Eigenvalue Problems

For an arbitrary ellipse in the x’-y’ plane we can write
\[ \lambda_1 x_1'^2 + \lambda_2 x_2'^2 = 1. \]

Rotating to the x-y axes
\[ x_1 = x_1' \cos \theta - x_2' \sin \theta \]
\[ x_2 = x_1' \sin \theta + x_2' \cos \theta. \]

In matrix form this is
\[
\begin{bmatrix}
    x_1 \\
    x_2 
\end{bmatrix}
= \begin{bmatrix}
    \cos \theta & -\sin \theta \\
    \sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
    x_1' \\
    x_2'
\end{bmatrix},
\]
or as \( \bar{x} = \bar{R} \bar{x}' \).

Note that the inverse of \( \bar{R} \) can be readily found using the inverse matrix:

1) \( |\bar{R}| = \cos^2 \theta + \sin^2 \theta = 1 \) is the determinant,

2) The co-factor matrix is \( \bar{C} = \begin{bmatrix}
    \cos \theta & -\sin \theta \\
    \sin \theta & \cos \theta
\end{bmatrix} = \bar{R} \), and

3) The inverse is given by
\[
\bar{R}^{-1} = \frac{\bar{C}^T}{|\bar{R}|} = \frac{\bar{R}^T}{1} = \bar{R}^T = \begin{bmatrix}
    \cos \theta & \sin \theta \\
    -\sin \theta & \cos \theta
\end{bmatrix}.
\]

This, of course, would be the result of changing \( \theta \) to \( -\theta \) in \( \bar{R} \) to get \( \bar{R}^{-1} \). Then
\[
\bar{R} \bar{R}^T = \bar{R}^T \bar{R} = I,
\]
and the matrix \( \bar{R} \) is orthogonal.
The equation for the ellipse can be written as

\[
(x_1', x_2')^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = 1,
\]

or in matrix form

\[
\bar{x}'^T \bar{A} \bar{x}' = 1.
\]

But \( \bar{x}' = \bar{R}^T \bar{x} \) and also \( \bar{x}'^T = \bar{x}^T \bar{R} \) so that in x-y coordinates

\[
\bar{x}^T \bar{R} \bar{A} \bar{R}^T \bar{x} = 1.
\]

Let \( \bar{A} = \bar{R} \bar{A} \bar{R}^T \) but \( \bar{A}^T = (\bar{R} \bar{A} \bar{R}^T)^T = \bar{R} \bar{A} \bar{R}^T = \bar{R} \bar{A} \bar{R}^T = \bar{A} \) and \( \bar{A} \) is symmetric.

We can now write \( \bar{x}^T \bar{A} \bar{x} = 1 \) which can be expanded to read

\[
a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_2x_1 + a_{22}x_2^2 = 1.
\]

Since \( \bar{A} = \bar{A}^T \) then \( a_{12} = a_{21} \) and the equation for the ellipse becomes

\[
a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 = 1.
\]

Finally we can transform to the X-Y system to get \((\bar{X} - \bar{C})^T \bar{A}(\bar{X} - \bar{C}) = 1\).

Let \( \bar{\phi}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \bar{\phi}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) be unit vectors along the x'-y' axes, then

\[
\bar{\phi}_1 = \bar{R} \bar{\phi}_1' = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix},
\]

and similarly

\[
\bar{\phi}_2 = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}.
\]
Note that \( \bar{\phi}_1 \cdot \bar{\phi}_1 = \cos^2 \theta + \sin^2 \theta = 1, \ bar{\phi}_2 \cdot \bar{\phi}_2 = \sin^2 \theta + \cos^2 \theta = 1 \) and \( \bar{\phi}_1 \cdot \bar{\phi}_2 = \bar{\phi}_2 \cdot \bar{\phi}_1 = -\cos \theta \sin \theta + \sin \theta \cos \theta = 0 \). The matrix \( \bar{R} \) consists of the unit \( \bar{\phi}_1 \) and \( \bar{\phi}_2 \) vectors stored as columns.

For the ellipse \( \bar{x}^T \bar{A} \bar{x} = 1 \) we can use \( \bar{x} = \bar{R} \bar{x}' \) to yield 
\[
\bar{x}^T \bar{R} \bar{A} \bar{R}^T \bar{x}' = 1 .
\]
But 
\[
\bar{x}^T \bar{R} \bar{x}' = 1
\]
for all \( \bar{x}' \) on the ellipse. Hence 
\[
\bar{R}^T \bar{A} \bar{R} = \bar{\lambda}
\]
and 
\[
\bar{R} \bar{R}^T \bar{A} \bar{R} \bar{R}^T = \bar{R} \bar{\lambda} \bar{R}^T
\]
or 
\[
\bar{A} = \bar{R} \bar{\lambda} \bar{R}^T .
\]

Now consider one of the vectors \( \bar{\phi}_1 \) & \( \bar{\phi}_2 \), the matrix-vector multiply 
\[
\bar{R} \bar{\phi}_k = \lambda_k \bar{\phi}'_k
\]
is just a scalar-vector multiply. But \( \bar{\phi}'_k = \bar{R}^T \bar{\phi}_k \) then 
\[
\bar{R} \bar{R}^T \bar{\phi}_k = \lambda_k \bar{R}^T \bar{\phi}_k .
\]
Pre-multiplying by \( \bar{R} \)
\[
\bar{R} \bar{R} \bar{R}^T \bar{\phi}_k = \bar{R} \lambda_k \bar{R}^T \bar{\phi}_k = \lambda_k \bar{\phi}_k .
\]
But \( \bar{A} = \bar{R} \bar{\lambda} \bar{R}^T \) then 
\[
\bar{A} \bar{\phi}_k = \lambda_k \bar{\phi}_k .
\]

Then multiplying \( \bar{A} \) by either \( \bar{\phi}_k \) produces a vector that is proportional to \( \bar{\phi}_k \) with constant of proportionality \( \lambda_k \). The \( \lambda_k \) values are the eigenvalues of \( \bar{A} \) and the \( \bar{\phi}_k \) vectors are the corresponding eigenvectors of \( \bar{A} \).

Hence the operation \( \bar{R}^T \bar{A} \bar{R} = \bar{\lambda} \) will diagonalize \( \bar{A} \).
Consider the spring-mass system, shown at the right, with spring constant $k$ and mass $m$ for each mass. The equations of motion are:

$$m \ddot{x}_1 = k(x_2 - x_1) \quad \text{and} \quad m \ddot{x}_2 = -k(x_2 - x_1)$$

where $(\ddot{ }) = \frac{d^2( )}{dt^2}$.

The equations of motion can be written in matrix form as

$$
\begin{bmatrix}
m & 0 \\
0 & m
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} +
\begin{bmatrix}
k & -k \\
-k & k
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = 0,
$$
or

$$M \ddot{x} + Kx = 0.$$  

The system of equations are linear second order equations with constant coefficients and so have a complimentary solution of the form $Ce^{rt}$ but they will vibrate so it would be better to change $r$ in the exponent and use $Ce^{i\omega t}$. Hence assume

$$\ddot{x} = \tilde{X}e^{i\omega t}$$

where $\tilde{X}$ is a constant vector with the amplitudes for each degree of freedom. We could have used $\ddot{x} = \tilde{X} \sin(\omega t)$ or $\ddot{x} = \tilde{X} \cos(\omega t)$ or a linear combination of these, but the exponential is the easiest to take derivatives. The

$$\ddot{x} = -\omega^2 \tilde{X}e^{i\omega t},$$

and the equations of motion are:

$$(-\omega^2 M\tilde{X} + K\tilde{X})e^{i\omega t} = 0 \quad \text{or} \quad (K - \omega^2 M)\tilde{X} = 0,$$

since $e^{i\omega t} \neq 0$. If we divide the equations of motion by $m$, then

$$
\begin{bmatrix}
k/m - \omega^2 & -k/m \\
-k/m & k/m - \omega^2
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix} = 0.
$$

Let $\bar{A} = \begin{bmatrix}
k/m & -k/m \\
k/m & k/m
\end{bmatrix}$ and $\bar{X} = \begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}$ then $(\bar{A} - \omega^2 I)\bar{X} = 0$ or

$$\bar{A} \bar{X} = \omega^2 \bar{X}.$$
Then $\omega^2$ represents the eigenvalues of $\overline{A}$ and the corresponding eigenvectors are represented by $\overline{X}$.

As before let the eigenvectors be $\bar{\phi}_k$ and store them as columns in the matrix $\Phi$, then

$$\Phi^T \overline{A} \Phi = \overline{\omega}^2$$

and

$$\Phi^T \Phi = \Phi \Phi^T = I.$$ 

**Diagonalizing a Matrix**

Let $\bar{x} = \Phi \bar{z}$ or $\bar{z} = \Phi^T \bar{x}$, where $\bar{z} = Z e^{i\alpha}$ and $Z$ is the amplitudes of the eigenvectors, then the original equations of motion become

$$\overline{M} \Phi \ddot{\bar{z}} + \overline{K} \Phi \bar{z} = 0.$$ 

Pre-multiplying by $\Phi^T$ and recall that $\overline{M} = mI$ yields

$$m \Phi^T I \Phi \ddot{\bar{z}} + \Phi^T \Phi \bar{z} = 0.$$ 

But $\Phi^T I \Phi = \Phi^T \Phi = I$, and dividing by $m$ yields

$$\ddot{\bar{z}} + \Phi^T \frac{\overline{K}}{m} \Phi \bar{z} = 0.$$ 

Since $\frac{\overline{K}}{m} = \overline{A}$

$$\ddot{\bar{z}} + \overline{A} \Phi \bar{z} = 0.$$ 

But $\Phi^T \overline{A} \Phi = \overline{\omega}^2$, hence

$$\ddot{\bar{z}} + \overline{\omega}^2 \bar{z} = 0,$$

which is now diagonalized, that is

$$\ddot{z}_j + \omega_j^2 z_j = 0.$$ 

The equations of motion can now be readily integrated exactly to yield

$$z_j = a_j e^{i\omega_j t} + b_j e^{-i\omega_j t} = A_j \cos(\omega_j t) + B_j \sin(\omega_j t)$$.
The eigenvalues satisfy \( A \hat{x} = \lambda \hat{x} \) or \((A - \lambda I) \hat{x} = 0\).

A non-trivial solution only results if the equations are linearly dependent. Then the determinant vanishes or

\[
|A - \lambda I| = 0.
\]

Consider the dynamics example the

\[
\begin{vmatrix}
\frac{k}{m} - \omega^2 & -\frac{k}{m} \\
-\frac{k}{m} & \frac{k}{m} - \omega^2
\end{vmatrix} = 0
\]

or

\[
\left(\frac{k}{m} - \omega^2\right)^2 - \left(\frac{k}{m}\right)^2 = 0
\]

is a quadratic in \( \omega^2 \).

The solution for the roots is

\[
\frac{k}{m} - \omega^2 = \pm \frac{k}{m} \quad \text{and} \quad \omega^2 = \frac{k}{m} \mp \frac{k}{m} = (0, 2 \frac{k}{m})
\]

and the corresponding eigenvectors satisfy

\((A - \lambda I) \hat{x} = 0\)

for each eigenvalue.

Using the dynamics example, for \( \omega^2 = \omega_1^2 = 0 \) the matrix equation \((K - \omega^2 M) \hat{\phi}_1 = 0\) gives

\[
\begin{bmatrix}
\frac{k}{m} - \frac{k}{m} \\
-\frac{k}{m} & \frac{k}{m}
\end{bmatrix}
\begin{Bmatrix}
\phi_{1,1} \\
\phi_{1,2}
\end{Bmatrix} = 0 \quad \text{or} \quad \frac{k}{m} (\phi_{1,1} - \phi_{1,2}) = 0, \quad \text{and} \quad -\frac{k}{m} (\phi_{1,1} - \phi_{1,2}) = 0.
\]

the two equations are linearly dependent. In fact they are identical. Hence \( \phi_{1,1} = \phi_{1,2} \).

But we would like to have \( \hat{\phi}_1 \cdot \hat{\phi}_1 = 1 \) then
Then \( \phi_{1,1}^2 + \phi_{1,2}^2 = 1 \), and \( \phi_{1,1} = \phi_{1,2} \) then \( 2\phi_{1,1}^2 = 1 \) making \( \phi_{1,1} = \pm \frac{1}{\sqrt{2}} \).

Take the plus sign the \( \phi_{1,1} = \phi_{1,2} = \frac{1}{\sqrt{2}} \) and \( \phi_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \) is the eigenvector for \( \omega^2 = \omega_1^2 = 0 \).

For \( \omega^2 = \omega_2^2 = \frac{2k}{m} \) the matrix equation \((\mathbf{\Phi} - \omega_2^2 \mathbf{I})\mathbf{\Phi}_2 = 0 \) is

\[
\begin{bmatrix}
-k & -k \\ m & m \\
-k & -k \\ m & m
\end{bmatrix}
\begin{bmatrix}
\phi_{2,1} \\ 
\phi_{2,2}
\end{bmatrix} = 0,
\]

and both equations are identical. Then \( \phi_{2,2} = -\phi_{2,1} \). But we would like to set \( \phi_{2,1}^2 + \phi_{2,2}^2 = 1 \) which yields \( 2\phi_{2,1}^2 = 1 \) and \( \phi_{2,1} = \pm \frac{1}{\sqrt{2}} \). We can take the plus sign then

\( \mathbf{\Phi}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \).

The eigenvalue matrix \( \mathbf{\Phi} \) and the eigenvector matrix \( \mathbf{\Phi}_2 \) become

\[
\mathbf{\Phi} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad \mathbf{\Phi}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \frac{k}{m} \end{bmatrix}.
\]

For the first eigenvalue, \( \omega_1^2 = 0 \) the equation of motion is \( \ddot{z}_1 = 0 \), which can be integrated to yield

\( z_1 = a_1 t + a_2 \),

and the two masses move as a rigid body at a constant speed (i.e., the length of the spring does not change from the original length).

For the first eigenvalue, \( \omega_2^2 = \frac{2k}{m} \) the equation of motion is \( \ddot{z}_2 + \frac{2k}{m} z_2 = 0 \), which can be integrated to yield

\( z_2 = b_1 \cos \left( \sqrt{\frac{2k}{m}} t \right) + b_2 \sin \left( \sqrt{\frac{2k}{m}} t \right) \),

and the spring alternates between extension and compression at frequency \( f = \frac{\omega_2}{2\pi} \).
Properties

1) If $\lambda$ is an eigenvalue of $A$ then $1/\lambda$ is an eigenvalue of $A^{-1}$
Proof: $A \bar{x} = \lambda \bar{x} \Rightarrow A^{-1} A \bar{x} = A^{-1}(\lambda \bar{x}) \Rightarrow I \bar{x} = \lambda A^{-1} \bar{x} \Rightarrow \frac{1}{\lambda} \bar{x} = A^{-1} \bar{x}$

2) For a normal matrix ($A^*A = AA^*$) the eigenvalues of $A^*$ are the complex conjugate of the eigenvalues of $A$.
Proof: Let $I - A\lambda = 0$ since $(A - \lambda I)x = 0$ then $Bx = 0 \Rightarrow (B\bar{x})^* = 0 \Rightarrow \bar{x}^*B^* = 0$, and $\bar{x}^*B^*B\bar{x} = 0$. Now $B^*B = (A - \lambda I)^*(A - \lambda I) = A^*A - A\lambda A^* - \lambda A^*A + \lambda^* \lambda I$. But $A^*A = A A^*$ hence $B^*B = A A^* - A\lambda A^* - \lambda A^*A + \lambda^* \lambda I = (A - \lambda I)(A - \lambda I)^* = B B^*$. Then $\bar{x}^*B^*B\bar{x} = \bar{x}^*B^*B\bar{x} = (B^*x)^* B^*x = 0$ or $B^*x = 0 = (A^* - \lambda^* A)\bar{x} = 0$ and $\lambda^*$ is an eigenvalue of $A^*$.

3) The non-equal eigenvalues of a normal matrix $A$ are orthogonal.
Proof: $A \bar{x}_i = \lambda_i \bar{x}_i$ and $A \bar{x}_j = \lambda_j \bar{x}_j$. Then $\bar{x}_i^* A \bar{x}_j = \lambda_j \bar{x}_i^* \bar{x}_j$, but $\bar{x}_i^* \bar{x}_j = (A^* \bar{x}_i)^* = (\lambda_i \bar{x}_i)^* = \bar{x}_i^* \lambda_i$ or $\bar{x}_i^* A = \lambda_i \bar{x}_i$. Substituting in $\bar{x}_i^* A \bar{x}_j = \lambda_j \bar{x}_i^* \bar{x}_j$ yields $\bar{x}_i^* \bar{x}_j = \lambda_i \bar{x}_i^* \bar{x}_j$. Then $(\lambda_i - \lambda_j) \bar{x}_i^* \bar{x}_j = 0$. Since $\lambda_i \neq \lambda_j$, $\bar{x}_i^* \bar{x}_j = 0$ or the eigenvectors are orthogonal.

Note that if two eigenvalues are equal (i.e., they are degenerate) then the corresponding eigenvectors can be made orthogonal. For example, given the vector $\vec{z}_i = \bar{x}_i$ then let $\vec{\xi}_2 = \vec{x}_2 - <\vec{z}_1 | \vec{x}_2> \vec{z}_1$ (where $<a | b> = a^* b$ is the scalar product). Then $<\vec{z}_1 | \vec{x}_2> = <\vec{z}_1 | \vec{x}_2> - <\vec{z}_1 | \vec{x}_1> <\vec{z}_1 | \vec{z}_2>$. But $<\vec{z}_1 | \vec{z}_1> = 1$, then $<\vec{z}_1 | \vec{x}_2> = <\vec{z}_1 | \vec{x}_2> = <\vec{z}_1 | \vec{x}_1> <\vec{z}_1 | \vec{z}_2>$. A third vector can be added by $\vec{z}_3 = \vec{x}_3 - <\vec{z}_2 | \vec{x}_3> \vec{z}_2 - <\vec{z}_1 | \vec{x}_3> \vec{z}_1$, and a fourth similarly.

4) An arbitrary vector can be expanded as a linear combination of orthogonal eigenvectors (actually we will consider orthonormal eigenvectors) $\bar{x}_i$.
Proof: Consider $\bar{y} = \sum_{i=1}^{N} a_i \bar{x}_i$, then $x_j^* \bar{y} = \sum_{i=1}^{N} a_i x_j^* \bar{x}_i = \sum_{i=1}^{N} a_i \delta_{ij} = a_j$, and we have the coefficients required for the expansion.

5) The eigenvalues of a Hermitian matrix are real.
Proof: Since \( \bar{A} \bar{x}_i = \lambda_i \bar{x}_i \) then \( \bar{x}_i^+ \bar{A}^+ = \lambda_i^* \bar{x}_i^+ \). But \( \bar{A}^+ = \bar{A} \) then \( \bar{x}_i^+ \bar{A} \bar{x}_i = \lambda_i^* \bar{x}_i^+ \bar{x}_i \) and \( \bar{x}_i^+ \bar{A} \bar{x}_i = \lambda_i \bar{x}_i^+ \bar{x}_i \). Subtracting yields \( 0 = (\lambda_i^* - \lambda_i) \bar{x}_i \). Since \( \bar{x}_i^+ \bar{x}_i \neq 0 \) then \( \lambda_i^* = \lambda_i \), and the eigenvalues are real (i.e., the imaginary part is zero).

In a similar manner it can be shown that the eigenvalues of a real symmetric matrix are all real. It can also be shown that the eigenvalues of an anti-symmetric are pure imaginary or zero.

6) If a matrix is unitary (i.e., \( \bar{A}^+ = \bar{A}^{-1} \)) then the eigenvalues have a unit modulus (i.e., \( \lambda^* \lambda = 1 = |\lambda|^2 \)).

Proof: Since \( \bar{A} \bar{x} = \lambda \bar{x} \) then \( \bar{x}^+ \bar{A}^+ = \lambda^* \bar{x}^+ \) and \( \bar{x}^+ \bar{A}^+ \bar{A} \bar{x} = \lambda^* \lambda \bar{x}^+ \). But \( \bar{A}^+ = \bar{A}^{-1} \) or \( \bar{A}^+ \bar{A} = I \) and \( \bar{x}^+ A^+ \bar{A} \bar{x} = \bar{x}^+ \bar{x} \). Then \( \bar{x}^+ \bar{x} = \lambda \lambda^* \bar{x}^+ \bar{x} \) or \( (\lambda^2 - \lambda - 1) \bar{x}^+ \bar{x} = 0 \). Since \( \bar{x}^+ \bar{x} \neq 0 \) then \( \lambda^2 = \lambda \).

7) A necessary and sufficient condition for two matrices \( \bar{A} \) and \( \bar{B} \) to have the same eigenvectors is that they commute under matrix multiplication (i.e., \( \bar{A} \bar{B} = \bar{B} \bar{A} \)).

Proof: \( \bar{A} \bar{x}_i = \lambda_i \bar{x}_i \) and \( \bar{B} \bar{x}_i = \mu_i \bar{x}_i \). Consider the expansion \( \bar{x} = \sum_i c_i \bar{x}_i \) then

\[
\bar{A} \bar{B} \bar{x} = \bar{A} \bar{B} \sum_i c_i \bar{x}_i = \bar{A} \sum_i c_i \mu_i \bar{x}_i = \sum_i c_i \mu_i \lambda_i \bar{x}_i \text{ and the eigenvectors are the same for } \bar{A}
\]

and \( \bar{B} \) then \( \bar{B} \bar{A} \bar{x} = \bar{B} \sum_i c_i \lambda_i \bar{x}_i = \bar{B} \sum_i c_i \lambda_i \mu_i \bar{x}_i = \sum_i c_i \lambda_i \mu_i \bar{x}_i = \bar{B} \bar{A} \bar{x} \). Then

\[
(\bar{A} \bar{B} - \bar{B} \bar{A}) \bar{x} = 0 \text{ for all } \bar{x} \text{. Hence } \bar{A} \bar{B} = \bar{B} \bar{A}.
\]

8) \( \sum_i \lambda_i = \text{Tr}(\bar{A}) = \sum_i A_{ii} \).

Proof: \( |\bar{A} - \lambda \bar{I}| = 0 \) is the polynomial \((-1)^N \lambda^N + (-1)^{N-1} \sum_{i=1}^N A_{ii} \lambda^{N-1} + ... = 0 \) and also

\[
|\bar{A} - \lambda \bar{I}| = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)...(\lambda_N - \lambda) = (-1)^N \lambda^N + (-1)^{N-1} \sum_{i=1}^N \lambda_i \lambda^{N-1} + ... = 0
\]

holds for all \( \lambda \), hence \( \sum_i A_{ii} = \sum_i \lambda_i \).

For a general non-normal matrix where \( \bar{A}^+ \neq \bar{A}^{-1} \), the eigenvectors are non-orthogonal but they are linearly independent. The eigenvalues need not be real and if the eigenvalues are degenerate they may not be linearly independent.
**Similarity Transformation**

Consider

\[ \overline{A} \overline{x} = \lambda \overline{x} \]

and let

\[ \overline{x} = \overline{S} \overline{x}' \]

and

\[ \overline{x}' = \overline{S}^{-1} \overline{x} . \]

Then

\[ \overline{A} \overline{S} \overline{x}' = \lambda \overline{S} \overline{x}' \]

and

\[ \overline{S}^{-1} \overline{A} \overline{S} \overline{x}' = \lambda \overline{S}^{-1} \overline{S} \overline{x}' = \lambda \overline{x}' . \]

Hence the matrix \( \overline{B} = \overline{S}^{-1} \overline{A} \overline{S} \) has the same eigenvalues as \( \overline{A} \).

This can be used to transform a matrix so that \( \overline{B} \) becomes diagonal.

**Norm of a Matrix**

We can write the norm of a matrix as \( \max \overline{x}^T \overline{A} \overline{x} \) where the norm of \( \overline{A}^T \) equals norm \( \overline{A} \) and \( \overline{x} \) is subject to \( \overline{x}^T \overline{x} = 1 \). Let \( F = \overline{x}^T \overline{A} \overline{x} \) and \( G = \overline{x}^T \overline{x} - 1 = 0 \) then we can maximize

\[ \mathcal{F} = F - \lambda G = \overline{x}^T \overline{A} \overline{x} - \lambda (\overline{x}^T \overline{x} - 1) . \]

Hence

\[ \mathcal{F} = \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij} x_i x_j - \lambda \sum_{i=1}^{N} x_i x_i - 1 . \]

Taking the derivative with respect to \( x_k \)

\[ \frac{\partial \mathcal{F}}{\partial x_k} = \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij} (\delta_{ik} x_j + x_i \delta_{jk}) - \lambda \sum_{i=1}^{N} 2 \delta_{ik} x_i . \]
Then
\[ \sum_{j=1}^{N} A_{kj} x_j + \sum_{i=1}^{N} A_{ik} x_i - 2\lambda x_k = 0. \]

But
\[ \sum_{i=1}^{N} A_{ik} x_i = \sum_{i=1}^{N} A_{ki} x_i = \sum_{j=1}^{N} A_{kj} x_j \]

and hence
\[ 2 \sum_{j=1}^{N} A_{kj} x_j - 2\lambda x_k = 0, \]
or
\[ \sum_{j=1}^{N} A_{kj} x_j = \lambda x_k. \]

This can be written as
\[ \bar{A} \bar{x} = \lambda \bar{x}. \]

Recall
\[ \| \bar{A} \| = \max \frac{x^T \bar{A} \bar{x}}{x^T x}, \]

but we set \( \bar{x}^T \bar{x} = 1 \) hence
\[ \| \bar{A} \| = \max \bar{x}^T \bar{A} \bar{x} = \max \bar{x}^T \lambda \bar{x} = \max \lambda \bar{x}^T \bar{x} = \max \lambda, \]
or
\[ \| \bar{x} \| = \max_k \lambda_k. \]

That is the 2-norm is the maximum eigenvalue.

**Singular Value Decomposition**

Any matrix \( \bar{A} \) can be written as the product of 3 matrices
\[ \bar{A} = \bar{U} \bar{S} \bar{V} \]

where
1) \( \bar{A} \) is MxN,
2) \( \bar{U} \) is MxM and \( \bar{U}^+ \bar{U} = \bar{I} \),
3) \( \bar{S} \) is MxN and is diagonal (if M>N the bottom M-N rows have zero elements and if M<N the right most N-M columns have elements that are all zero, and
4) \( \bar{V} \) is NxN and \( \bar{V}^+ \bar{V} = \bar{I} \).
Consider

\[ \overline{A^+ A} = \overline{V^+ \Phi V} = \overline{V^+ V V^+ V} = \overline{V^+ \Phi V} = \overline{V^+ \Phi V}. \]

The matrix \( \overline{\Phi V} \) is diagonal. Since \( \overline{V^+} = \overline{V}^{-1} \) then

\[ \overline{A^+ A} = \overline{V^+ \Phi V} \]

or

\[ \overline{V^+ A^+ A^+ V} = \overline{\Phi V} \]

and \( \overline{\Phi V} \) contains the eigenvalues of \( \overline{A^+ A} \).

Similarly

\[ \overline{A^+ A} = \overline{U^+ \Phi U} = \overline{U^+ \Phi U} = \overline{U^+ \Phi U} = \overline{U^+ \Phi U} = \overline{U^+ \Phi U}. \]

The matrix \( \overline{\Phi U} \) is diagonal. Since \( \overline{U^+} = \overline{U}^{-1} \) then

\[ \overline{U^+ A^+ A^+ U} = \overline{\Phi U} \]

and \( \overline{\Phi U} \) contains the eigenvalues of \( \overline{A^+ A} \).

**Functions of a Matrix**

For example, consider

\[ e^{\overline{A}} = \sum_{k=0}^{\infty} \overline{A}^k / k! \]

where

\[ \overline{A}^k = \overline{A} \overline{A} \overline{A} \]

k times. Of course, \( \overline{A} \) must be square. Instead consider the eigenvector matrix \( \overline{\Phi} \) and the eigenvalues \( \lambda \) of the real symmetric matrix \( \overline{A} \).
Then $\Phi^T \Phi = I$ and $\Phi^T A \Phi = \bar{\Phi}$

Hence

$$\Phi^T e^{\bar{\Phi}} = \sum_{k=0}^{\infty} \frac{1}{k!} \Phi^T \bar{\Phi}^k \Phi = \sum_{j=1}^{k} \frac{1}{k!} \Phi^T \left\{ \prod_{j=1}^{k} \bar{A} \right\} \Phi$$

or

$$\Phi^T e^{\bar{A}} \Phi = \sum_{k=0}^{\infty} \frac{1}{k!} \Phi^T \bar{A} \Phi \Phi^T \bar{A} \Phi \cdots \Phi^T \bar{A} \Phi = \sum_{k=0}^{k} \frac{1}{k!} \bar{\Phi} \bar{\Phi} \cdots \bar{\Phi} = \sum_{k=0}^{k} \frac{1}{k!} \bar{\Phi} = e^{\bar{\Phi}}$$

Note that

$$e^{\bar{\Phi}} = \begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_M} \end{bmatrix}$$

Finally

$$e^{\bar{A}} = \Phi e^{\bar{\Phi}} \Phi^T$$

**Natural Frequencies**

Consider a general discrete dynamic system

$$\bar{M} \ddot{x} + \bar{K} \dot{x} = 0.$$ 

Let

$$\ddot{x} = \bar{X} e^{i \omega t}.$$ 

Then

$$(\bar{K} - \omega^2 \bar{M}) \bar{X} = 0$$

since $e^{i \omega t} \neq 0$.

In finite element codes $\bar{M}$ is not generally diagonal and the natural frequencies satisfy the more general eigenvalue problem

$$\bar{K} \bar{X} = \omega^2 \bar{M} \bar{X}.$$ 

Pre-multiplying by $\bar{M}^{-1}$

$$\bar{M}^{-1} \bar{K} \bar{X} = \omega^2 \bar{X}.$$
Hence, the natural frequencies are the eigenvalues of $\overline{M}^{-1} \overline{K}$, but $\overline{M}^{-1} \overline{K}$ is not symmetric and most efficient eigenvalue extraction software assumes the matrix is symmetric.

Instead let

$$\overline{M} = \overline{M}^{1/2} \overline{M}^{1/2}$$

where

$$\overline{M} = \Psi \overline{\Xi} \Psi^T,$$

$\overline{\Xi}$ contains the eigenvalues of $\overline{M}$ and $\Psi$ contains the eigenvectors stored columnwise.

Then from the equations of motion

$$M^{-1/2} \overline{M}^{1/2} \overline{M}^{1/2} \overline{M}^{-1/2} M^{-1/2} \ddot{x} + \overline{M}^{-1/2} K \overline{M}^{-1/2} M^{1/2} \ddot{x} = 0$$

Let $\ddot{y} = \overline{M}^{1/2} \ddot{x}$ then $\ddot{y} = \overline{M}^{1/2} \ddot{x}$ and the governing equation is now

$$\ddot{\overline{y}} + \overline{K}' \overline{y} = 0,$$

and the matrix $\overline{K}'$ is symmetric!

Setting $\ddot{y} = \ddot{\overline{y}} e^{i\omega \tau} = \overline{M}^{1/2} \ddot{x} e^{i\omega \tau}$ we find that

$$\overline{K}' \ddot{y} = \omega^2 \overline{y}$$

is an eigenvalue problem and since $\overline{K}'$ is symmetric the natural frequencies $\omega^2$ are real.
Homework No. 9

The dynamic system

\[
\begin{bmatrix}
    m & 0 \\
    0 & m
\end{bmatrix}
\begin{bmatrix}
    \ddot{x}_1 \\
    \ddot{x}_2
\end{bmatrix}
+ \begin{bmatrix}
    2k & -k \\
    -k & 2k
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix} = 0.
\]

Assume

\[
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix} = \begin{bmatrix}
    X_1 \\
    X_2
\end{bmatrix} e^{i\omega t},
\]

and find the eigenvalues, $\omega^2$, and the corresponding eigenvectors (i.e., mode shapes).