Boundary Layer Theory

BY

Dr HERMANN SCHLICHTING

Professor at the Engineering University of Braunschweig
Director of the Aerodynamische Versuchsanstalt Göttingen
Head of the Institute for Aerodynamics of
the Deutsche Forschungsanstalt für Luft- und Raumfahrt, Braunschweig, Germany

Translated by

Dr J. KESTIN

Professor at Brown University in Providence, Rhode Island

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CHAPTER XXIII

Free turbulent flows; jets and wakes

a. General remarks

In the preceding chapters we have considered turbulent flows along solid walls and we propose to continue the study of turbulent streams with the discussion of several examples of so-called free turbulent flow. Turbulent flows will be termed free if they are not confined by solid walls. We shall discern three kinds of turbulent flows, Fig. 21.1: free jet boundaries, free jets, and wakes.

A jet boundary occurs between two streams which move at different speeds in the same general direction. Such a surface of discontinuity in the velocity of flow is unstable and gives rise to a zone of turbulent mixing downstream of the point, where the two streams first meet. The width of this mixing region increases in a downstream direction, Fig. 23.1a.

![Diagram a) jet boundary, b) free jet, c) wake](image)

Fig. 23.1. Examples of free turbulent flows; a) jet boundary, b) free jet, c) wake

A free jet occurs when a fluid is discharged from a nozzle or orifice, Fig. 23.1b. Disregarding very small velocities of flow, it is found that the jet becomes completely turbulent at a short distance from the point of discharge. Owing to turbulence, the emerging jet becomes partly mixed with the surrounding fluid at rest. Particles of fluid from the surroundings are carried away by the jet so that the mass-flow increases in a downstream direction. Concurrently the jet spreads out and its velocity decreases, but the total momentum remains constant. A comprehensive account of the problems encountered in the study of free jets was recently given by S. I. Pai [15].
A wake is formed behind a solid body which is being dragged through fluid at rest, Fig. 23.1 c, or behind a solid body which has been placed in a stream of fluid. The velocities in a wake are smaller than those in the main stream and the losses in the velocity in the wake amount to a loss of momentum which is due to the drag on the body. The spread of the wake increases as the distance from the body is increased and the differences between the velocity in the wake and that outside become smaller.

Qualitatively such flows resemble similar flows in the laminar region (Chaps. IX and X), but there are large quantitative differences which are due to the very much larger turbulent friction. Free turbulent flows are much more amenable to mathematical analysis than turbulent flows along walls because turbulent friction is much larger than laminar friction in the whole region under consideration. Consequently laminar friction may be wholly neglected in problems involving free turbulent flows, which is not the case in flows along solid walls. It will be recalled that in the latter case, by contrast, laminar friction must always be taken into account in the immediate neighbourhood of the wall (i.e. in the laminar sub-layer), and that causes great mathematical difficulties.

Furthermore, it will be noted that problems in free turbulent flow are of a boundary layer nature, meaning that the region of space in which a solution is being sought does not extend far in a transverse direction, as compared with the main direction of flow, and that the transverse gradients are large. Consequently it is permissible to study such problems with the aid of the boundary layer equations. In the two-dimensional case these are

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \]  
(23.1)

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \]  
(23.2)

Here \( \tau \) denotes the turbulent shearing stress and the pressure term has been dropped in the equation of motion because in all problems to be considered it is permissible to assume, at least to a first approximation, that the pressure remains constant. In the case of wakes this assumption is satisfied only from a certain distance from the body onwards.

In order to be in a position to integrate the system of equations (23.1) and (23.2), it is necessary to express the turbulent shearing stress in terms of the parameters of the main flow. At present such an elimination can only be achieved with the aid of semi-empirical assumptions. These have already been discussed in Chap. XIX. In this connexion it is possible to make use of Prandtl's mixing length theory, eqn. (19.6):

\[ \tau = \rho \ell^2 \left| \frac{\partial u}{\partial y} \right| \frac{\partial v}{\partial y} \]  
(23.3)

or of its extension in eqn. (19.8)

\[ \tau = \rho \ell^2 \frac{\partial u}{\partial y} \sqrt{\left( \frac{\partial u}{\partial y} \right)^2 + \ell^4 \left( \frac{\partial^2 u}{\partial y^2} \right)^2} \]  
(23.4)
where the mixing lengths \( l \) and \( l_1 \) are to be regarded as purely local functions. They must be suitably dealt with in each particular case. Further, it is possible to use Prandtl's hypothesis in eqn. (19.10), namely

\[
\tau = \rho \varepsilon \frac{\partial u}{\partial y} = \rho \kappa_1 b (u_{\text{max}} - u_{\text{min}}) \frac{\partial u}{\partial y},
\]

(23.5)

where \( b \) denotes the width of the mixing zone and \( \kappa_1 \) is an empirical constant. Moreover

\[
\varepsilon = \kappa_1 b (u_{\text{max}} - u_{\text{min}})
\]

(23.5a)

is the virtual kinematic viscosity, assumed constant over the whole width and, hence, independent of \( x \). In addition it is possible to use von Kármán's hypothesis, eqn. (19.23) and that due to G. I. Taylor, eqn. (19.15).

When either of the assumptions (23.3), (23.4) or (23.5) is used it is found that the results differ from each other only comparatively little. The best measure of agreement with experimental results is furnished by the assumption in eqn. (23.5) and, in addition, the resulting equations are more convenient to solve. For these reasons we shall express a preference for this hypothesis. Nevertheless, some examples will be studied with the aid of the hypothesis in eqns. (23.3) and (23.4) in order to exhibit the differences in the results when different hypotheses are used. Moreover, the mixing length formula, eqn. (23.3), has rendered such valuable service in the theory of pipe flow that it is useful to test its applicability to the type of flow under consideration. It will be recalled that, among others, the universal logarithmic velocity distribution law has been deduced from it.

b. Estimation of the increase in width and of the decrease in velocity

Before proceeding to integrate eqns. (23.1) and (23.2) for several particular cases we first propose to make estimations of orders of magnitude. In this way we shall be able to form an idea of the type of law which governs the increase in the width of the mixing zone and of the decrease in the 'height' of the velocity profile with increasing distance \( x \). The following account will be based on one first given by L. Prandtl [17].

When dealing with problems of turbulent jets and wakes it is usually assumed that the mixing length \( l \) is proportional to the width of jet, \( b \), because in this way we are led to useful results. Hence we put

\[
\frac{l}{b} = \beta = \text{const.}
\]

(23.6)

† The assumption that the virtual kinematic viscosity is approximately constant in the \( z \)-direction was first discussed by H. Reichardt. In his critical review of the phenomenological theory of turbulent flows, Reichardt [19] provided a comparison of the results which follow from the alternative assumptions \( A(y) = \text{const.} \), \( l(y) = \text{const.} \), and \( A(y) = \text{const.} \). See also Sec. XXIII d.
b. Estimation of the increase in width and of the decrease in velocity

In addition, the following rule has withstood the test of time: The rate of increase of the width, $b$, of the mixing zone with time is proportional to the transverse velocity $v'$:

$$\frac{Db}{Dt} \sim v'. \tag{23.5}$$

Here $D/Dt$ denotes, as usual, the substantive derivative, so that $D/Dt = u \partial/\partial x + v \partial/\partial y$. According to a previous estimate, eqn. (19.4), we have $v' \sim l \partial u/\partial y$, and thus

$$\frac{Db}{Dt} \sim l \frac{\partial u}{\partial y}. \tag{23.5a}$$

Further, the mean value of $\partial u/\partial y$ taken over half the width of the jet may be assumed to be approximately proportional to $u_{max}/b$. Consequently

$$\frac{Db}{Dt} = \text{const} \times \frac{l}{b} u_{max} = \text{const} \times \beta u_{max}. \tag{23.7}$$

Jet boundary: With the use of the preceding relations we shall now estimate the rate at which the width of the mixing zone which accompanies a free jet boundary increases with the distance, $x$. For the jet boundary we have

$$\frac{Db}{Dt} \sim u_{max} \frac{db}{dx}. \tag{23.8}$$

On comparing eqns. (23.8) and (23.7) we obtain

$$\frac{db}{dx} = \text{const} \times \frac{l}{b} = \text{const},$$

or

$$b = \text{const} \times x,$$

which means that the width of the mixing zone associated with a free jet boundary is proportional to the distance from the point where the two jets meet. The constant of integration which must, strictly speaking, appear in the above equation can be made to vanish by a suitable choice of the origin of the co-ordinate system.

Two-dimensional and circular jet: Equation (23.8) remains valid in the case of a two-dimensional and of a circular jet, $u_{max}$ denoting now the velocity at the centre-line. Thus in such cases we also have

$$b = \text{const} \times x. \tag{23.9}$$

The relation between $u_{max}$ and $x$ can be obtained from the momentum equation. Since the pressure remains constant the integral of the $x$-component of momentum taken over the whole cross-sectional area must remain constant and independent of $x$, i.e.

$$J = \int u^2 \, dA = \text{const}.$$
In the case of a two-dimensional jet we have \( J' = \text{const} \times \rho \, u_{\text{max}}^2 \, b \), where \( J' \) denotes momentum per unit length, and hence \( u_{\text{max}} = \text{const} \times b^{-1/2} \sqrt{J'/\rho} \). In view of eqn. (23.9) we have, further,
\[
u_{\text{max}} = \text{const} \times \frac{1}{\sqrt{z}} \sqrt{\frac{J'}{\rho}} \text{ (two-dimensional jet).} \tag{23.10}\]

In the case of a circular jet the momentum is
\[ J = \text{const} \times \rho \, u_{\text{max}}^2 \, b^2 \]
and hence
\[ u_{\text{max}} = \text{const} \times \frac{1}{b} \sqrt{\frac{J}{\rho}}. \]

In view of eqn. (23.9) we now have
\[ u_{\text{max}} = \text{const} \times \frac{1}{z} \sqrt{\frac{J}{\rho}} \text{ (circular jet).} \tag{23.11}\]

**Two-dimensional and circular wake:** Instead of eqn. (23.8) we now have
\[ \frac{\partial V}{\partial t} = U_\infty \frac{\partial b}{\partial z}, \]
and eqn. (23.7) is replaced by
\[ \frac{\partial V}{\partial t} = \text{const} \times \frac{1}{b} \, u_1 = \text{const} \times \beta \, u_1, \]
where \( u_1 = U_\infty - u \). On equating the two expressions, we obtain
\[ U_\infty \frac{\partial b}{\partial z} \sim \frac{1}{b} \, u_1 = \beta \, u_1 \]
or
\[ \frac{\partial b}{\partial z} \sim \beta \frac{u_1}{U_\infty} \text{ (two-dimensional and circular wake).} \tag{23.12}\]

The calculation of momentum in problems involving wakes differs from that for the case of jets, because now there is a direct relationship between momentum and the drag on the body. As already mentioned, eqn. (9.40), the momentum integral is
\[ D = J = \rho \int u \, (U_\infty - u) \, dA, \]
powered that the control surface has been placed so far behind the body that the static pressure has become equal to that in the undisturbed stream. At a large distance behind the body \( u_1 = U_\infty - u \) is small compared with \( U_\infty \) so that we may put \( u \, (U_\infty - u) = (U_\infty - u_1) \, u_1 \approx U_\infty \, u_1 \). Thus for two-dimensional and circular wakes
\[ J = D \approx \rho \, U_\infty \int u_1 \, dA. \tag{23.13} \]
b. Estimation of the increase in width and of the decrease in velocity

Two-dimensional wake: Let \( b \) denote the height of the cylindrical body and \( d \) its diameter; its drag will then be \( D = \frac{1}{2} c_D \rho U_{\infty}^2 \frac{b^2}{d} \) and the momentum, eqn. (23.13), is \( J \sim \rho U_{\infty} u_1 b h \). Equating, according to eqn. (23.13), we have

\[
\frac{u_1}{U_{\infty}} \sim \frac{c_D}{2b} \cdot \quad (23.14)
\]

Inserting eqn. (23.12) for the rate of increase in width, we obtain

\[
2b \frac{db}{dz} \sim \beta c_D d \]

or

\[
b \sim (\beta x c_D d)^{1/2} \quad \text{(two-dimensional wake)}.
\]

Inserting this value into eqn. (23.14) we find that the rate at which the 'depression' in the velocity curve decreases downstream is represented by

\[
\frac{u_1}{U_{\infty}} \sim \left(\frac{c_D}{\beta x}\right)^{1/2} \quad \text{(two-dimensional wake).}
\]

In other words, the width of a two-dimensional wake increases as \( \sqrt{x} \) and the velocity decreases as \( 1/\sqrt{x} \).

Circular wake: Denoting the frontal area of the body by \( A \) we can write its drag as \( D = \frac{1}{2} c_D A \rho U_{\infty}^2 \) and the momentum, eqn. (23.13), becomes \( J \sim \rho U_{\infty} u_1 b^2 \). Equating \( D \) and \( J \), we obtain

\[
\frac{u_1}{U_{\infty}} \sim \frac{c_D A}{b^2} \cdot \quad (23.17)
\]

Inserting this value into eqn. (23.12), we find that the increase in width is given by

\[
2b \frac{db}{dz} \sim \beta c_D A
\]

or

\[
b \sim (\beta c_D A x)^{1/3} \quad \text{(circular wake).}
\]

Inserting eqn. (23.18) into (23.17) we find for the decrease in the depression in the velocity profile the expression

\[
\frac{u_1}{U_{\infty}} \sim \left(\frac{c_D A}{\beta x^2}\right)^{1/3} \quad \text{(circular wake).}
\]

Thus for a circular wake we find that the width of the wake increases in proportion to \( x^{1/3} \) and that the velocity decreases in proportion to \( x^{-2/3} \).

The power-laws for the width and for the velocity in the centre have been summarized in Table 23.1. The corresponding laminar cases, which were partly considered in Chaps. IX and X, have been added for completeness.
Table 23.1. Power-laws for the increase in width and for the decrease in the centre-line velocity in terms of distance $x$ for problems of free turbulent flow

<table>
<thead>
<tr>
<th>Laminar</th>
<th>Turbulent</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>width $b$</td>
</tr>
<tr>
<td>Free jet boundary</td>
<td>$x^{1/3}$</td>
</tr>
<tr>
<td>Two-dimensional jet</td>
<td>$x^{1/3}$</td>
</tr>
<tr>
<td>Circular jet</td>
<td>$x$</td>
</tr>
<tr>
<td>Two-dimensional wake</td>
<td>$x^{1/3}$</td>
</tr>
<tr>
<td>Circular wake</td>
<td>$x^{1/3}$</td>
</tr>
</tbody>
</table>

**c. Examples**

The preceding estimates give in themselves a very good idea of the essential features encountered in problems involving free turbulent flows. We shall, however, now go one step further and shall examine several particular cases in much greater detail deducing the complete velocity distribution function from the equations of motion. In order to achieve this result it is necessary to draw on one of the hypotheses in eqns. (23.3) to (23.5). The examples which have been selected here for consideration all have the common feature that the velocity profiles which occur in them are similar to each other. This means that the velocity profiles at different distances $x$ can be made congruent by a suitable choice of a velocity and a width scale factor.

1. **The smoothing out of a velocity discontinuity.** As our first example we shall consider the problem of the smoothing out of a velocity discontinuity which was first treated by L. Prandtl [17]. At time $t = 0$ there are two streams moving at two different velocities $U_1$ and $U_2$ respectively, their boundary being at $y = 0$ (Fig. 23.2). As already mentioned, the boundary across which the velocity varies discontinuously is unstable and the process of turbulent mixing smoothes out the transition so that it becomes continuous. The width of the zone over which this continuous transition from velocity $U_1$ to velocity $U_2$ takes place increases with increasing time. We are here concerned with a problem in non-steady parallel flow for which

$$u = u(y,t); \quad v = 0.$$  \hspace{1cm} (23.20)

The convective terms in eqn. (23.1) vanish identically. Making use of Prandtl's mixing theory, eqn. (23.3), we can transform eqn. (23.1) to give

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2}.$$  \hspace{1cm} (23.21)
but it is usually introduced artificially anyway. There is no characteristic velocity $U$ and no axial length scale $L$ either, since we are supposedly far from the entrance or exit. The proper scaling of Eq. (3-30) should include $\mu$, $d\hat{p}/dx$, and some characteristic duct width $h$, as suggested in Fig. 3-6. Thus the dimensionless variables are

$$
y^* = \frac{y}{h} \quad z^* = \frac{z}{h} \quad u^* = \frac{\mu u}{h^2 (-d\hat{p}/dx)}$$

(3-31)

where the negative pressure gradient is needed to make $u^*$ a positive quantity. In terms of these variables, Eq. (3-30) becomes

$$\nabla^2 u^* = -1$$

(3-32)

subject to $u^* = 0$ at all points on the boundary of the duct cross section.

### 3-3.1 The Circular Pipe: Hagen–Poiseuille Flow

The circular pipe is perhaps our most celebrated viscous flow, first studied by Hagen (1839) and Poiseuille (1840). The single variable is $r^* = r/r_0$, where $r_0$ is the pipe radius. The Laplacian operator reduces to

$$\nabla^2 = \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right)$$

and the solution of Eq. (3-32) is

$$u^* = -\frac{1}{4} r^{*2} + C_1 \ln r^* + C_2$$

(3-33)

Since the velocity cannot be infinite at the centerline, on physical grounds, we reject the logarithm term and set $C_1 = 0$. The no-slip condition is satisfied by setting $C_2 = +\frac{1}{4}$. The pipe-flow solution is thus

$$u = \frac{-d\hat{p}/dx}{4\mu} (r_0^2 - r^2)$$

(3-34)

Thus the velocity distribution in fully developed laminar pipe flow is a paraboloid of revolution about the centerline (the Poiseuille paraboloid). The total volume rate of flow $Q$ is of interest, as defined for any duct by

$$Q = \int_{\text{section}} u \, dA$$

where the element of area is $2\pi r \, dr$ for this pipe case. The integration is simple and yields

$$Q_{\text{pipe}} = \frac{\pi r_0^4}{8\mu} \left( -\frac{d\hat{p}}{dx} \right)$$

(3-35)
The mean velocity is defined by \( \bar{u} = \frac{Q}{A} \) and gives, in this case

\[
\bar{u} = \frac{r_0^2 (-d\rho/dx)}{8\mu} = \frac{1}{2} u_{\text{max}}
\]  

(3-36)

Finally, the wall shear stress is constant and is given by

\[
\tau_w = \mu \left( -\frac{du}{dr} \right)_w = \frac{1}{2} r_0 \left( -\frac{d\rho}{dx} \right) = \frac{4\mu\bar{u}}{r_0}
\]  

(3-37)

Even though \( \tau_w \) is proportional to mean velocity (laminar flow), it is customary, anticipating turbulent flow, to nondimensionalize wall shear with the pipe *dynamic pressure*, \( \rho u^2/2 \), by analogy with Eq. (3-8). Two different friction factor definitions are in common use in the literature:

\[
\lambda = \frac{8\tau_w}{\rho u^2} = \text{Darcy friction factor}
\]  

(3-38)

\[
C_f = \frac{2\tau_w}{\rho u^2} = \frac{1}{4} \lambda = \text{Fanning friction factor, or skin-friction coefficient}
\]

By substituting into Eqs. (3-38), we obtain the classic relations

\[
\lambda = \frac{64}{Re_D}
\]

\[
C_f = \frac{16}{Re_D}
\]  

(3-39)

![Friction factor vs. Reynolds number](image)

**Figure 3-7.**

Comparison of theory and experiment for the friction factor of air flowing in small-bore tubes. [After Senecal and Rodilhas (1953).]