Lecture 23: NP completeness

Our next example is the independent set problem. Given a graph, find the largest subset of nodes of $V$ s.t. no two nodes in the set are connected by an edge of $G$. Again, we can convert this to a search problem, and it is easy to see the search problem is in NP. Now we look at the vertex cover problem. Given graph $G$, does there exist $k$ nodes in $G$ s.t. each edge in $G$ is touched (covered) by one of the $k$ nodes. This again, is in NP.

Now is one of the most important concepts in this chapter: polynomial-time reduction. Suppose we have a black box efficient code that can solve problem $X$, and we can use it to solve problem $Y$. Which is harder: $X$ or $Y$? Since $X$ lets solve $Y$, so $X$ is at least as hard as $Y$. More precisely: can we solve any instance of $Y$ (say a graph) using poly-time to prepare input and read output for black box for $X$ and make polynomial number of calls to the black box of $X$? If yes, we say $Y$ is poly-time reducible to $X$, denoted $Y \leq_p X$. Consequence: if $X$ does have a poly-time algorithm, then $Y$ also has a polynomial time algorithm. Conversely, if $Y \leq_p X$, and $Y$ can not be solved in poly-time, then $X$ can not be solved in poly-time either.

We now show our first reduction for two problems in NP: independent set ($IS$) problem and vertex cover ($VC$) problem. Note we will use $IS$ to denote both the IS problem and some independent set for a graph. Same usage of $VC$. We will show $IS \leq_p VC$ and $VC \leq_p IS$. Here is a useful observation. Given a graph $G = (V,E)$, a subset $S$ is an IS of $G$ iff $V - S$ is a VC. This can be easily verified by the definition of IS and VC. Check this out if you do not understand it. Thus, given a black box code for the VC problem, we will find solution for the following question on the IS problem: given a graph $G = (V,E)$, does $G$ have an IS with $k$ nodes or more? Based on our previous observation, we note that this question can be easily answered by passing this question to the VC black box: does $G$ (the same graph) contain a VC with $n - k$ nodes or smaller? Here $n = |V|$. Clearly the reduction is done in polynomial-time: we actually just make one call of the black box and the cost of preparing the input is none. I omit the other reduction, which is similar.

NP-complete: the hardest problems in NP. A problem is NP-complete if (1) it is in NP and (2) every problem in NP is polynomial-time reducible to it. In class, we went over the implication of NP completeness (1) if problem $X$ is NP complete, and $X$ has a polynomial time algorithm, then every (yes every) problem in NP has polynomial-time algorithm. (2) if any problem in NP has no polynomial time algorithm, the each NP complete problem has no polynomial algorithm.

Does NP complete problem exist? The 1971 Cook-Levin theorem says yes: circuit satisfiability problem ($CIRUIT$-$SAT$) is NP complete. Our textbook has a nice introduction to $CIRUIT$-$SAT$. Read it if you still do not understand it.

Lecture 24: NP completeness proof

From now on, we have one starting problem: $CIRUIT$-$SAT$. Now how to prove problem $X$ is NP complete? The recipe for NP-completeness: (1) show $X$ is in NP (2) Find a known NP complete problem $Y$, (3) describe how to convert instances for $Y$ into an instance for $X$ (4) Prove the instance for $Y$ can be efficient solved iff $X$ can be solved. (5) Ensure the conversion can be done in poly-time.

We then went over the proof of $3$-CNF-$SAT$ (or simply $3$SAT) is NP complete (where each clause has exactly 3 literals). Read the textbook for more details. But here is the important points. We need to first show 3SAT is in NP. Then, we will reduce from $CIRUIT$-$SAT$. The basic idea is to construct equivalent formula for each of three types of logical gates in the circuits. Read the textbook for more details. Finally, some of reductions will lead to clauses with only two literals. But in class, I showed how to convert such clauses into equivalent clauses with 3 literals each. Again, read the book if you still miss something.

We now move on to graph-theoretic problems. Recall to show problem $X$ is NPC: (1) $X$ is in NP, (2) find a NPC problem that is polynomial time reducible to $X$. Our first example is CLIQUE problem. A clique is a fully connected subgraph. The CLIQUE problem (the decision version) is: does given graph $G$ have a clique of size $k$ or more? Clearly, CLIQUE is in NP: its certificate is a set of nodes and we can easily check whether this set of nodes forms a clique or not. We now reduce the 3SAT problem to CLIQUE. I will not repeat the proof which is well documented in the textbook. Refer to the textbook for more details.

Our second example is VERTEX-COVER (VC). A vertex cover is a subset of nodes of a graph $G$ such that all edges in $G$ have at least one node in the subset. The VC problem is: does $G$ have a VC of size $k$ or fewer?
It is again easy to see VC is in NP. We can directly reduce CLIQUE to VC, as done in the textbook. But it is slightly easier to first show a related problem, INDEPENDENT-SET (IS), is NP complete. Since we have seen before VC and IS can be reduced to each other, we will then know VC is also NP complete. The basic idea is to complement G: when there is an edge between two nodes, remove it; otherwise add an edge. We call the resulting graph \( G' \). Claim: \( G \) has a clique of size \( k \) iff \( G' \) has IS of size \( k \). First suppose \( G \) has clique \( S \) of size \( k \). Then \( S \) is an IS for \( G' \). This is because no edges in \( S \) are between two nodes of \( G' \) (since \( S \) is a clique in \( G \)). The other direction is also easy to see. With this observation, we can easily decide the answer for the CLIQUE by asking the black box of IS. Since we have established before IndependentSet \( \leq_{p} \) VertexCover, we know VERTEX-COVER is also NP complete (remember you should verify VERTEX-COVER is in NP, which we have done before).

Now HAM-CYCLE. This reduction is more complex. In class, I briefly explained the most important part of the reduction. Read the textbook for more details.