First Ordinary Differential Equations

Consider the differential equation

$$\frac{dy}{dx} = H(x)$$

We can integrate immediately to find

$$y = \int_{0}^{x} H(x_1) \, dx_1 + C_1$$

where $C_1$ is an arbitrary constant.

The constant can be found if we know $y$ at a given $x$. For example, if $y(x_0) = y_0$, then

$$y_0 = \int_{0}^{x_0} H(x_1) \, dx_1 + C_1$$

or

$$C_1 = y_0 - \int_{0}^{x_0} H(x_1) \, dx_1$$

and

$$y = y_0 - \int_{0}^{x_0} H(x_1) \, dx_1 + \int_{0}^{x} H(x_1) \, dx_1$$

Since $\int_{a}^{b} f(z) \, dz = -\int_{b}^{a} f(z) \, dz$ then

$$-\int_{0}^{x_0} H(x_1) \, dx_1 = \int_{x_0}^{0} H(x_1) \, dx_1$$

Also $\int_{a}^{b} f(z) \, dz + \int_{c}^{d} f(z) \, dz = \int_{a}^{d} f(z) \, dz$

Then

$$y = y_0 + \int_{0}^{x_0} H(x_1) \, dx_1 + \int_{x_0}^{x} H(x_1) \, dx_1$$

or

$$y = y_0 + \int_{x_0}^{x} H(x_1) \, dx_1$$
Integrating Factors

We can readily generalize the result to first order linear ordinary differential equations, by considering the differential equation

\[
\frac{d[F(x)y]}{dx} = H(x)
\]

Replacing \(y\) by \(F(x)y\) in our previous solution we get

\[
F(x)y = F(x_0)y_0 + \int_{x_0}^{x} H(x_1)dx_1
\]

or

\[
y = \frac{F(x_0)}{F(x)}y_0 + \frac{1}{F(x)} \int_{x_0}^{x} H(x_1)dx_1
\]

The original differential equation can be written as

\[
F(x) \frac{dy}{dx} + F'(x)y = H(x)
\]

or

\[
\frac{dy}{dx} + \frac{F'(x)}{F(x)}y = \frac{H(x)}{F(x)}
\]

Of course we would not write the equation as shown, instead it could be written as

\[
\frac{dy}{dx} + P(x)y = Q(x)
\]

Where \(P(x) = \frac{F'(x)}{F(x)}\) and \(Q(x) = \frac{H(x)}{F(x)}\). The function \(F\) is given in terms of \(P\) by noting

\[
\frac{dF}{dx} = P \quad \text{or} \quad \frac{dF}{P} = Fdx.
\]

The variables are separated since \(F\) is alone on one side and \(x\) on the other. Integrating

\[
\ln F = C_2 + \int_{0}^{x} P(x_1)dx_1 \quad \text{where} \ C_2 \ \text{is an arbitrary constant. Then}
\]

\[
F = e^{\int_{0}^{x} P(x_1)dx_1} = C_3 e^{\int_{0}^{x} P(x_1)dx_1} = C_3 e^{C_2} = C_3 e^{C_2 + \int_{0}^{x} P(x_1)dx_1}
\]

where \(C_3 = e^{C_2}\) then the “integrating factor, \(F\), is

\[
F = C_3 e^{\int_{0}^{x} P(x_1)dx_1}
\]
Finally we can find \( H(x) \) from \( H(x) = C_3 Q(x) e^{\int P(x) \, dx} \). Now the solution becomes

\[
y = \frac{y_0 C_3 e^{\int_0^x P(x) \, dx}}{C_3 e^{\int_0^x P(x) \, dx}} + \frac{1}{C_3 e^{\int_0^x P(x) \, dx}} \int_{x_0}^x C_3 e^{\int_0^z P(x) \, dx} Q(x) \, dx
\]

This can be simplified first to yield

\[
y = y_0 e^{\int_0^x P(x) \, dx} + e^{\int_0^x P(x) \, dx} \int_{x_0}^x e^{\int_0^z P(x) \, dx} Q(x) \, dx
\]

and again to yield

\[
y = y_0 e^{\int_0^x P(x) \, dx} + \int_{x_0}^x e^{\int_0^z P(x) \, dx} Q(x) \, dx
\]

But \( \int_{x_1}^x P(x_2) \, dx_2 = \int_{x_0}^x P(x_2) \, dx_2 + \int_{x_0}^{x_1} P(x_2) \, dx_2 \) or \( \int_{x_1}^x P(x_2) \, dx_2 = \int_{x_0}^x P(x_2) \, dx_2 - \int_{x_0}^{x_1} P(x_2) \, dx_2 \).

Then

\[
y = y_0 e^{\int_0^x P(x) \, dx} + e^{\int_0^x P(x) \, dx} \int_{x_0}^x e^{\int_0^z P(x) \, dx} Q(x) \, dx
\]

This is the general solution of a non-homogeneous linear first order ordinary differential equation. The differential equation is homogeneous if \( Q(x) = 0 \). The solution for the homogeneous equation is then

\[
y = y_0 e^{\int_0^x P(x) \, dx}
\]

This is sometimes referred to as a complimentary solution, and appears twice in the non-homogeneous solution. Hence we can write the solution as

\[
y = y_0 e^{\int_0^x P(x) \, dx} + U(x) e^{\int_0^x P(x) \, dx}
\]
where
\[ U(x) = \int_{x_0}^{x} e^{-\int_{x_0}^{x} P(s) ds} Q(x_1) dx_1 \]

Hence if we have the general solution of the homogeneous equation (i.e., the complimentary solution) the complete solution for the non-homogeneous can be assumed to be
\[ y = C_1 y_c(x) + y_p(x) \]
where \( y_p(x) = U(x) y_c(x) \) is the particular solution, and \( y_c(x) \) is the general solution of
\[ \frac{dy_c}{dx} + P(x) y_c = 0. \]

Consider the original differential equation: \( \frac{dy}{dx} + P(x) y = Q(x) \).
Assuming \( y = C_1 y_c(x) + y_p(x) \), we get
\[ C_1 \frac{dy_c}{dx} + \frac{dy_p}{dx} + C_1 P(x) y_c + P(x) y_p = Q(x). \]
Rearranging the terms
\[ C_1 \left( \frac{dy_c}{dx} + P(x) y_c \right) + \frac{dy_p}{dx} + P(x) y_p = Q(x) \]
The quantity in the parenthesis is zero from the definition of the complimentary solution. Then
\[ \frac{dy_p}{dx} + P(x) y_p = Q(x) \]
Any \( y_p(x) \) that satisfies the above equation can be used since the arbitrary constant is included in the complimentary part.

Note that if \( y(x_0) = y_0 \) then \( y_0 = C_1 y_c(x_0) + y_p(x_0) \), and we can solve for \( C_1 \) as
\[ C_1 = \frac{y_0 - y_p(x_0)}{y_c(x_0)}. \]
**Second Order Linear Differential Equations with Constant Coefficients**

The general case is

\[
\frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = Q(x) .
\]

We first should find the homogeneous solution, then \( y_c \) satisfies

\[
\frac{d^2 y_c}{dx^2} + a_1 \frac{dy_c}{dx} + a_0 y_c = 0 .
\]

From the general solution for a first order differential equation with \( P(x) \) constant, try

\[
y_c = Ce^{rx} .
\]

Since \( \frac{dy_c}{dx} = rCe^{rx} \) and \( \frac{d^2 y_c}{dx^2} = r^2Ce^{rx} \) then

\[
r^2Ce^{rx} + a_1 rCe^{rx} + a_0 Ce^{rx} = 0
\]

or

\[
Ce^{rx} (r^2 + a_1 r + a_0) = 0 .
\]

Then either \( C=0 \) or \( e^{rx} = 0 \) or \( r^2 + a_1 r + a_0 = 0 \). But \( C \neq 0 \), since if \( C=0 \) there would be no complimentary solution, and \( e^{rx} \neq 0 \) for any value of \( x \), then the only possibility is

\[
r^2 + a_1 r + a_0 = 0 .
\]

There are two solutions

\[
r = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2} .
\]

Let \( r_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2} \) and \( r_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2} \).

There are two possible complimentary solutions. Therefore set

\[
y_1 = C_1 e^{r_1 x} \quad \text{and} \quad y_2 = C_2 e^{r_2 x} ,
\]
Since the differential equation is linear the complete solution to the homogeneous equation is

\[ y = y_1 + y_2 = C_1 e^{r_1 x} + C_2 e^{r_2 x}. \]

Then \( \frac{dy}{dx} = r_1 C_1 e^{r_1 x} + r_2 C_2 e^{r_2 x} \) and \( \frac{d^2y}{dx^2} = r_1^2 C_1 e^{r_1 x} + r_2^2 C_2 e^{r_2 x} \).

Substituting into the original differential equation

\[ r_1^2 C_1 e^{r_1 x} + r_2^2 C_2 e^{r_2 x} + a_1 r_1 C_1 e^{r_1 x} + a_1 r_2 C_2 e^{r_2 x} + a_0 C_1 e^{r_1 x} + a_0 C_2 e^{r_2 x} = 0. \]

Then

\[ \left( r_1^2 + a_1 r_1 + a_0 \right) C_1 e^{r_1 x} + \left( r_2^2 + a_1 r_2 + a_0 \right) C_2 e^{r_2 x} = 0, \]

and each quantity in the parentheses vanishes since each contains the roots of the original quadratic equation for \( r \). It checks!

There are three possibilities:

1) \( a_1^2 > 4a_0 \),
2) \( a_1^2 = 4a_0 \), and
3) \( a_1^2 < 4a_0 \).

For the first case the solution is simply

\[ y_c = C_1 e^{\frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2} x} + C_2 e^{\frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2} x}. \]

For the third case let

\[ \alpha = -\frac{a_1}{2} \text{ and } \beta = \frac{\sqrt{4a_0 - a_1^2}}{2}. \]

Then

\[ r_1 = \alpha + i\beta, \ r_2 = \alpha - i\beta, \text{ where } i = \sqrt{-1}. \]

Hence,

\[ y_c = \overline{C_1} e^{(\alpha+i\beta)x} + \overline{C_2} e^{(\alpha-i\beta)x} = e^{\alpha x} \left( \overline{C_1} e^{i\beta x} + \overline{C_2} e^{-i\beta x} \right). \]
Recall that $e^{i\theta} = \cos \theta + i \sin \theta$ and, hence, $e^{-i\theta} = \cos \theta - i \sin \theta$. Then

$$y_c = e^{\alpha x} \left\{ \overline{C}_1 [\cos(\beta x) + i \sin(\beta x)] + \overline{C}_2 [\cos(\beta x) - i \sin(\beta x)] \right\}$$

or

$$y_c = e^{\alpha x} \left\{ \left( \overline{C}_1 + \overline{C}_2 \right) \cos(\beta x) + i \left( \overline{C}_1 - \overline{C}_2 \right) \sin(\beta x) \right\}.$$

Let $C_1 = \overline{C}_1 + \overline{C}_2$ and $C_2 = i(\overline{C}_1 - \overline{C}_2)$ then

$$y_c = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x),$$

and recall that

$$\alpha = -\frac{a_1}{2}, \quad \beta = \frac{\sqrt{4a_0 - a_1^2}}{2}.$$

Now for the case where $r_1 = r_2 = -\frac{a_1}{2}$ and there is only the one complimentary solution

$$y_c = C_1 e^{\alpha x}.$$

To get the second solution let $\alpha_0 = \frac{1}{4} \left( a_1^2 - e^2 \right)$ where $e^2 << a_1^2$ and $0 < e << 1$, then

$$n_{1,2} = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^2 - 4a_0}{2}} = \frac{1}{2} \left( a_1 \pm e \right)$$

or

$$n_1 = \frac{-a_1 + e}{2}, \quad r_2 = \frac{-a_1 - e}{2}.$$

Then

$$y_c = \overline{C}_1 e^{n_1 x} + \overline{C}_2 e^{n_2 x} = \overline{C}_1 e^{rac{(a_1 - e)x}{2}} + \overline{C}_2 e^{rac{(a_1 + e)x}{2}}$$

or

$$y_c = \overline{C}_1 e^{-\frac{a_1 x}{2}} e^{-\frac{ex}{2}} + \overline{C}_2 e^{-\frac{a_1 x}{2}} e^{-\frac{ex}{2}}.$$

Since $e << 1$, $e^x \approx 1 + \frac{1}{2} ex$, and $e^{-\frac{1}{2}ex} \approx 1 - \frac{1}{2} ex$. 

Then

\[ y_c = C_1 e^{-\frac{a_1 x}{2}} \left(1 + \frac{e^x}{2}\right) + C_2 e^{-\frac{a_1 x}{2}} \left(1 - \frac{e^x}{2}\right), \]

or

\[ y_c = (C_1 + C_2) e^{-\frac{a_1 x}{2}} + \frac{1}{2} (C_1 - C_2) xe^{-\frac{a_1 x}{2}}. \]

Let \( C_1 = C_1 + C_2 \), and \( C_2 = \frac{1}{2} (C_1 - C_2) e \), then

\[ y_c = C_1 e^{-\frac{a_1 x}{2}} + C_2 xe^{-\frac{a_1 x}{2}} \]

Summarizing the solution of

\[ \frac{d^2 y_c}{dx^2} + a_1 \frac{dy_c}{dx} + a_0 y_c = 0 \]

is

<table>
<thead>
<tr>
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<th>Solution</th>
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<tr>
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<td>( y_c = C_1 e^{-\frac{a_1 x}{2}} + C_2 e^{-\frac{a_1 x}{2}} )</td>
</tr>
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<tr>
<td>3</td>
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<td>( y_c = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x) )</td>
</tr>
</tbody>
</table>

Where

\[ \alpha = -\frac{a_1}{2} \]

and

\[ \beta = \frac{\sqrt{4a_0 - a_1^2}}{2}. \]
Second Order Equidimensional Equation (Euler Equation)

Now consider
\[ x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = f(x). \]

Note that the x-dimension cancels out in each term.

The complimentary solution must have the form
\[ y_c = Cx^r. \]

Then substituting in the differential equation
\[ r(r-1)Cx^r + a_1 rCx^r + a_0 Cx^r = 0. \]

But \( Cx^r \neq 0 \) for all possible \( x \), hence
\[ r(r-1) + a_1 r + a_0 = 0, \]
or
\[ r^2 + (a_1 - 1)r + a_0 = 0, \]
and
\[ r_{1,2} = \frac{- (a_1 - 1) \pm \sqrt{(a_1 - 1)^2 - 4a_0}}{2}. \]

We can use the same procedure as for the constant coefficient case. Again there are three cases. The solutions are summarized in the table.

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<td>(y_c = C_1 x^\alpha \cos[\beta \ln x] + C_2 x^\alpha \sin[\beta \ln x])</td>
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where \( \alpha = -\frac{a_1 - 1}{2}, \beta = \sqrt{a_0 - \left(\frac{a_1 - 1}{2}\right)^2} \).
Nonlinear Ordinary Differential Equations

Nonlinear differential equations are not well behaved. Consider as an example,

$$\left( \frac{dy}{dx} \right)^2 - 4y + 4 = 0.$$  

Then

$$\frac{dy}{dx} = \pm 2\sqrt{y - 1},$$

or

$$\pm \frac{dy}{2\sqrt{y - 1}} = dx.$$  

If \( y \neq 1 \) then since \( \frac{d(y-1)}{2\sqrt{y-1}} = d\left(\sqrt{y-1} + C\right) \), we can integrate the equation to yield

$$\sqrt{y-1} + C = \pm x$$

or

$$y = 1 + (x - C)^2.$$  

Note that \( \frac{dy}{dx} = 2(x - C) \) and hence \( \frac{dy}{dx} = 0 \) \( @ \ x = C \) and \( \frac{d^2y}{dx^2} = 2 > 0. \)  

Hence \( x=2 \) is a minimum.

Note also that if \( y=1 \) that the differential equation is identically satisfied.

If the condition is \( y = 1 @ x = x_0 \) then there are two possible solutions:

1) \( y=1 \) for all \( x \), or

2) \( y = 1 + (x - x_0)^2. \)
**Single Degree of Freedom System Example**

The force due to the spring is, \( F_k \) is: \[ F_k = -kx. \]

The force due to the damper, \( F_c \), is: \[ F_c = -c\dot{x}. \]

The initial conditions are: \( x = x_0, \dot{x} = 0 @ t = 0. \)

Note that \( \ddot{x} = \frac{dx}{dt}, \) and \( \dddot{x} = \frac{d^2x}{dt^2}. \)

Newton’s 2\textsuperscript{nd} Law yields:

\[ m\dddot{x} = -c\dddot{x} - kx + F_0 \sin \omega t \]

or

\[ m\dddot{x} + c\dddot{x} + kx = F_0 \sin \omega t \]

A: Consider as a first example, \( F_0 = 0 \) (i.e., the homogeneous solution). Then

\[ m\dddot{x} + c\dddot{x} + kx = 0, \quad x(0) = x_0 \text{ and } \dot{x}(0) = 0. \]

This is a second order differential equation with constant coefficients, hence try

\[ x = C e^{\alpha t} \]

Before proceeding we can remove one parameter by dividing by \( m \).

\[ \dddot{x} + \frac{c}{m} \dddot{x} + \frac{k}{m} x = 0. \]
Let \( \omega_0^2 = \frac{k}{m} \) be the square of the undamped natural frequency (in radians), and let \( 2\zeta \omega_0 = \frac{c}{m} \) be the fraction of critical damping. Then

\[
\ddot{x} + 2\zeta \omega_0 \dot{x} + \omega_0^2 x = 0 .
\]

Substituting \( x = Ce^{\alpha t} \) we get

\[
\left( r^2 + 2\zeta \omega_0 r + \omega_0^2 \right)Ce^{\alpha t} = 0 ,
\]

and

\[
r = -\zeta \omega_0 \pm \omega_0 \sqrt{\zeta^2 - 1} = \omega_0 \left( -\zeta \pm \sqrt{\zeta^2 - 1} \right) .
\]

Again there are three cases:

1) \( \zeta > 1 \): Over damped,
2) \( \zeta = 1 \): Critically damped, and
3) \( \zeta < 1 \): Under damped.

Proceeding we can find the solution for all three cases:

1) Case 1 Over Damped:

\[
x = \left[ C_1 e^{(-\zeta \sqrt{\zeta^2 - 1})t} + C_2 e^{(-\zeta \sqrt{\zeta^2 - 1})t} \right] e^{-\zeta \omega_0 t} .
\]

The initial condition on the displacement yields

\[
x(0) = x_0 = C_1 + C_2 .
\]

While for the initial we need

\[
\dot{x}(0) = \left[ -\zeta + \sqrt{\zeta^2 - 1} \right] C_1 e^{(-\zeta \sqrt{\zeta^2 - 1})t} + \left[ -\zeta - \sqrt{\zeta^2 - 1} \right] C_2 e^{(-\zeta \sqrt{\zeta^2 - 1})t} \right] e^{-\zeta \omega_0 t} = 0 , \text{ or}
\]

\[
\left( -\zeta + \sqrt{\zeta^2 - 1} \right) C_1 + \left( -\zeta - \sqrt{\zeta^2 - 1} \right) C_2 = 0 , \text{ and}
\]

\[
-\zeta (C_1 + C_2) + \sqrt{\zeta^2 - 1}(C_1 - C_2) = 0 .
\]

But from the condition on \( x(0) \), \( C_1 + C_2 = x_0 \), hence

\[
\sqrt{\zeta^2 - 1}(C_1 - C_2) = \zeta x_0 .
\]
The two equations are now just the sum and the difference of the unknowns, i.e.

\[ C_1 + C_2 = x_0, \]
\[ (C_1 - C_2) = \frac{\zeta}{\sqrt{\xi^2 - 1}} x_0. \]

Adding \(2C_1 = \left(1 + \frac{\zeta}{\sqrt{\xi^2 - 1}}\right)x_0\), or \(C_1 = \frac{1}{2}\left(1 + \frac{\zeta}{\sqrt{\xi^2 - 1}}\right)x_0\),

and subtracting \(2C_2 = \left(1 - \frac{\zeta}{\sqrt{\xi^2 - 1}}\right)x_0\) or \(C_2 = \frac{1}{2}\left(1 - \frac{\zeta}{\sqrt{\xi^2 - 1}}\right)x_0\).

Hence the solution is

\[ x = \frac{x_0}{2} \left[ \left(1 + \frac{\zeta}{\sqrt{\xi^2 - 1}}\right) e^{\sqrt{\xi^2 - 1} \omega t} + \left(1 - \frac{\zeta}{\sqrt{\xi^2 - 1}}\right) e^{-\sqrt{\xi^2 - 1} \omega t} \right] e^{-\zeta \alpha t}. \]

In terms of hyperbolic functions

\[ x = x_0 e^{-\zeta \alpha t} \cosh \left[ \sqrt{\xi^2 - 1} \ \alpha t \right] + \frac{\zeta x_0}{\sqrt{\xi^2 - 1}} e^{-\zeta \alpha t} \sinh \left[ \sqrt{\xi^2 - 1} \ \alpha t \right]. \]

Recall \( \cosh z = \frac{e^z + e^{-z}}{2} \), and \( \sinh z = \frac{e^z - e^{-z}}{2} \).

2) Case 3 Under Damped:

Since \( \cosh iz = \frac{e^{iz} + e^{-iz}}{2} = \cos z \) and \( \sinh iz = \frac{e^{iz} - e^{-iz}}{2} = i \sin z \). Then

\[ x = x_0 e^{-\zeta \alpha t} \cos \left[ \sqrt{1 - \xi^2} \ \alpha t \right] + \frac{\zeta x_0}{i\sqrt{\xi^2 - 1}} e^{-\zeta \alpha t} i \sin \left[ \sqrt{1 - \xi^2} \ \alpha t \right], \] or

\[ x = x_0 e^{-\zeta \alpha t} \cos \left[ \sqrt{1 - \xi^2} \ \alpha t \right] + \frac{\zeta x_0}{\sqrt{1 - \xi^2}} e^{-\zeta \alpha t} \sin \left[ \sqrt{1 - \xi^2} \ \alpha t \right]. \]
3) Case 2 Critically Damped: \( \zeta^2 = 1 \)

\[
x = x_0 e^{-\alpha t} \cos 0 + \zeta x_0 e^{-\alpha t} \lim_{t \to -\infty} \frac{\sin[\epsilon \omega_0 t]}{\epsilon}.
\]

Using \( \cos(0) = 0 \) and applying a Taylor series for the sine

\[
\sin(az) = az - \frac{a^3 z^3}{3!} + \frac{a^5 z^5}{5!} - \ldots
\]

Then

\[
\frac{\sin(az)}{z} = a - \frac{a^3 z^2}{3!} + \frac{a^5 z^4}{5!} - \ldots
\]

Hence

\[
\lim_{z \to 0} \frac{\sin(az)}{z} = a,
\]

and

\[
\lim_{z \to 0} \frac{\sin(\epsilon \omega_0 t)}{\epsilon} = \omega_0 t.
\]

Therefore

\[
x = x_0 e^{-\alpha t} + x_0 e^{-\alpha t} \omega_0 t
\]

**B:** As a second example, consider the case \( c=0 \) and \( F_0 \neq 0 \)

\[
\ddot{x} + \frac{k}{m} x = \frac{F_0}{m} \sin \omega t \quad \text{or} \quad \ddot{x} + \frac{k}{m} x = \left(\frac{F_0}{k}\right) \left(\frac{k}{m}\right) \sin \omega t.
\]

Let \( \delta_0 = \frac{F_0}{k} \) then \( \ddot{x} + \omega_0^2 x = \delta_0 \omega_0^2 \sin \omega t \). For the homogeneous (or complimentary) solution we can use Case 3 above with \( \zeta = 0 \), or

\[
x_c = C_1 e^{-\zeta \omega_0 t} \cos[\omega_0 t] + C_2 e^{-\zeta \omega_0 t} \sin[\omega_0 t]
\]

For the particular solution assume that \( x_p = A \sin \omega t \). Then \( \ddot{x}_p = -\omega^2 A \sin \omega t \) and

\[
-\omega^2 A \sin \omega t + \omega_0^2 A \sin \omega t = \delta_0 \omega_0^2 \sin \omega t
\]

or

\[
A = \frac{\delta_0 \omega_0^2}{\omega_0^2 - \omega^2}.
\]
The complete solution is

\[ x = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{\delta \omega_0^3}{\omega_0^2 - \omega^2} \sin(\omega t). \]

Recall the initial conditions are \( x(0) = x_0 \) and \( \dot{x}(0) = 0 \) then

\[ x_0 = C_1, \]

and since

\[ \dot{x}(t) = -\omega_0 C_1 \sin(\omega_0 t) + \omega_0 C_2 \cos(\omega_0 t) + \frac{\delta \omega_0^2}{\omega_0^2 - \omega^2} \cos(\omega t), \]

the second initial condition is

\[ 0 = \omega_0 C_2 + \frac{\delta \omega_0^2}{\omega_0^2 - \omega^2} \omega. \]

Then

\[ C_2 = -\frac{\delta \omega_0^2}{\omega_0^2 - \omega^2} \omega. \]

The complete solution is then

\[ x = x_0 \cos(\omega_0 t) - \frac{\delta \omega_0^2}{\omega_0^2 - \omega^2} \sin(\omega_0 t) + \frac{\delta \omega_0^2}{\omega_0^2 - \omega^2} \sin(\omega t). \]

The last term is due to the forcing function \( F_0 \sin ωt \) and is referred to as the steady state while the first two terms are the complimentary part and is referred to as the transient part. The labeling is obvious if the system is under damped (i.e., \( 0 < \zeta < 1 \)). In this case the first two terms would decay away and the last would remain (with a slight modification due to damping).
Homework No. 1

1. Use a Taylor Series to show

\[ e^{ix} = \cos x + i \sin x, \]

where

\[
e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \ldots
\]
\[
\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots
\]
\[
\sin z = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots
\]

Then using \( e^{-ix} = \cos x - i \sin x \) show \( \cos x = \frac{e^{ix} + e^{-ix}}{2} \) and \( \sin x = \frac{e^{ix} - e^{-ix}}{2i} \).

2. Solve

\[ x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = 2 \]

for \( y(x) \) where \( y(0) \) is finite and \( y(1) = 0 \). Note you should guess the particular solution. Recall any solution, but a complimentary solution, is good enough.

3. Solve

\[ \frac{dy}{dx} - 2y = 3x + 1 \]

for \( y = y(x) \) where \( y(0) = 0 \).