Linear Algebra

Vectors

A general N-dimensional vector consists of N values. They can be arranged as a column or a row and can be real or complex.

Recall a 3-dimensional vector can represent position, velocity, or acceleration. Let \( \hat{i}, \hat{j}, \& \hat{k} \) be unit vectors along \( x, y, \& z \) respectively and let

\[
\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}
\]

have the components \( \{a_x, a_y, a_z\} \) of the vector, \( \vec{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \). Then the unit vectors

\[
\hat{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

and we can write

\[
\vec{a} = a_x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a_y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + a_z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}.
\]

Vector Spaces

A set of vectors \( \vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots \) forms a linear vector space if:

1. \( \vec{a}_1 + \vec{a}_2 = \vec{a}_2 + \vec{a}_1 \) and \( \vec{a}_1 + (\vec{a}_2 + \vec{a}_3) = (\vec{a}_1 + \vec{a}_2) + \vec{a}_3 \) (i.e., the set is closed under commutative and associative addition).

2. \( \lambda(\hat{a} + \hat{b}) = \lambda\hat{a} + \lambda\hat{b}, (\lambda + \mu)\hat{a} = \lambda\hat{a} + \mu\hat{a} \) and \( \lambda(\mu\hat{a}) = (\lambda\mu)\hat{a} \) (i.e., the set is closed under scalar multiplication and is both associative and distributive).

3. The vector \( \vec{0} \) exists such that \( \vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a} \).

4. Multiplication by 1 leaves the vector the same the original (i.e., \( 1 \cdot \vec{a} = \vec{a} \)).

5. Each vector, \( \vec{a} \), has a negative vector, \( -\vec{a} \) such that \( \vec{a} + (-\vec{a}) = \vec{0} \) or \( \vec{a} - \vec{a} = \vec{0} \). We can write \( -\vec{a} = -1 \cdot \vec{a} \) then \( (1 - 1)\vec{a} = \vec{0} \) holds.
New vectors, $\vec{x}$, that can be constructed from $\vec{a}_1, \vec{a}_2, \vec{a}_3, ..., \vec{a}_n$ is the span of the set. For a choice of $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ we can write $\vec{x} = \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \alpha_3 \vec{a}_3 + ... + \alpha_n \vec{a}_n$. If for a choice of $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ we find that $\alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \alpha_3 \vec{a}_3 + ... + \alpha_n \vec{a}_n = \vec{0}$, then the set $\vec{a}_1, \vec{a}_2, \vec{a}_3, ..., \vec{a}_n$ is linearly dependent, but if no combination of $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ produces $\vec{0}$ then the set is linearly independent.

**Basis Vectors**

The $N$ vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3, ..., \vec{e}_N$ form an $N$-dimensional basis if they are linearly independent. Then any other vector added to the set (in the same $N$-dimensional space) must satisfy

$$\sum_{k=1}^{N} \alpha_k \vec{e}_k + \lambda \vec{e} = 0$$

where $\lambda \neq 0$, and we could write

$$\vec{x} = -\frac{1}{\lambda} \sum_{k=1}^{N} \alpha_k \vec{e}_k .$$

This leads to the equality of two vectors. Let $\vec{x} = \sum_{k=1}^{N} x_k \vec{e}_k$ and $\vec{y} = \sum_{k=1}^{N} y_k \vec{e}_k$ then $\sum_{k=1}^{N} (x_k - y_k) \vec{e}_k = 0$ if and only if $x_k = y_k$ for $k = 1, 2, 3, ..., N$.

**Inner Product**

The inner product of has the following properties:

1) $<\vec{a} | \vec{b}> = <\vec{b} | \vec{a}>$, and
2) $<\vec{a} | \lambda \vec{b} + \mu \vec{c}> = \lambda <\vec{a} | \vec{b}> + \mu <\vec{a} | \vec{c}>$

The basis set, $\vec{e}_1, \vec{e}_2, \vec{e}_3, ..., \vec{e}_N$, is orthonormal if $<\vec{e}_i | \vec{e}_j > = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$ (The quantity $\delta_{ij}$ is the Kronecker delta.) Hence, if $\vec{a} = \sum_{k=1}^{N} a_k \vec{e}_k$ and $\vec{b} = \sum_{k=1}^{N} b_k \vec{e}_k$ then

$$<\vec{e}_j | \vec{a}> = \sum_{i=1}^{N} <\vec{e}_j | a_i \vec{e}_i> = \sum_{i=1}^{N} a_i <\vec{e}_j | \vec{e}_i> = \sum_{i=1}^{N} a_i \delta_{ij} = a_j$$

Also
\[ <a | b > = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3 + ... a_N \hat{e}_N | b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3 + ... b_N \hat{e}_N \geq \sum_{i=1}^{N} \sum_{j=1}^{N} a_i b_j < \hat{e}_i | \hat{e}_j > \]

or

\[ <a | b > = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i b_j \delta_{ij} = \sum_{i=1}^{N} a_i b_i = a \cdot b . \]

Note that these definitions can be extended to complex vectors.

Other properties of the inner product include:
1) The Schwartz inequality: \[ |a | b > | \leq |a||b|. \]
This is sometimes written as \[ \|a \cdot b \| \leq |a||b|. \]
2) Triangle inequality: \[ |a + b| \leq a + b . \]

**Linear Operators**

A linear operator, \( A \), associates a vector \( \tilde{y} \) with a vector \( \tilde{x} \) where \( y = Ax \) so that

\[ A(\lambda \tilde{a} + \mu \tilde{b}) = \lambda A \tilde{a} + \mu A \tilde{b} . \]

If we use a basis vector \( A \) must result in a linear combination of the basis vectors or

\[ A \tilde{e}_j = \sum_{i=1}^{N} A_{ij} \tilde{e}_i \]

where \( A_{ij} \) is the component of \( A \tilde{e}_j \), then

\[ y = \sum_{i=1}^{N} y_i \tilde{e}_i = A \sum_{j=1}^{N} x_j \tilde{e}_j = \sum_{i=1}^{N} x_i \sum_{j=1}^{N} A_{ij} \tilde{e}_i = \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij} \tilde{x}_j \tilde{e}_i \]

or

\[ y_i = \sum_{j=1}^{N} A_{ij} x_j . \]

If \( \tilde{y} \) belongs to the basis set \( \tilde{f}_i \), \( i = 1,2,3,...,M \) then

\[ A \tilde{e}_j = \sum_{i=1}^{M} A_{ij} \tilde{f}_i . \]
Properties of Linear Operators

1) \((\vec{A} + \vec{B})\vec{x} = \vec{A}\vec{x} + \vec{B}\vec{x}\), \((\lambda \vec{A})\vec{x} = \lambda (\vec{A}\vec{x})\), \((\vec{A} \vec{B})\vec{x} = \vec{A}(\vec{B}\vec{x})\)
2) \(\vec{O}\vec{x} = \vec{0}\), \(\vec{I}\vec{x} = \vec{x}\)
3) \(\vec{A}\vec{A}^{-1} = \vec{A}^{-1}\vec{A} = \vec{I}\)

Matrices

For the components of \(A\) can be arranged in a rectangular array

\[
\vec{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{MN} \end{bmatrix}
\]

is the MxN matrix \(A\) has elements \(A_{ij}\). If \(M = N\) it is square matrix of order \(N\). Vectors can also be represented as a 1xN matrix.

\[
\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}
\]

Note:
1) \(\vec{x}^T = (x_1, x_2, \ldots, x_N)\) is the transpose. The transpose of \(A\) would have the rows and columns reversed.

2) If a different basis is used the components of a vector become \(x'_i\) but it is still the same vector.

Matrix Algebra

For a vector \(\vec{x}\) we can write

\[
\sum_j (\vec{A} + \vec{B})_{ij} x_j = \sum_j A_{ij} x_j + \sum_j B_{ij} x_j \Rightarrow (\vec{A} + \vec{B})_{ij} = A_{ij} + B_{ij}
\]

\[
\sum_j (\lambda \vec{A})_{ij} x_j = \lambda \sum_j A_{ij} x_j \Rightarrow (\lambda \vec{A})_{ij} = \lambda A_{ij}
\]

and
\[
\sum_j (\overline{A} \; \overline{B})_j x_j = \sum_k A_{ik} (B_i \overline{x})_k = \sum_j \sum_k A_{ik} B_{kj} x_j \Rightarrow (\overline{A} \; \overline{B})_j = \sum_k A_{ik} B_{kj}
\]

These are the rules for matrix addition, multiplication by a scalar, and matrix multiplication, respectively.

For matrix addition

\[
\overline{S} = \overline{A} + \overline{B} \text{ or } S_{ij} = A_{ij} + B_{ij}.
\]

If

\[
\overline{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix} = \overline{A} + \overline{B}
\]

or

\[
\overline{S} = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} & A_{13} + B_{13} \\ A_{21} + B_{21} & A_{22} + B_{22} & A_{23} + B_{23} \end{bmatrix}
\]

For scalar multiplication

\[
\lambda \overline{A} = \lambda \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} = \begin{bmatrix} \lambda A_{11} & \lambda A_{12} & \lambda A_{13} \\ \lambda A_{21} & \lambda A_{22} & \lambda A_{23} \end{bmatrix}
\]

Subtraction

\[
\overline{D} = \overline{A} - \overline{B} = \overline{A} + (-1) \cdot \overline{B} \Rightarrow D_{ij} = A_{ij} - B_{ij}
\]

Matrix multiplication is more difficult

Consider matrix-vector multiplication \( y_i = \sum_j A_{ij} x_j \), \( i = 1, 2, \ldots, M \)

\[
\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{MN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}
\]
For a unit basis vector $\vec{e}_j$ where $\vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ where the $j^{th}$ component is one all the other components are zero. Then

$$\bar{A}\vec{e}_j = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1N} \\ A_{21} & A_{22} & \ldots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{M1} & A_{M2} & \ldots & A_{MN} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{Nj} \end{bmatrix}$$

If we put all the $\vec{e}_j$ vectors in a matrix in order as columns we would have an identity matrix (defined below) and pre-multiplying by $A$ should produce $A$. Consider now the matrix multiplication example

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}$$

where

$$P_{11} = A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31}$$
$$P_{21} = A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31}$$
$$P_{12} = A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32}$$
$$P_{22} = A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32}$$

It can readily be shown that:

1) Matrix multiplication is associative: $\bar{A} (\bar{B} \bar{C}) = (\bar{A} \bar{B}) \bar{C}$.
2) In general matrix multiplication is not commutative: $\bar{A} \bar{B} \neq \bar{B} \bar{A}$.
3) Matrix multiplication is distributive:

$$(\bar{A} + \bar{B}) \bar{C} = \bar{A} \bar{C} + \bar{B} \bar{C}, \text{ and } \bar{C} (\bar{A} + \bar{B}) = \bar{C} \bar{A} + \bar{C} \bar{B}.$$  

The null matrix, $\bar{O}$, has all of its elements equal to zero. Hence,

$$\bar{A} \bar{O} = \bar{O} \bar{A} = \bar{O}, \text{ and } \bar{A} + \bar{O} = \bar{O} + \bar{A} = \bar{A}$$  

where $\bar{O} = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}$.

There is also the identity matrix, $\bar{I}$, where $\bar{I} \bar{A} = \bar{A} \bar{I} = \bar{A}$.
The matrix $\mathbf{I}$ has elements

$$
\delta_{ij} = \begin{cases} 
0, & i \neq j \\
1, & i = j 
\end{cases}.
$$

Recall this is the Kronecker delta.

Note that $\mathbf{I}$ must be square and

$$
\mathbf{I} = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix}.
$$

**Matrix Functions**

We can define powers by $\mathbf{A}^n = \mathbf{A} \mathbf{A} \mathbf{A} \ldots \mathbf{A}$ $n$ times.

Of course $\mathbf{A}$ must be square. Now a Taylor series can be use to define functions. They would have the form:

$$
\mathbf{S} = \sum_{n=0}^{\infty} a_n \mathbf{A}^n.
$$

which is in general a function. For example,

$$
\exp(\mathbf{A}) = e^{\mathbf{A}} = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!}.
$$

In a similar manner other functions such as $\sin(\mathbf{A})$ or $\cos(\mathbf{A})$. Finally we can include

$$
\ln(\mathbf{I} + \mathbf{A}) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{\mathbf{A}^n}{n}
$$

**Transpose of a Matrix**

The transpose of $\mathbf{A}$ will be denoted as $\mathbf{A}^T$ where the elements are related by

$$
A_{ij}^T = A_{ji}.
$$
For example, if $\bar{A} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 4 & 1 \end{bmatrix}$ then $\bar{A}^T = \begin{bmatrix} 3 & 0 \\ 1 & 4 \\ 2 & 1 \end{bmatrix}$.

Now consider

$$(\bar{A} \ B)^T = \left[ \sum_{k=1}^{N} A_{ik} B_{kj} \right]^T = \sum_{k=1}^{N} A_{jk} B_{ki} = \sum_{k=1}^{N} B_{ki} A_{jk} = \sum_{k=1}^{N} B_{ik}^T A_{kj}^T = B^T \bar{A}^T.$$

Then

$$(\bar{A} \ B \ C)^T = C^T B^T \bar{A}^T,$$

and

$$(\bar{A} \ B \ C \ldots \bar{C})^T = C^T \ldots C^T B^T \bar{A}^T.$$  

**Trace of a Matrix**

The trace of a matrix, $\bar{A}$, is the sum of the diagonal components, that is

$$Tr(\bar{A}) = A_{11} + A_{22} + \ldots + A_{NN} = \sum_{i=1}^{N} A_{ii}.$$  

Of course, $\bar{A}$, must be square.

Note: $Tr(\bar{A} + \bar{B}) = Tr(\bar{A}) + Tr(\bar{B})$, $Tr(\bar{A} - \bar{B}) = Tr(\bar{A}) - Tr(\bar{B})$ and $Tr(\lambda \bar{A}) = \lambda Tr(\bar{A})$.

Also

$$Tr(\bar{A} \ B) = \sum_{i=1}^{N} (\bar{A} \ B)_{ii} = \sum_{i=1}^{N} \left( \sum_{j=1}^{N} A_{ij} B_{ji} \right) = \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij} B_{ji} = \sum_{i=1}^{N} \sum_{j=1}^{N} B_{ji} A_{ij} = \sum_{j=1}^{N} \sum_{i=1}^{N} (B \bar{A})_{ij} = \sum_{j=1}^{N} (B \bar{A})_{jj}$$

or

$$Tr(\bar{A} \ B) = Tr(\bar{B} \ A).$$

Similarly

$$Tr(\bar{A} \ B \ C) = Tr(\bar{C} \ B \ \bar{A}) = Tr(\bar{A} \ \bar{C} \ B),$$

that is any cyclic permutation of the matrices in the product will keep the trace the same.
**Determinant of a Matrix**

The determinant applies only to square matrices and is defined as “the sum of all possible permutations of the products of the elements of $A$ where the sign of each term is positive or negative according to whether the permutations positive or negative respectively.”

For example for a 3x3 matrix

<table>
<thead>
<tr>
<th>Product</th>
<th>Row Order</th>
<th>Row Permutation</th>
<th>Column Order</th>
<th>Column Permutation</th>
<th>Final Permutation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{11}a_{22}a_{33}$</td>
<td>123</td>
<td>+1</td>
<td>123</td>
<td>+1</td>
<td>$(+1)(+1)=+1$</td>
</tr>
<tr>
<td>$a_{11}a_{23}a_{32}$</td>
<td>123</td>
<td>+1</td>
<td>132</td>
<td>-1</td>
<td>$(+1)(-1)=-1$</td>
</tr>
<tr>
<td>$a_{21}a_{32}a_{13}$</td>
<td>231</td>
<td>+1</td>
<td>123</td>
<td>+1</td>
<td>$(+1)(+1)=+1$</td>
</tr>
<tr>
<td>$a_{21}a_{33}a_{12}$</td>
<td>231</td>
<td>+1</td>
<td>132</td>
<td>-1</td>
<td>$(+1)(-1)=-1$</td>
</tr>
<tr>
<td>$a_{31}a_{12}a_{23}$</td>
<td>312</td>
<td>+1</td>
<td>123</td>
<td>+1</td>
<td>$(+1)(+1)=+1$</td>
</tr>
<tr>
<td>$a_{31}a_{13}a_{22}$</td>
<td>312</td>
<td>+1</td>
<td>132</td>
<td>-1</td>
<td>$(+1)(-1)=-1$</td>
</tr>
</tbody>
</table>

and

$$|A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{21}a_{32}a_{13} - a_{21}a_{33}a_{12} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22}.$$  

But we can also write it as

$$A = a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} + a_{21} a_{32} a_{13} - a_{21} a_{33} a_{12} + a_{31} a_{12} a_{23} - a_{31} a_{13} a_{22}.$$  

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$  

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$  

$$\text{Sign} = (-1)^{1+7} = -1 \quad \text{Sign} = (-1)^{2+7} = -1 \quad \text{Sign} = (-1)^{3+7} = +1$$  

This is an expansion by minors (Laplace expansion), and uses the cofactor matrix and the minor.
The minor, $M_{ij}$, of the element $A_{ij}$ from the matrix $\mathbf{A}$ is defined as the determinant that remains after row $i$ and column $j$ are removed.

The cofactor, $C_{ij}$, of element $A_{ij}$ is the minor matrix, $M_{ij}$, multiplied by $(-1)^{i+j}$.

We can now write the expansion of the determinant by expanding along a column as

$$|\mathbf{A}| = \sum_{k=1}^{N} (-1)^{i+j} A_{jk} M_{jk}.$$ 

Since $|\mathbf{A}| = |\mathbf{A}^T|$ we could also sum over $j$ then we can expand along a row also

$$|\mathbf{A}| = \sum_{k=1}^{N} (-1)^{i+j} A_{kj} M_{kj}.$$ 

In order to expand an $N\times N$ by minors we need to continue from the three $2\times 2$ determinants we had. Expanding a $2\times 2$ by minors

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = a_{11}a_{22} - a_{12}a_{21}.$$ 

Where, of course a $1\times 1$ is $|a_{11}| = a_{11}$.

**Geometric Interpretation of a 3x3 Determinant**

Consider the parallelepiped at the right. The area of the parallelogram $(\vec{b}, \vec{c})$ is the absolute value of the cross product $|\vec{b} \times \vec{c}| = bc \sin \theta$ and the vector $\vec{b} \times \vec{c}$ points perpendicular, in a right hand sense, to the plane of the parallelogram. Projecting the area, with its direction, on the vector $\vec{a}$ gives the volume as $\vec{a} \cdot (\vec{b} \times \vec{c})$. This exactly the $3\times 3$ determinant, or

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$
**Properties of Determinants**

The following properties hold:

1) \( |\mathbf{A}| = |\mathbf{A}^T| \). This also implies that whatever holds for the rows holds for the columns.

where \( \mathbf{A} \) is an \( NxN \).

2) Interchanging two rows (or columns) changes the sign only.

3) If two rows (or columns) are the same the determinant is zero.

4) \( |\mathbf{A} \mathbf{B}| = |\mathbf{A}||\mathbf{B}| \) if both matrices are square. Hence for any number of square matrices

\[
\mathbf{A} \mathbf{B} \ldots \mathbf{G} = |\mathbf{A}||\mathbf{B}||\ldots||\mathbf{G}|.
\]

5) Adding a constant multiple of one row (or column) to another leaves the determinant unchanged.

6) Multiplying a row (or column) by a constant multiplies the determinant by the constant.

**Example**

<table>
<thead>
<tr>
<th>Step</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>\mathbf{A}</td>
<td>= \begin{vmatrix} 1 &amp; 0 &amp; 2 &amp; 3 \ 0 &amp; 1 &amp; -2 &amp; 1 \ 3 &amp; -3 &amp; 4 &amp; -2 \ -2 &amp; -1 &amp; 2 &amp; -1 \end{vmatrix} = 2 \begin{vmatrix} 1 &amp; 0 &amp; 1 &amp; 3 \ 0 &amp; 1 &amp; -1 &amp; 1 \ 3 &amp; -3 &amp; 2 &amp; -2 \ -2 &amp; -1 &amp; 1 &amp; -1 \end{vmatrix} = 2 \begin{vmatrix} 1 &amp; 0 &amp; 1 &amp; 3 \ 0 &amp; 1 &amp; 0 &amp; 0 \ 3 &amp; -3 &amp; -1 &amp; -2 \ -2 &amp; -1 &amp; 0 &amp; 0 \end{vmatrix} = 2 )</td>
<td></td>
</tr>
</tbody>
</table>

Steps:
1) factoring out 2 from column 3,
2) col. 3 \( \rightarrow \) col. 2+col. 3
3) col. 4 \( \rightarrow \) col. 4- col. 2

Expanding by row 2

\[
|\mathbf{A}| = 2 \cdot 1 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 1 & 3 \\ -1 & 1 & 1 \\ -2 & 0 & 0 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{vmatrix}
\]

Expanding by row 3 and then expanding the 2x2 determinants.

\[
|\mathbf{A}| = 2 \begin{vmatrix} -2 & (-1)^{1+3} \frac{1}{3} \\ -1 & 1 \end{vmatrix} = 2[-2 \cdot (1 + 3)] = 2(-2 \cdot 4) = -16.
\]

You might want to check this! I did using Macsyma. I think I got it.
Complex Vectors & Matrices

Let \((\ )^*\) be the complex conjugate then if \(z = x + iy\), where \(i = \sqrt{-1}\), \(z^* = x - iy\).

Note that \(z^* z = (x + iy)(x - iy) = x(x - iy) + iy(x - iy) = x^2 - ixy + ixy - i^2 y^2 = x^2 + y^2\).

The inner product is defined by \(\langle a | b \rangle = a^* b\). With real numbers it would just be the dot product. The properties of the inner product include: 1) \(\langle a | b \rangle = \langle b | a \rangle\), and 2) \(\langle a | \lambda b + \lambda c \rangle = \lambda \langle a | b \rangle + \lambda \langle a | c \rangle\). Hence, for example,

\[
\langle \lambda a + \mu b | c \rangle = \lambda^* \langle a | c \rangle + \mu^* \langle b | c \rangle.
\]

Matrices with complex values need a better definition for the transpose. This is the Hermitian conjugate (adjoint) of \(A\), defined by

\[
A^+ = (A^*)^T = (A^T)^*.
\]

Hence

\[(A B \ldots G)^+ = G^+ \ldots B^+ A^+\]

Example:

\[
A = \begin{bmatrix} 1 & 2 & 3i \\ 1+i & 1 & 0 \end{bmatrix}
\]

Then

\[
A^* = \begin{bmatrix} 1 & 2 & -3i \\ 1-i & 1 & 0 \end{bmatrix}
\]

and

\[
A^+ = \begin{bmatrix} 1 & 1-i \\ 2 & 1 \\ -3i & 0 \end{bmatrix}
\]
Homework No 7

1. For the square matrix $\bar{A}$ with element $A_{ij} = i + j - 1$, $i = 1, 2, 3, ..., N$. What is the value of the determinant for all orders $N > 2$.
   Answer: $|A| = 0$.

2. For
   \[
   \bar{A} = \begin{bmatrix}
   -2 & 3 & 1 & 5 \\
   2 & 0 & -4 & -1 \\
   1 & 6 & -3 & 2 \\
   \end{bmatrix}, \quad \bar{B} = \begin{bmatrix}
   4 & -2 & 1 \\
   1 & -3 & -4 \\
   2 & 5 & 7 \\
   \end{bmatrix}
   \]
   Find: a) $\bar{B} \bar{A}$, b) $\bar{B}^2$, c) $\bar{A} \bar{A}^T$, d) $|\bar{B}|$, and e) $Tr(\bar{B})$. 