Automatic Control

Root Locus

Dr. Aly Mousaad Aly
Effect of pole position on the transient response

\[ T.F. = \frac{X(s)}{U(s)} = \frac{K}{(s+\lambda_1)(s+\lambda_2)(s+\lambda_3)((s+\lambda_4)^2 + \beta^2)} \]

under unit impulse of input

\[ X(t) = a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t} + a_3 e^{-\lambda_3 t} + c e^{-\lambda_4 t} \sin(\beta t - \phi) \]
General case for an underdamped second order system

ch. eqn., \[ s^2 + 2\zeta \omega_n s + \omega_n^2 = 0 \quad \Rightarrow \quad s_1, s_2 = -\zeta \omega_n \pm j\omega_n \sqrt{1-\zeta^2} \]

\[ OP = \sqrt{(-\zeta \omega_n)^2 + \left(\omega_n \sqrt{1-\zeta^2}\right)^2} \quad \Rightarrow \quad OP = \omega_n \]

\[ \cos \beta = \frac{|-\zeta \omega_n|}{\omega_n} = \zeta \]
Notes

• The basic characteristic of the transient response of a closed-loop system is closely related to the location of the closed-loop poles.

• If the system has a variable loop gain, then the location of the closed-loop poles depends on the value of the loop gain chosen.

• It is important, therefore, that the designer know how the closed-loop poles move in the s-plane as the loop gain is varied.
Root Locus

• A simple method for finding the roots of the characteristic equation has been developed by W. R. Evans and used extensively in control engineering. This method, called the root-locus method, is one in which the roots of the characteristic equation are plotted for all values of a system parameter.
By using the root-locus method the designer can predict the effects on the location of the closed-loop poles by varying the gain value (K).
Example

open loop T.F., \[ G(s)H(s) = \frac{K}{s(s+1)} \]

Ch. eqn., \[ 1 + G(s)H(s) = 0 \]

\[ 1 + \frac{K}{s(s+1)} = 0 \]

\[ s = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4K} \]

<table>
<thead>
<tr>
<th>K</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.25</td>
<td>-0.5</td>
</tr>
<tr>
<td>&gt;0.25</td>
<td>-0.5</td>
</tr>
</tbody>
</table>
Sketching root locus, example 1

open loop T.F., \[ G(s)H(s) = \frac{K}{s(s+4)(s+16)} \]

1. Mark open-loop poles and zeros.
   Number of poles, \( n = 3 \); \( s= 0, -4, -16 \)
   Number of zeros, \( m = 0 \)

2. Find asymptotes
   \( n - m = \) number of asymptotes
   The asymptotes intersect the real axis at \( \sigma_A \)
   and depart at angle \( \phi_A \) given by
\[ \sigma_A = \frac{\sum P - \sum Z}{n-m} = \frac{(0-4-16)-(0)}{3-0} = -6.6 \]

\[ \phi_l = \frac{180^\circ + (l-1)360^\circ}{n-m}, \quad l = 1, 2, \ldots n-m \]

\[ \phi_1 = 60^\circ; \quad \phi_2 = 180^\circ; \quad \phi_3 = 300^\circ \]

where:
\( \sum P \) is the sum of all the locations of the poles,
\( \sum Z \) is the sum of all the locations of the explicit zeros.
3. Compute breakaway/break-in points

The breakaway points are located at the roots of the following equation:

\[
\text{ch. eqn. } 1 + G(s)H(s) = 0; \quad 1 + \frac{K}{s(s + 4)(s + 16)} = 0
\]

\[
K = -s^3 - 20s^2 - 64s
\]

\[
\frac{dK}{ds} = 0; \quad -3s^2 - 40s - 64 = 0 \quad \Rightarrow \quad 3s^2 + 40s + 64 = 0
\]

\[
\Rightarrow \quad s = -1.86, -11.47
\]

*take only the value that exist on the root locus; i.e. \( s = -1.86 \)
Notes:

- The break-away (break-in) points are obtained by solving a polynomial equation.
- Once you solve for $s$, the real roots give you the breakaway/reentry points. Complex roots correspond to a lack of breakaway/reentry.

4. Intersection of the root locus with the imaginary axis

\[
ch. eqn. \quad 1 + G(s)H(s) = 0; \\
\quad s^3 + 20s^2 + 64s + K = 0
\]
**Routh method**

\[ s^3 + 20s^2 + 64s + K = 0 \]

<table>
<thead>
<tr>
<th>( s^3 )</th>
<th>1</th>
<th>64</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^2 )</td>
<td>20</td>
<td>K</td>
<td></td>
</tr>
<tr>
<td>( s^1 )</td>
<td>( \frac{20 \times 64 - K}{20} )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( s^0 )</td>
<td>K</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\frac{20 \times 64 - K}{20} = 0 \quad \Rightarrow \quad K = 1280
\]

auxiliary equation,

\[
20s^2 + 1280 = 0 \quad \Rightarrow \quad s = \pm j8
\]
HW (submit to aly.mousaad@pua.edu.eg )

1. Solve using MATLAB one item of problems 1 and 2 of Sheet 1. (according to your ID)
2. Solve using MATLAB problem 3 of Sheet 1.
3. Solve using MATLAB one of problems 4, 5 of Sheet 2. (assume $H_1 = 1$; $H_2 = 1/s$; $G_1 = 1/(2s+1)$; $G_2 = (s+3)/(s^2+5)$; $G_3 = s/(s^2+3s)$; $H_3 = H_1$; $G_4 = G_2$. (according to your ID)
4. Solve using MATLAB problems 7 of Sheet 5.
5. Solve using MATLAB one of problems 3 and 4 of Sheet 7. (according to your ID)
6. Solve problems, 2,4,5,6 of Sheet 8.