine matrix, \( A \), is one for a right handed system (Note that the determinant of \( A \) is sometimes called the Jacobian), then

\[ \varepsilon'_{ijk} = \varepsilon_{ijk} \]

and \( \varepsilon_{ijk} \) is an isotropic rank three tensor. It can be shown it is the ‘only’ rank three tensor.

The \( \varepsilon-\delta \) Identity

Recall that

\[ \varepsilon_{ijk} \delta_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \]

then contracting on the first index in each \( \varepsilon \)

\[ \varepsilon_{ijk} \delta_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} = \delta_{il} \delta_{jm} \delta_{kn} + \delta_{im} \delta_{jn} \delta_{kl} + \delta_{in} \delta_{jl} \delta_{km} - \delta_{il} \delta_{jm} \delta_{kn} - \delta_{im} \delta_{jn} \delta_{kl} - \delta_{in} \delta_{jl} \delta_{km} \]

or

\[ \varepsilon_{ijk} \delta_{lmn} = 3 \delta_{jm} \delta_{kn} + \delta_{km} \delta_{jn} + \delta_{jn} \delta_{km} - \delta_{km} \delta_{jn} - \delta_{jn} \delta_{km} - 3 \delta_{km} \delta_{jn} \]

and \( \varepsilon_{ijk} \delta_{lmn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} \) is the \( \varepsilon-\delta \) identity.
**Vector Dot Product**

Recall that \( \vec{A} \cdot \vec{B} = AB \cos \theta \) to prove this transform the vector \( A_i \) and \( B_i \) to \( A'_i \) and \( B'_i \) where \( A'_i \) is along the \( x'_1 \) and \( B'_i \) is in the \( x'_1 \cdot x'_2 \) plane. Then

\[
A'_i = a_{ij} A_j
\]

and

\[
B'_i = a_{ij} B_j.
\]

Then

\[
\vec{A} \cdot \vec{B} = A_i B_i = A'_i B'_i = \vec{A}' \cdot \vec{B}'
\]

since

\[
A'_2 = A'_3 = 0.
\]

But \( B'_1 = B \cos \theta \) and \( A'_1 = A \), hence

\[
\vec{A} \cdot \vec{B} = AB \cos \theta.
\]

We have assumed that the scalar resulting from \( \vec{A} \cdot \vec{B} \) does not depend on orientation of the axes. In order to show this consider

\[
\vec{A}' \cdot \vec{B}' = A'_i B'_i = a_{ij} A'_j a_{ik} B'_k = a_{ij} a_{ik} A_j B_k = \delta_{jk} A_j B_k = A_j B_j = A_i B_i = \vec{A} \cdot \vec{B}.
\]

Since

\[
a_{ij} a_{ik} = \delta_{jk}
\]
Cross Products

Consider the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ along $x_1, x_2, x_3$ respectively, then by definition

\[
\mathbf{i} \times \mathbf{i} = 0 \quad \mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \\
\mathbf{j} \times \mathbf{i} = -\mathbf{k} \quad \mathbf{j} \times \mathbf{j} = 0 \quad \mathbf{j} \times \mathbf{k} = \mathbf{i} \\
\mathbf{k} \times \mathbf{i} = \mathbf{j} \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \quad \mathbf{k} \times \mathbf{k} = 0
\]

is a consistent definition.

The cross product of two vectors is then

\[
\mathbf{A} \times \mathbf{B} = \left( a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \right) \times \left( b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k} \right)
\]

Applying the distributive law

\[
\mathbf{A} \times \mathbf{B} = a_1 b_2 \mathbf{k} - a_1 b_3 \mathbf{j} - a_2 b_1 \mathbf{k} + a_2 b_3 \mathbf{i} + a_3 b_1 \mathbf{j} - a_3 b_2 \mathbf{i}
\]

this can be written as

\[
\mathbf{A} \times \mathbf{B} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{vmatrix}
\]

Using the permutation symbol

\[
\mathbf{A} \times \mathbf{B} = \epsilon_{ijk} a_j b_k.
\]

Triple-Vector (Scalar) Product

We can now write the triple scalar product as

\[
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \epsilon_{ijk} a_i b_j c_k = \begin{vmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{vmatrix}
\]
Moments and Vector Products

Let

$$\vec{C} = \vec{A} \times \vec{B}$$

then

$$C_i = \varepsilon_{ijk} A_j B_k .$$

Hence,

$$C_i A_i = \varepsilon_{ijk} A_j A_i B_k = \varepsilon_{ijk} A_j A_i B_k = \varepsilon_{ijk} A_i A_j B_k$$

and since $\varepsilon_{ijk} = -\varepsilon_{ijk}$,

$$C_i A_i = -\varepsilon_{ijk} A_j A_i B_k = -C_i A_i$$

or

$$2C_i A_i = 0 .$$

Then $C_i A_i = 0$ or $\vec{C} \cdot \vec{A} = 0$ and $\vec{C} = \vec{A} \times \vec{B}$ is perpendicular the plane of $\vec{A} - \vec{B}$, and is positive in a right hand sense by our definitions.

We need to show that the amplitude satisfies $|\vec{A} \times \vec{B}| = AB \sin \theta$ where $\theta$ is the angle from $\vec{A}$ to $\vec{B}$. As before, transform the vector $A_i$ and $B_j$ to $A'_i$ and $B'_j$ where $A'_i$ is along the $x'_1$ and $B'_j$ is in the $x'_1 x'_2$ plane, then

$$\vec{C} = \vec{A} \times \vec{B} = A_i \vec{\imath} \times (B_i \vec{\imath} + B_j \vec{j}) = A_i B_2 \vec{k} .$$

But $A_i = |\vec{A}| = A$

and $B_2 = |\vec{B}| \sin \theta = B \sin \theta$,

then $|\vec{C}| = C = AB \sin \theta$.

Hence, $\vec{C}$ is the moment of $\vec{B}$ about $\vec{A}$. 

![Diagram](image-url)
Derivatives of Tensors

Let

\[ T_{ij} = T_{ij}(x_1, x_2, x_3, t) = T_{ij}(\bar{x}, t), \]

\[ V_i = V_i(x_1, x_2, x_3, t) = V_i(\bar{x}, t), \]

and

\[ S = S(x_1, x_2, x_3, t) = S(\bar{x}, t). \]

then

\[ \frac{\partial \phi}{\partial x_i} = \partial_i \phi = \phi_{,i} \]

and

\[ \frac{\partial \phi}{\partial x'_i} = \frac{\partial \phi}{\partial x_j} \frac{\partial x_j}{\partial x'_i}. \]

Since \( x_j = a_{ij} x'_i \) then

\[ \frac{\partial x_j}{\partial x'_i} = a_{ij}, \]

and

\[ \frac{\partial \phi}{\partial x'_i} = a_{ij} \frac{\partial \phi}{\partial x_j}. \]

(i.e., \( \Phi'_i = a_{ij} \Phi_j \)).

Hence, \( \Phi_{,j} \) transforms like a rank one tensor and must be a rank one (Cartesian) tensor.

For a vector,

\[ V_{i,j} = \frac{\partial V'_i}{\partial x'_j} = \frac{\partial V'_j}{\partial x_k} \frac{\partial x_k}{\partial x'_j} = a_{jk} V'_{i,k} = a_{il} a_{jk} V_{l,k}. \]

Hence, \( V_{i,j} \) is a rank two tensor.

Similarly,

\[ \frac{\partial T'_{ij}}{\partial x'_k} = a_{il} a_{jm} \frac{\partial T_{km}}{\partial x_n} \frac{\partial x_n}{\partial x'_k} = a_{il} a_{jm} a_{kn} \frac{\partial T_{km}}{\partial x_n} \]

and \( T_{ij,k} \) is a rank three tensor.
Continuing, $T_{ij,kl}$ is a rank four tensor, and so on.

Finally note that we can write

$$\tilde{\nabla} \phi = \phi_{,i}$$
$$\tilde{\nabla} \cdot \tilde{V} = V_{i,i}$$
$$\tilde{\nabla} \times \tilde{V} = \epsilon_{ijk} V_{k,j}$$

and

$$\nabla^2 \phi = \phi_{,ii}.$$
Symmetric and Antisymmetric Tensors

A tensor is symmetric in \( i \) and \( j \) if

\[
S_{...ij...} = S_{...ji...}.
\]

A tensor is anti-symmetric or skew symmetric in \( i \) and \( j \) if

\[
A_{...ij...} = -A_{...ji...}.
\]

For a general tensor \( T_{ij} \)

We can write a general tensor as the sum of symmetric and anti-symmetric parts

\[
T_{ij} = S_{ij} + A_{ij}
\]

where

\[
S_{ij} = \frac{1}{2} (T_{ij} + T_{ji}) \quad \text{and} \quad A_{ij} = \frac{1}{2} (T_{ij} - T_{ji}).
\]

There are six different components in \( S_{ij} \) and three in \( A_{ij} \), that is

\[
S_{ij} = \begin{bmatrix}
S_{11} & S_{12} & S_{13} \\
S_{12} & S_{22} & S_{23} \\
S_{13} & S_{23} & S_{33}
\end{bmatrix}
\quad \text{and} \quad
A_{ij} = \begin{bmatrix}
0 & A_{12} & -A_{31} \\
-A_{12} & 0 & A_{23} \\
A_{31} & -A_{23} & 0
\end{bmatrix}.
\]

Note that the dual of a tensor is sometimes mentioned. For a rank two tensor in three dimensions it is given

\[
t_i = \epsilon_{ijk} T_{jk}
\]

and if

\[
T_{ij} = S_{ij} + A_{ij}
\]

where \( S_{ij} = S_{ji} \) and \( A_{ij} = -A_{ji} \)

then because of the symmetries

\[
t_i = \epsilon_{ijk} A_{jk}.
\]
**Principal Values and Principals Directions of Symmetric Rank Two Tensors**

The vector \( v_i \) from tensor \( T_{ij} \) along unit normal, \( n_i \), can be written

\[
v_i = T_{ij} n_j
\]

If \( v_i \) points along \( n_i \) then

\[
v_i = \lambda n_i = \lambda \delta_{ij} n_j
\]

and

\[
(T_{ij} - \lambda \delta_{ij}) n_j = 0.
\]

(i.e., \( T_{ij} n_j = \lambda n_j \)) is an eigenvalue.

Expanding

\[
(T_{11} - \lambda)n_1 + T_{12}n_2 + T_{13}n_3 = 0
\]
\[
T_{21}n_1 + (T_{22} - \lambda)n_2 + T_{23}n_3 = 0
\]
\[
T_{31}n_1 + T_{32}n_2 + (T_{33} - \lambda)n_3 = 0
\]

In matrix form

\[
\begin{bmatrix}
T_{11} - \lambda & T_{12} & T_{13} \\
T_{21} & T_{22} - \lambda & T_{23} \\
T_{31} & T_{32} & T_{33} - \lambda
\end{bmatrix}
\begin{bmatrix}
n_1 \\
n_2 \\
n_3
\end{bmatrix} = 0
\]

Then \( \lambda \) contains the eigenvalues of \( T \) and \( n_i \) are the associated eigenvectors. For symmetric tensors \( (T_{ij} = T_{ji}) \) all the eigenvalues are real and the different \( \bar{n} \) are orthogonal.

The cubic for the eigenvalues satisfy

\[
\frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} (T_{il} - \lambda \delta_{il})(T_{jm} - \lambda \delta_{jm})(T_{kn} - \lambda \delta_{kn}) = 0,
\]

or

\[
\frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} (T_{il} T_{jm} - \lambda \delta_{il} T_{jm} - \lambda T_{il} \delta_{jm} + \lambda^2 \delta_{il} \delta_{jm})(T_{kn} - \lambda \delta_{kn}) = 0.
\]
Then

\[ \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} [T_{il} T_{jm} T_{kn} - \lambda T_{il} T_{jm} \delta_{kn} - \lambda T_{il} \delta_{jm} T_{kn} - \lambda \delta_{il} T_{jm} T_{kn} ] + \lambda^2 T_{il} \delta_{jm} \delta_{kn} + \lambda^2 \delta_{il} T_{jm} \delta_{kn} + \lambda^2 \delta_{il} \delta_{jm} T_{kn} - \lambda^3 \delta_{il} \delta_{jm} \delta_{kn} = 0. \]

Bringing in the permutation symbols

\[ \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} T_{il} T_{jm} T_{kn} - \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} [\lambda T_{il} T_{jm} \delta_{kn} + \lambda T_{il} \delta_{jm} T_{kn} + \lambda \delta_{il} T_{jm} T_{kn} ] + \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} [\lambda^2 T_{il} \delta_{jm} \delta_{kn} + \lambda^2 \delta_{il} T_{jm} \delta_{kn} + \lambda^2 \delta_{il} \delta_{jm} T_{kn} - \lambda^3 \delta_{il} \delta_{jm} \delta_{kn} = 0. \]

First consider

\[ a_3 = \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} \delta_{il} \delta_{jm} \delta_{kn} = \frac{1}{6} \epsilon_{ijk} \epsilon_{ijk} = (\frac{1}{6})6=1. \]

Now consider

\[ a_2 = \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} [T_{il} \delta_{jm} \delta_{kn} + \delta_{il} T_{jm} \delta_{kn} + \delta_{il} \delta_{jm} T_{kn} ] = \frac{1}{6} \epsilon_{ijk} \epsilon_{ijk} T_{il} + \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} T_{jm} + \frac{1}{6} \epsilon_{ijk} \epsilon_{ijn} T_{kn} \]

Since \( \epsilon_{ijk} \epsilon_{ijl} = 2 \delta_{kl} \) then

\[ a_2 = \frac{1}{3} \delta_{il} T_{il} + \frac{1}{3} \delta_{jm} T_{jm} + \frac{1}{3} \delta_{kn} T_{kn} = \frac{1}{3} (T_{ii} + T_{jj} + T_{kk}) = T_{ii}. \]

The coefficient of \( \lambda \) is

\[ a_1 = \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} T_{il} T_{jm} + \frac{1}{6} \epsilon_{ijk} \epsilon_{ijn} T_{il} T_{kn} + \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} T_{jm} T_{kn} = \frac{1}{2} \epsilon_{ijk} \epsilon_{lmn} T_{il} T_{jm} \]

Since \( \epsilon_{ijk} \epsilon_{lmn} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \) then

\[ a_1 = \frac{1}{2} \left( \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right) T_{il} T_{jm} = \frac{1}{2} (T_{il} T_{jj} - T_{ij} T_{ji}). \]

The last coefficient is the constant

\[ a_0 = \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} T_{il} T_{jm} T_{kn} = |T| \]

The characteristic equation is

\[ |T| - \frac{1}{2} (T_{il} T_{jj} - T_{ij} T_{ji}) \lambda + T_{il} \lambda^2 - \lambda^3 = 0 \]

Each of the coefficients (\(|T|, \frac{1}{2} T_{il} T_{jj} - \frac{1}{2} T_{ij} T_{ji}, T_{ii} \)) are scalars and hence do not depend on the orientation of the axes. Therefore they have the same value regardless of the Cartesian coordinate system chosen. They are invariants with changes in orientation.
Homework 10

1. In two dimensions show that the direction cosine matrix, \( a_{ij} \), satisfies \( |a_{ij}| = 1 \).

2. Show \( \vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \).

3. Show \( \epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl} \) and \( \epsilon_{ijk}\epsilon_{ijk} = 6 \).

4. Prove for any number of dimensions that if the direction cosine matrix, \( \vec{A} = [a_{ij}] \), satisfies \( \vec{A}^{-1} = \vec{A}^T \) then \( |\vec{A}| = \pm 1 \).