Lecture 24: NP Completeness

First, I showed the traveling salesman problem (TSP) is NP complete. Given a complete weighted graph (cost function \( c \)), we want a tour of all cities (each city visited exactly once and return to the starting city) with cost at most \( k \). Now now give the NP-complete proof. First TSP is in NP: check a proposed tour can be easily done in polynomial-time (make sure you understand what it means). We now show HAM-CYCLE \( \leq_p \) TSP. Given a graph \( G \) (may not be complete), we form a complete weighted graph \( G'(V, E') \): \( E' \) contain all \((i, j)\), where we assign \( c(i, j) = 0 \) if \((i, j)\) in \( G \) an 1 otherwise. Claim: \( G \) has a Hamiltonian cycle iff \( G' \) has TSP tour of cost 0. First, if \( G \) has a Hamiltonian cycle, then all the edge cost is 0, and so the tour costs zero. On the other hand, if \( G' \) has TSP cost of 0, then each edge will have cost 0 (and thus the edge is in \( G \)). So this tour is Hamiltonian cycle. Therefore, we simply ask the black box of TSP one question: does the input \( G' \) has TSP tour of cost 0 or less? And we can then determine the solution for the HAM-CYCLE problem.

We now move to numerical problems (recall Knapsack problem). We start with the SUBSET-SUM problem. Given finite set \( S \) of positive integers, and an integer \( t \), we ask whether there exists a subset of integers of \( S \) that sum to \( t \). Clearly, the SUBSET-SUM is in NP. We now show VERTEX-COVER is polynomial-time reducible to SUBSET-SUM.

In VC, we have a graph \( G \) and an integer \( k \), and the question is: does \( G \) has a vertex cover of size \( k \) or smaller? To use a black box of SUBSET-SUM, we will construct integers \( S \), and \( t \) from \( G \) s.t. \( G \) has a VC of size \( k \) or smaller iff subset of \( S \) add up to \( t \). Let nodes in graph labeled by 1, 2, 3, \ldots, \( n \). We will create one integer \( a_i \) per node \( v_i \) and one integer \( b_{ij} \) for edge \((v_i, v_j)\). Then selected \( a_i \) correspond to a vertex cover (denoted \( C \)), and selected \( b_{ij} \) are those with precisely one node in the vertex cover. We define integers in a matrix, where a row corresponds to an integer. Integers are treated as base-4. We add a special column as the first column (the most significant bit), which is 1 if the row is for \( a_i \), and 0 for \( b_{ij} \). Then we have one column for each edge \((u, v)\). For integer \( a_i \), it has value 1 if node \( v_i \) is one of the two nodes for the edge corresponding to the column. \( b_{ij} \) has precisely one 1 at the column corresponding to the edge \((v_i, v_j)\).

Finally, \( t = k*4^{|E|} + 2 \sum_{i=0}^{|E|-1} 4^i \).

To see why it works, we show two things. First, given a vertex cover \( C \), we select the following integers. We pick all corresponding \( a_i \) for \( v_i \in C \), and \( b_{ij} \) if exactly one of \( v_i, v_j \) is in \( C \). The selected \( k \) integers (corresponding to \( a_i \)) have MSB of 1, which add to \( k4^{|E|} \). For the rest of edges, we have exactly two 1 selected for that column. Check this out if you do not understand it yet. On the other hand, suppose we have a subset of integers summing to \( t \). We note that there is carry in all columns (except the MSB). It follows easily that the corresponding nodes (for the selected \( a_i \) integers) must form a vertex cover. Check it if you do not yet understand.

Now recall the KNAPSACK problem. Can we have a polynomial time algorithm? Note \( O(nW) \) (from the dynamic programming) is not a polynomial-time algorithm since \( W \) can be very large. Suppose KNAPSACK has a black box polynomial-time algorithm. Now we can solve SUBSET-SUM using this KNAPSACK black box. How? Given a list of \( n \) integers (denoted as \( x_i \)) from the SUBSET-SUM, we create items from integers. We have: \((v_i, w_i) = (x_i, x_i)\). Note we want to find out whether a subset of \( x_i \) sum to \( t \). That is, value/weight are the same as the given integer \( x_i \) for the item corresponding to \( x_i \). Now it is easy to see the SUBSET-SUM problem has a solution iff KNAPSACK problem has a solution for the constructed items where the total selected weights is no more than \( t \) and the value is at least \( t \). Check to understand why this is the case.

I have also shown that a formulation of the Minesweeper is NP complete. Please refer to the posted slides for more information.

Lecture 23: Algorithms for NP completeness problems

Often we must work on NP-complete problems, even when polynomial-time algorithms are unlikely obtainable. We can take several approaches.
• Faster (although not polynomial time) algorithm than brute force algorithm.

• Efficient algorithm for special case of the problem.

• Approximation algorithm: we may not find optimal solutions, but we can find a solution not very far from it.

We now revisit the traveling salesman problem, where we will give an exponential time algorithm. Recall the problem is given a complete weighted graph and needs a tour of all nodes exactly once with min-cost. The naive algorithm would take $O(n!)$ time. We are going to use dynamic programming on this problem. First let us start tour at node 1 (and will return to node 1). Here is the DP subproblem: $OPT[S, i]$ for all subset $S$ of 2, 3, ..., $n$ and some $i \in S$, which is equal to the minimum total cost of the part of simple path from city 1 and visit nodes in $S$ in some order and end at city $i$. Here, $S$ is non-empty. First, $OPT[\{i\}, i] = dist(1, i)$ (where $dist(i, j)$ is the distance between node $i$ and node $j$). Then, $OPT[S, i] = \min_{j \in S - \{i\}} OPT[S - \{i\}, j] + dist(i, j)$. Suppose all $OPT[S, i]$ is computed, the final solution is: $\min_i OPT[\{2, 3, \ldots, n\}, i] + dist(1, i)$. How to compute $OPT[S, i]$? In the increasing size of $S$: start with $S$ of size 1, then size 2, and so on. This works since in the recurrence $OPT[S, i]$ only depends on subsets of smaller cardinality. Time: $O(n \cdot 2^n)$ is the DP table size, and each table cell takes $O(n)$ time. So total time is $O(n^2 2^n)$. Although this is still an exponential time algorithm, it is much faster than the naive algorithm.

Now, some NP complete problems become easier when we restrict the problem instances on special cases. As an example, we consider the problem of finding independent set on a tree. We first note that if there is an isolated node $u$ (i.e. has no edge incident on $u$), we can add $u$ to the independent set. A key property of a tree is that there exists a leaf $u$ (only connects to a node $v$). For INDEPENDENT-SET problem, we prefer $u$. To see this, we consider an independent set $C$. If neither $u$ nor $v$ is in $C$, we can simply add $u$ and this will lead to a (larger) independent set (why the enlarged set remains an independent set?). On the other hand, if $v \in C$, we can have another independent set $C’ = (C - \{v\}) \cup \{u\}$ which remains an independent set (why?). Thus, we know for a given tree $T(V, E)$ and $u$ is a leaf, then there exists an IS includes $u$. This gives a greedy algorithm: each time, we look for a leaf $u$ (which must exist when there is no cycle), include $u$ in the independent set, and remove $u$ and $v$ (where there is an edge $(u, v)$). We repeat until the graph is empty. Note: what we got becomes a forest (not a single tree but several disjoint trees). But what we proved still work: there still exists a leaf in a set of trees.

Now a quick tour of approximation algorithm which can help when exact optimal solution (say min) is hard to find. and we can find a solution that is not very far from optimal. Approximation ratio is: for a solution $C$, and optimal solution $C^*$, if we show $C/C^* \leq p$, we say approximation ratio is $p$. Our example is the VERTEX-COVER problem, where we give a 2-approximation algorithm. We omit the details here. See the textbook (chapter 35) for details on correctness proof.