Off-Line and Real-Time Methods for ML-PDA

Target Validation

Wayne R. Blanding*, Member, IEEE, Peter K. Willett, Fellow, IEEE and Yaakov Bar-Shalom, Fellow, IEEE

Abstract

We present two procedures for validating targets whose track estimates are obtained using the Maximum Likelihood Probabilistic Data Association (ML-PDA) algorithm. The ML-PDA, developed for Very Low Observable (VLO) target tracking, always provides a track estimate that must then be tested for target existence by comparing the value of the Log Likelihood Ratio (LLR) at the track estimate to a threshold. Using extreme value theory, we show that in the absence of a target the LLR at the track estimate obeys approximately a Gumbel distribution rather than the Gaussian distribution previously ascribed to it in the literature. The off-line target validation procedure relies on extensive off-line simulations to obtain a set of target validation thresholds that are then used by the tracking system. The real-time procedure uses the data set that produced the track estimate to also determine the target validation threshold. The performance of these two procedures is investigated through simulation of two active sonar tracking scenarios by comparing the false true target/track acceptance probabilities. These techniques have potential for use in a broader class of maximum likelihood estimation problems with similar structure.

Keywords: Target tracking, track validation, maximum likelihood, extreme value theory, probabilistic data association.

EDICS: SSP-TRAC

Portions of an earlier version of this paper were presented at the IEEE Aerospace Conference, March 2006. This research is supported by the Office of Naval Research under the University/Laboratory Initiative program.

Authors’ address: University of Connecticut, Department of Electrical and Computer Engineering, 371 Fairfield Rd. U-2157, Storrs, CT 06269-2157; Phone: 860-486-2195; E-mail: wayne.blanding@uconn.edu, willett@engr.uconn.edu, ybs@engr.uconn.edu.

June 21, 2006 DRAFT
I. INTRODUCTION

Target tracking in a high clutter or low signal-to-noise (SNR) environment, i.e., for tracking Very Low Observable (VLO) targets, presents many difficulties for typical single or multi hypothesis tracking systems. By making frame-by-frame track formation and track estimate decisions based on associating specific measurements to tracks, the number of possible tracks quickly becomes overwhelming and an operator is presented either with a multitude of false tracks mingled with the true tracks or a picture showing only high SNR targets—neither of which is desirable. As a result the class of Track-Before-Detect (TBD) algorithms have been developed and have been shown to be effective in tracking VLO targets.

TBD algorithms share similar characteristics. In general they use either unthresholded measurement data, or thresholded measurement data with significantly lower detection thresholds (resulting in many more measurements per frame of data) than commonly used in non-TBD systems. TBD algorithms typically operate on multiple frames of data simultaneously in order to average out the effects of noise in the measurement data and to accentuate the effect of target related measurements although there are Bayesian single-frame recursive TBD techniques as well. A key feature of these algorithms (giving this class of algorithms its name) is that they simultaneously perform target detection/validation and track estimation functions. The present paper deals with the target validation problem for the Maximum-Likelihood Probabilistic Data Association (ML-PDA) algorithm, one of the class of TBD algorithms.

Three general classes of TBD algorithms can be found in the literature. Particle Filtering TBD techniques (PF-TBD) [17], [18] use unthresholded data using a recursive technique based on particle filters. Most recently, target validation has been addressed through the appending of a “target existence” state to the kinematic state vector used to find the track estimate [18]. Dynamic Programming TBD techniques (DP-TBD) [5], [12], [21] also use unthresholded data but in a multi-frame batch algorithm. In order to reduce the computational complexity of the algorithm the track state space is discretized and the track estimate is obtained using dynamic programming techniques. Target validation is performed using hypothesis testing based on the value of a cost function evaluated at the track estimate. The statistical distributions used in the hypothesis test have recently been based on Extreme Value Theory (EVT) techniques [12]. Maximum Likelihood TBD techniques (ML-TBD) [11], [13], [19], [20] have used either thresholded or unthresholded data in which the track estimate is based on maximizing a log likelihood ratio (LLR). Both multi-frame and single-frame recursive techniques exist. Target validation is accomplished using hypothesis testing based on the value of the LLR at the track estimate.
The ML-PDA algorithm, one of the early TBD algorithms developed [11], is one of the class of ML-TBD trackers. In the ML-PDA algorithm, a log likelihood ratio (LLR) is formed that uses thresholded measurement data from a set of sensor scans. The track estimate is given by the location of the global LLR maximum in the target state space. A sliding window is used such that when a new frame of data arrives, the oldest frame of data is dropped and the ML-PDA algorithm is applied to this new set of windowed data. Measurement data includes kinematic information (e.g., positions and velocities) as well as an amplitude “feature”. The inclusion of amplitude was shown to significantly improve the performance of this estimator [13]. A limitation to the use of ML-PDA is that it is formulated as a single-target tracking algorithm and has not been extended to multi-target applications. Consequently the scope of this paper is limited target validation procedures for the single-target case.

To validate a target the LLR global maximum must be tested to determine if it is from a target or the result of noise measurements (clutter). Previous researchers have used hypothesis testing for this target validation in which the value of the LLR global maximum is compared to a threshold. This threshold is determined based on the probability density function (pdf) of the LLR global maximum for the “target present” hypothesis [11], [13]. This hypothesis test is generally considered suboptimal from a Neyman-Pearson lemma sense in that the most powerful test is based on the pdf of the LLR global maximum for the “no target” hypothesis and represents the basis for the hypothesis testing in this paper. First, we will show that the Gaussian distribution for the pdf of the LLR global maximum for the “no target” hypothesis derived by the previous researchers [11], [13] is incorrect and that a better approximation for this distribution (a Gumbel distribution) based on EVT is more appropriate. This represents the first major contribution of this paper.

Next, we present two procedures for obtaining a target validation threshold based on a maximum allowable probability of accepting a false target/track ($P_{FT}$) and this represents the second major contribution of this paper. The first procedure relies on off-line simulations of the tracking problem to obtain the Gumbel distribution parameters necessary to set the target validation threshold. While it will be shown that this procedure yields good results, it may not be practical except in limited circumstances since the Gumbel distribution parameters vary as a function of the specific tracking scenario (e.g., measurement false alarm probability ($P_{fa}$), assumed target SNR, and the number of frames used in the sliding window ($N_W$)).

The second procedure provides a real-time method for estimating the Gumbel distribution parameters. Real-time in this context means that this procedure can be performed within the time interval between successive frames of data. A key feature of this procedure is that it uses the same measurement data
set that was used to produce the track estimate to also set the target validation threshold. While this procedure yields less accurate results than the off-line procedure, it does not suffer from the off-line procedure’s disadvantages.

The performance of these target validation procedures is evaluated through Monte Carlo simulations of two active sonar tracking scenarios. The scenarios differ in the shape and dimensionality of the measurement space. Within each scenario, four cases are simulated to illustrate the performance of these procedures as $P_{fa}$, SNR, and $N_W$ are varied.

While this paper is focused on solving the ML-PDA target validation problem, the techniques presented here can be applied to any maximum likelihood estimation problem where the data contains noise and the likelihood surface contains a similar “noisy” structure and one must either validate or reject the estimate. Examples include image processing algorithms focused on detecting specific structures or features, text search algorithms and pattern classification algorithms.

The remainder of this paper is organized as follows. Section II presents a formulation of the ML-PDA algorithm and analyzes the LLR surface. Section III presents the theoretical basis for the assumed form of the pdf of the LLR global maximum under the “no target” ($H_0$) and “target present” ($H_1$) hypotheses. Section IV presents the two procedures for obtaining the pdf of the LLR global maximum under $H_0$. Simulation results are presented in Section V followed by conclusions in Section VI.

II. ML-PDA PROBLEM FORMULATION

A. ML-PDA Algorithm

A detailed derivation of the ML-PDA algorithm incorporating amplitude information in a 2D measurement space can be found in [13]. A summary of the key equations and results is presented here, applied to an active sonar problem with a 3D measurement space.\textsuperscript{1} The ML-PDA algorithm uses the following assumptions in constructing the likelihood function:

1) A single target is present in each data frame with a given detection probability ($P_d$), and detections are independent across frames.
2) At most one measurement per frame corresponds to the target.
3) The target moves according to deterministic kinematics (i.e., no process noise).
4) False detections are distributed uniformly in the search volume ($U$).

\textsuperscript{1}The dimensionality of the measurement space is based on the kinematic measurement dimensionality with amplitude considered to be a separate “feature”.

June 21, 2006 DRAFT
5) The number of false detections is Poisson distributed according to probability mass function $\mu_f(m)$, with parameter $\lambda$ (spatial density), a function of the detector false alarm probability ($P_{fa}$) in a resolution cell.

6) The amplitudes of target originated and false detections are Rayleigh distributed according to pdf $p_1(a)$ and $p_0(a)$ respectively. The target SNR, which affects $p_1(a)$, is either known or accurately estimated in real time.

7) Target originated measurements are corrupted with additive white zero-mean Gaussian noise.

8) Measurements obtained at different times are, conditioned on the target state, independent.

The measurement data set used by the ML-PDA algorithm is given by

$$\begin{align*}
(Z, a) &= \{(Z_{ij}, a_{ij})\} = \{(\beta_{ij}, r_{ij}, \dot{r}_{ij}, a_{ij})\} \\
i &= 1, 2, \ldots, N_W \quad \text{frame number} \\
j &= 1, 2, \ldots, m_i \quad \text{measurement number}
\end{align*}$$

where measurements consist of bearing ($\beta$), range ($r$), range rate ($\dot{r}$) and amplitude ($a$). The measurement space contains $N_c$ sensor resolution cells. A window of $N_W$ frames is used to compute track estimates.

The target parameter to be estimated is the target kinematic state (in Cartesian coordinates) at the last frame of the windowed data set assuming constant velocity motion,

$$x = \begin{bmatrix} x & y & v_x & v_y \end{bmatrix}$$

where $x$ ($v_x$), $y$ ($v_y$) are the position (velocity) components of the state respectively.

The maximum likelihood approach finds the target parameter that maximizes the likelihood function of the parameter $x$, i.e., $p(Z, a|x)$. When incorporating the amplitude (received signal strength associated with a measurement) into the likelihood function, it is convenient to define the amplitude likelihood ratio as

$$\rho_{ij} = \frac{p_1(a_{ij}|a_{ij} > \tau)}{p_0(a_{ij}|a_{ij} > \tau)}$$

where $\tau$ is the detector threshold (in each resolution cell).

The pdf $p(Z_{ij})$ given that it is target originated is a multivariate Gaussian $N(Z_{ij}; M_i, \Sigma)$ where

$$M_i = [\beta_i(x) \quad r_i(x) \quad \dot{r}_i(x)]^T$$

$$\Sigma = \text{diag} \begin{bmatrix} \sigma^2_{\beta} & \sigma^2_{r} & \sigma^2_{\dot{r}} \end{bmatrix}$$
From these assumptions and definitions, the likelihood function of \(x\) is [13]

\[
p(Z, a|x) = \prod_{i=1}^{N_W} \left[ 1 - \frac{P_d}{U m_i} \mu_f(m_i) \prod_{j=1}^{m_i} p_0(a_{ij}|a_{ij} > \tau) + \frac{P_d \mu_f(m_i - 1)}{U m_i - 1} \prod_{j=1}^{m_i} p_0(a_{ij}|a_{ij} > \tau) \sum_{j=1}^{m_i} p(Z_{ij}) \rho_{ij} \right]
\]

(6)

The above equation represents the weighted sum of all the likelihoods of associating a specific measurement (or no measurement) to the target with all other measurements considered as false detections. This is obtained using the total probability theorem and is the essence of the PDA approach [2].

Dividing (6) by the likelihood function given that all measurements are false detections, namely

\[
\prod_{i=1}^{N_W} \left[ \frac{1}{U m_i} \mu_f(m_i) \prod_{j=1}^{m_i} p_0(a_{ij}|a_{ij} > \tau) \right]
\]

(7)

and by taking the logarithm of the resulting function, a more compact form (the LLR) is obtained as

\[
\Lambda'(Z, a|x) = \sum_{i=1}^{N_W} \ln \left(1 - \frac{P_d}{\mu_f(m_i)} \prod_{j=1}^{m_i} p_0(a_{ij}|a_{ij} > \tau) \right)
\]

(8)

Finally, the value \(N_W \ln(1 - P_d)\) is subtracted from both sides, which normalizes the LLR so that it has a minimum value of zero regardless of the value of \(N_W\) and \(P_d\). This will aid in analyzing threshold values as these problem-specific values change. The final version of the LLR used in this paper is therefore,

\[
\Lambda(Z, a|x) = \sum_{i=1}^{N_W} \ln \left[1 + \frac{P_d}{\lambda(1 - P_d)} \sum_{j=1}^{m_i} \rho_{ij} p(Z_{ij}) \right]
\]

(9)

The LLR global maximum defines the parameter estimate, \(\hat{x}\), for a single iteration of the ML-PDA tracking algorithm. A separate test must be performed to determine if the track estimate is the result of noise or target originated measurements (a test for target existence).

### B. LLR Characterization

The LLR surface is non-convex with a large number of local maxima distributed throughout the four dimensional parameter space. Additionally, there are extended flat regions where no measurements influence the LLR, creating a “floor” of zero on the LLR surface. Fig. 1 shows a representative region of the LLR surface at the velocity maximizing the central peak in this figure over a 500 m by 500 m positional area. The measurement data used to construct this figure is a representative data set from the 3D active sonar scenario to be presented in detail later. The area displayed in this figure represents only 10% of the positional search area for this scenario at a single velocity. A more detailed analysis of the LLR surface for realistic clutter densities reveals that about 7/10 of the parameter space for this problem contains LLR values near the “floor” value (less than 0.1% of the height of the LLR global maximum).
A second feature of the LLR surface occurs when there is a one-to-many mapping of points in the measurement space to values in the parameter space. Any single measurement in any frame will generate a “ridge” in the LLR surface. Taking the 3D measurement space described in (1) as an example, for a given parameter matching the measured bearing, range, and range rate (radial velocity), a local LLR maximum will exist for all values of tangential velocity. Fig. 2 illustrates this by plotting the LLR height contours as a function of velocity for a given position. This illustrates the long ridges that characterize the LLR for this problem.

C. The Number of LLR Maxima

The average number of local maxima for a given tracking problem, required for the real-time target validation procedure, can be estimated as follows. For purposes of illustration, the measurement and parameter spaces described in (1) and (2) will be used. Further consider the parameter space to be constrained in position to the search area described by the measurement space and in velocity by a maximum target speed.

Any single measurement in any frame will generate a local maximum (a “ridge”) on the LLR surface. The expected number of such single-measurement maxima is given by the expected number of detections in the data set,

\[ N_1 = N_W N_c P_{fa} \]  \hspace{1cm} (10)

Under certain conditions, a combination of two measurements in different frames will generate a single
LLR maximum. Since measurements are provided by the detector as the center of resolution cells in the measurement space, one can compute the number of resolution cell centers from the second data frame that can be associated with a measurement in the first frame to form a single LLR maximum. This is done by testing all of the resolution cells in the second frame using the procedure outlined in Appendix A. Call this quantity the number of associable cells, \( N_{AC}(l) \) which is a function of the frame difference, \( l \), between the two measurements.

For two given frames of data, the expected number of two-measurement associations given one measurement in the first frame is \( N_{AC}(l) P_{fa} \). This quantity is multiplied by the expected number of measurements in the first frame to obtain the number of two-measurement associations for a given set of two frames. To estimate the total number of LLR peaks that are formed from two-measurement associations, \( N_2 \), one must consider measurement associations between all possible combinations of two frames,

\[
N_2 = \sum_{l=1}^{N_W-1} (N_W - l) N_{AC}(l) P_{fa} N_c P_{fa}
\]

In general, the probability of \( n \) noise measurements associating to form a single LLR maximum is proportional to \( (P_{fa})^n \). As a result, the probability that an association of three or more noise measurements would form a single LLR maximum is low unless \( P_{fa} \) is very high; the number of such three (or more) measurement associations will therefore be assumed zero for purposes of estimating the number of local LLR maxima. Note that this does not imply that maxima formed from three or more measurements do not exist or do not affect the distribution of the global LLR maximum, but rather that the number of them is small compared to that of one and two measurement maxima.

The total number of LLR maxima, \( M \), can therefore be estimated using (10) and (11) as

\[
M = N_1 + N_2
\]

In the simulations to be presented later and for typical values of \( P_{fa} \) (1% to 2%), \( N_W \) (5–10 frames), and assumed target SNR (6–12 dB) use of this procedure results in estimates of 300–1800 LLR local maxima resulting from the noise-due measurements.

\section*{D. The Distribution of LLR Local Maxima}

In the absence of a target, one can view the pdf of an LLR local maximum as a mixture distribution. Let random variable \( y \) represent the value of an LLR local maximum with pdf \( f_y(y) \). Each component of the mixture distribution is distributed according to \( f_y^i(y) \) with the superscript indicating the number of measurements that associate to form the LLR local maximum. The probability of an LLR local maximum
consisting of \( i \) associated measurements is denoted \( p_i \). Theoretically \( i \) can take on values from 1 to the total number of measurements in the data set. The distribution \( f_y(y) \) can therefore be expressed by,

\[
f_y(y) = \sum_i p_i f^i_y(y)
\]  

(13)

### III. THE PDF OF THE LLR GLOBAL MAXIMUM

#### A. The LLR Global Maximum under \( H_0 \) (no target)

In previous ML-PDA implementations, the target validation test was based on Gaussian approximations [11], [13] for both \( H_0 \) and \( H_1 \). Since the LLR is comprised of the sum over the \( N_W \) data frames of single-frame LLRs, it was postulated that the LLR distribution was approximately Gaussian by invoking the Central Limit Theorem (CLT) as one was essentially summing over i.i.d. random variables. The single-frame LLR mean and variance were derived analytically [11], [13] as multi-fold integrals. As these equations do not have closed form solutions, they were solved using numerical techniques.

The analytic equations from [11], [13] are incorrect in that the mean and variance being solved for are not the mean and variance of the LLR global maximum but rather the mean and variance of the LLR at an arbitrary location in parameter space. The target validation test requires an evaluation of the statistics of the LLR global maximum. It should also be noted that in the simulations accompanying [11], [13], the test statistic used to set the validation threshold was based not on a given limiting false target/track acceptance probability (\( P_{FT} \)) derived from the \( H_0 \) statistics, but rather on a desired true target/track detection probability (\( P_{DT} \)) based on the \( H_1 \) statistics. As a result no confirmation of the accuracy of the Gaussian assumption was made for the pdf of the LLR global maximum under \( H_0 \).

To estimate the pdf of the LLR global maximum under \( H_0 \), we use Extreme Value Theory (EVT) instead of invoking the Central Limit Theorem, similar to [12] in the DP-TBD case. The LLR global maximum can be viewed as the maximum from the set of all LLR local maxima. Define the random variable \( y \) with cumulative distribution function (cdf) \( F_y(y) \) to be the value of an LLR local maximum and the random variable \( w \) with cdf \( F_w(w) \) to be the value of the LLR global maximum. Then using the formula for the distribution of the maximum order statistic from a set of \( M \) LLR local maxima (samples) [15],

\[
F_w(w) = [F_y(w)]^M
\]  

(14)

Implicit is the assumption that LLR local maxima are i.i.d. The independence assumption is not strictly valid in that LLR local maxima which share measurements will be correlated to some extent. However, it can be considered a good approximation because the maxima are generally well separated.
in the parameter space (see Fig. 1) and noise-related maxima will principally result from single- or two-measurement maxima. The small number of measurements contributing to an LLR maximum limits the correlation between maxima for a given measurement data set. These assumptions remain valid over a wide range of problem formulations and typical values of $P_{fa}$. Absent conditioning on the number of measurements associated with an LLR local maxima, the LLR local maxima can be considered to be identically distributed according to the mixture distribution described in (13).

EVT describes the asymptotic (large sample size) behavior of the largest value from an i.i.d. sample of size $M$ from a distribution with a cdf $F_y(y)$, and is well developed in the statistical literature [6], [10]. EVT can also be applied to some dependent sequences (see e.g., [8]). Let

$$w = \max\{y_1, y_2, \ldots, y_M\}$$

(15)

EVT states that if a limiting cdf of $w$ exists as $M \to \infty$, then that distribution must belong to one of three forms (Gumbel, Weibull, or Frechet). The distribution appropriate to a specific application is based on the support of the underlying distribution of $F_y(y)$. The Gumbel cdf is the appropriate distribution in our application because the support of the distribution of the LLR local maximum is restricted to $0 < y < \infty$, and is of the form

$$F_w(w) = \exp\{-\exp[-a_n(w - u_n)]\}$$

(16)

where $a_n$ and $u_n$ are the scale and location parameters for the distribution and which depend on the number of samples used in (15).

In addition to their interpretation as scale and location parameters, $a_n$ and $u_n$ have specific meanings [10] derived from the underlying distribution $F_y(y)$. The parameter $u_n$ is called the characteristic largest value in a set of $n$ samples where the expected number of measurements greater than or equal to $u_n$ is one, or

$$u_n = F_y^{-1}(1 - 1/n)$$

(17)

The parameter $a_n$ is called the extremal intensity function. The intensity function, $\mu(y)$ of an arbitrary distribution $F(y)$ with density $f(y)$ is defined as

$$\mu(y) = \frac{f(y)}{1 - F(y)}$$

(18)

and, when evaluated at the characteristic largest value ($u_n$), it becomes the extremal intensity function such that for the Gumbel distribution

$$a_n = n f_y(u_n)$$

(19)
The level of accuracy to which the Gumbel distribution approximates the distribution of the LLR global maximum is affected by two important issues:

1) There is no guarantee that an asymptotic distribution exists for the given $f_y(y)$. Specifically the structure of $f_y(y)$ as a mixture distribution described in (13) may preclude the existence of an asymptotic distribution.

2) While the number of LLR local maxima is large $F_w(w)$ may not have reached its asymptotic distribution. It has been noted for example that while the maximum from samples of an exponential distribution attains the asymptotic distribution with a relatively small number of samples (fast convergence), for a Gaussian distribution a much larger sample size is required to attain the asymptotic distribution (slow convergence) [10].

In Section V we show that the Gumbel distribution does approximate the pdf of the LLR global maximum with a high degree of accuracy and in particular is far more accurate than the previously ascribed Gaussian distribution.

B. The LLR Global Maximum under $H_1$

The statistical distribution of the LLR global maximum under $H_1$ is considered next. As was the case for the statistics under $H_0$, in [11], [13], integral expressions for the first two moments of the pdf of the LLR at the true values of the target parameter were derived under hypothesis $H_1$. By applying the CLT and using the same logic as described in the $H_0$ case, the distribution of the LLR value corresponding to the target-related maximum was assumed to be Gaussian.

In this case, since target-related measurements are distributed about the true target track with Gaussian measurement noises and since target-related measurements are present with a single frame detection probability, $P_d$, one is able to reasonably apply the CLT to the LLR at the LLR global maximum and approximate the LLR global maximum as a Gaussian random variable. One should expect some deviation from the Gaussian distribution when small values of window length ($N_W$) are used.

IV. PROCEDURES TO DETERMINE TARGET VALIDATION THRESHOLDS

According to the Neyman–Pearson Lemma [14], the most powerful test of $H_0$ versus $H_1$ is given by comparing the likelihood ratio (or log likelihood ratio) to a threshold. In the present case we consider a binary hypothesis test of target existence. If a valid track estimate exists, then by using ML principles it is given by the location of the LLR global maximum. Therefore the test becomes one of determining if the LLR global maximum is more likely to have been formed from only noise-originated measurements ($H_0$)
or target- plus noise-originated measurements \((H_1)\). The threshold for this test is selected to maximize the power of the test, \(P_{DT}\) (true target/track detection probability), at a given size or level of significance, \(P_{FT}\) (false target/track acceptance probability). Thus the threshold value, \(\gamma\), is chosen based on the statistics of the global LLR maximum under \(H_0\),

\[
\gamma = F_w^{-1}(1 - P_{FT})
\]  

(20)

If \(F_w(w)\) is known exactly, this hypothesis test becomes the optimal test as it obeys all conditions required by the Neyman–Pearson Lemma. Since in this application \(F_w(w)\) is only estimated, optimality of this test is not guaranteed. Two methods for estimating the pdf of the LLR global maximum, \(F_w(w)\), under \(H_0\) are presented next.

A. Off-line Method to Estimate \(F_w(w)\)

The off-line method to estimate the Gumbel parameters \(\alpha_n\) and \(\mu_n\) is to conduct Monte Carlo simulations of the specific tracking problem intended for system use under \(H_0\), and for specific values of the detection and tracking parameters \(P_{fa}\), SNR and \(N_W\). These simulations yield a set of LLR global maxima, \(\{w_i\}\). This set of LLR global maxima are then used to estimate the Gumbel distribution parameters \(\alpha_n\) and \(\mu_n\) for the distribution of the LLR global maximum, \(F_w(w)\). We use a maximum likelihood estimation technique [10]. These simulations would be performed before the tracking system is used so that the validation thresholds are available in real-time during tracking operations.

Given a set of \(c\) samples \(\{w_i\}\) with mean value \(\bar{w}\), the log likelihood function under the Gumbel distribution becomes

\[
p(w|\alpha_n, \mu_n) = c \ln \alpha_n - \sum_{i=1}^{c} \alpha_n (w_i - \mu_n) - \sum_{i=1}^{c} \exp[-\alpha_n (w_i - \mu_n)]
\]

(21)

By taking the partial derivatives with respect to \(\alpha_n\) and \(\mu_n\), setting them equal to zero the parameter estimates can be found from the relations

\[
\hat{\alpha}_n = \left[ \bar{w} - \frac{\sum_{i=1}^{c} w_i \exp(-\hat{\alpha}_n w_i)}{\sum_{i=1}^{c} \exp(-\hat{\alpha}_n w_i)} \right]^{-1}
\]

(22)

\[
\hat{\mu}_n = \frac{1}{\hat{\alpha}_n} \ln \left[ \frac{c}{\sum_{i=1}^{c} \exp(-\hat{\alpha}_n w_i)} \right]
\]

(23)

These equations are solved iteratively to obtain the maximum likelihood estimates of the Gumbel parameters.
This method has the advantage of yielding an optimal (in the ML sense) estimate of the Gumbel distribution parameters, although as has been previously stated the Gumbel distribution is only an approximation to the true distribution of $F_w(w)$. This approach may be impractical due to the extensive off-line simulations required (we use $c = 5000$). For a general-purpose tracking system using this methodology, separate sets of Gumbel distribution parameters must be estimated for the full range of possible $P_{fa}$, target SNR, and $N_W$ as well as for variations in the boundaries and volumes of the measurement and parameter spaces since each of these factors affect either the number of local LLR maxima or the distribution $f_y(y)$ or both. If the system were designed for a single special purpose use this method may be advantageous.

B. Real-Time Method to Estimate $F_w(w)$

The real-time method to compute $a_n$ and $u_n$, summarized in Table I, consists of estimating the distribution of LLR local maxima $F_y(y)$. Then for a suitable choice of $n$, determine the parameters of the Gumbel distribution of the LLR global maximum, $F_w(w)$ by applying (17) and (19). The following assumptions and simplifications, some of which are explained in more detail in the following paragraphs, are used:

1) The number of LLR local maxima is estimated using the method outlined in Section II-C.
2) The data set is censored by editing out measurements associated with the LLR global maximum and this does not significantly affect the estimates for $F_y(y)$ or $F_w(w)$.
3) The pdf of LLR local maxima over the parameter space for a single (large) data set is the same as the pdf of LLR local maxima over an ensemble of data sets (an ergodic property).
4) The shape of an LLR local maximum can be approximated as a linear function in each dimension of the parameter space (i.e., triangular shaped).
5) The pdf of the LLR local maxima can be approximated as a weighted mixture of Exponential distributions.

To estimate the pdf of the LLR local maxima, the real-time method uses the same measurement data set that was used by the ML-PDA algorithm to compute the track estimate. Measurements that are associated with the LLR global maximum are edited out of the data set as part of this procedure to prevent the presence of possible target-related measurements from influencing the estimate of $F_y(y)$ under $H_0$.

Ergodicity is a required assumption in order to use a single realization of the LLR surface to estimate the distribution of local LLR maxima, $F_y(y)$, for an arbitrary realization of data. We make the assumption of ergodicity in this case and justify it through the observation that the LLR surface obeys the ergodic...
TABLE I

REAL-TIME TARGET VALIDATION PROCEDURE

<table>
<thead>
<tr>
<th>Step</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Find global LLR maximum</td>
</tr>
<tr>
<td>2</td>
<td>Edit out measurements associated with global LLR maximum</td>
</tr>
<tr>
<td>3</td>
<td>Obtain $S$ samples of LLR surface</td>
</tr>
<tr>
<td>4</td>
<td>Use EM algorithm to compute $F_y(y)$</td>
</tr>
<tr>
<td>5</td>
<td>Compute $u_n, a_n$ to obtain $F_w(w)$</td>
</tr>
<tr>
<td>6</td>
<td>Compute threshold from $F_w(w)$</td>
</tr>
</tbody>
</table>

property that within the bounds of the allowable parameter space, the statistics of the LLR value are invariant to shift transformations in the parameter space (see e.g., [9]). The invariance to shift transformations results from the spatially uniform distribution of false detections and of the independence of the amplitude of false detections over parameter space. Based on this ergodic assumption, and analogous to the one dimensional case of a stationary random process where time averages/distributions are equivalent to ensemble averages/distributions, one can treat the statistics of the LLR local maxima for a single realization of data as equivalent to the statistics of the LLR local maxima over an ensemble of data sets. As a result, the estimated distribution of the LLR local maxima from a single data set will be treated as an estimate of $F_y(y)$.

If it were feasible to identify all LLR local maxima then an estimate of $F_y(y)$ would be straightforward to obtain. However it is not feasible to exhaustively search the parameter space to identify all LLR local maxima. One can easily obtain the values of local maxima resulting from a single measurement. However one must also identify all two- or more measurement maxima, which could not be performed in a real-time tracking system (real-time being defined here as the ability to compute a track estimate and validate a target in the time interval between successive data frames) without resorting to parallel computation.

As a consequence, a sampling approach is used whereby we infer the distribution $F_y(y)$ based on sampling the LLR surface at $S$ random points uniformly distributed in parameter space. An individual sample, $q_s (s \in \{1, \ldots, S\})$, will either be in the vicinity of a specific LLR local maximum, $y_m (m \in \{1, \ldots, M\})$, or will correspond to the “floor” of the LLR surface. Those samples that lie within an $\epsilon$ of the floor, where the specific value of $\epsilon$ is given in Appendix B, are excluded from the set of $S$ samples such that each sample used in subsequent steps of this procedure lies within the vicinity of a specific
One can relate a specific $q_s$ to a specific $y_m$ through the relation

$$q_s = \alpha_s y_m$$  \hspace{1cm} (24)$$

where $0 < \alpha_s \leq 1$ can also be considered a random variable whose distribution is based on the method in which the $q_s$ samples are chosen as well as by the shape of the LLR surface in the vicinity of the peak. By approximating the shape of the LLR surface in the vicinity of an LLR local maximum to linearly fall from peak to floor in each dimension of parameter space (i.e., an $n_d$-dimension triangular shape) as well as the uniform distribution in parameter space used to obtain the LLR samples, the $\alpha_s$ are from a distribution $F_{\alpha}(\alpha)$ given by

$$F_{\alpha}(\alpha) = 1 - (1 - \alpha)^{n_d}$$  \hspace{1cm} (25)$$

which represents the cdf of the joint distribution of $n_d$ independent uniform random variables.

Based on knowledge of the values of the $S$ LLR samples and the distribution $F_{\alpha}(\alpha)$ one must infer the distribution $F_y(y)$. This can be viewed as a missing data problem where the specific values of $\alpha_s$ are missing as well as the association of a specific LLR sample to its corresponding LLR local maximum and can be solved using the Expectation-Maximization (EM) algorithm (see, e.g. [16]). The distribution $F_y(y)$ is estimated by assuming that it is composed of a mixture density consisting of Exponentially distributed components. From the EM algorithm one obtains the Exponential distribution parameters for each component in the mixture as well as the weights associated with each mixture component. Because this is a straightforward application of the EM algorithm, details of its implementation are found in Appendix B.

Next, $u_n$ and $a_n$ are determined according to (17) and (19) using the estimate of $F_y(y)$ given by the EM algorithm. The value of $n$ should be selected to equal the expected number of LLR local maxima, $M$, computed from Section II-C. This yields the Gumbel distribution parameters for $F_w(w)$, the distribution of the global LLR maxima under $H_0$.

In order to use the real-time target validation procedure, the following user-selected parameters are needed: $M$ (the number of LLR local maxima), $S$ (the number of samples of the LLR surface), $K$ (the number of terms in the Exponential mixture density of $F_y(y)$), and $P$ (the number discrete values of $\alpha$ used by the EM algorithm to represent the pdf $f_\alpha(\alpha)$). In general, these parameters are selected to minimize execution time while also minimizing the variations in the resulting threshold values.

1) **The number of LLR local maxima**: $M$ is determined using the procedure outlined in Section II-C and represents the expected number of LLR maxima for a given measurement data set and given specific
values of $P_{fa}$, $N_W$, and SNR. As $M$ itself is a random variable, the actual number of LLR local maxima for a given data set will vary from this number. This difference will introduce variations in the threshold calculation from one data set to the next as $M$ is used in the final step to obtain $F_w(w)$ from $F_y(y)$.

2) **The number of samples of the LLR surface:** $S$ is the key value that determines the speed at which the procedure can be executed. As $S$ increases, the number of LLR evaluations increases, as does the computation time of the EM algorithm weights. However, as $S$ increases, a more accurate estimation of $F_y(y)$ is obtained. Selecting $S$ to be about an order of magnitude greater than $M$ gives a good balance between speed and accuracy.

3) **The number of terms in the Exponential mixture density of $F_y(y)$:** $K$ is chosen to provide a good fit between the actual (unknown) density, $F_y(y)$ and the mixture density approximation, and should be at least as large as the number of terms in (13) whose associated weighting probability is significantly different from zero. Since at least two processes are involved in creating local LLR peaks (single-measurement peaks and two-measurement peaks), a minimum of 2 modes is necessary in the mixture. Algorithm testing has shown that the EM algorithm does converge to two dominant modes.

4) **The number of discrete values of $\alpha$:** $P$ is selected to balance solution accuracy with execution time. Given the non-linear shape of $f_\alpha(\alpha)$, if too few values are selected these values would not be representative of the underlying pdf. As the number of increases beyond a certain point, the gains in terms of accurately representing $f_\alpha(\alpha)$ diminish and execution time increases. Selecting $P$ to be about 100 gives a good balance between speed and accuracy.

**V. Simulation Results**

The simulation results presented below highlight the two major contributions of this paper. First, using the off-line target validation method, we show that the pdf of the LLR global maximum is best approximated by a Gumbel distribution instead of the Gaussian distribution that was assumed in previous papers. Next, the relative effectiveness of the off-line and real-time target validation methods will be illustrated using the System Operating Characteristic (SOC) and by how close actual false target/track acceptance ($P_{FT}$) matches with the desired value. The SOC (following the nomenclature used in [2]) combines the effects of the detection and track processing subsystems to give a tracking system performance characterization for track detection/acceptance by relating $P_{DT}$ to $P_{FT}$. This plot is also referred to as a Tracker Operating Characteristic (TOC) in the literature.

Results are presented for two active sonar scenarios that use different dimensionalities of the measurement space. Within each scenario, four cases are examined that show the performance of the off-line and
TABLE II
SIMULATION PARAMETER VALUES

<table>
<thead>
<tr>
<th>Parameter</th>
<th>3D Scenario</th>
<th>2D Scenario</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ping Interval</td>
<td>$T = 4,\text{s}$</td>
<td>$T = 40,\text{s}$</td>
</tr>
<tr>
<td>Measurement Space</td>
<td>$\beta = \pm 18^\circ$</td>
<td>$\beta = \pm 60^\circ$</td>
</tr>
<tr>
<td></td>
<td>$r = 0 - 3,000,\text{m}$</td>
<td>$r = 0 - 30,000,\text{m}$</td>
</tr>
<tr>
<td></td>
<td>$\dot{r} = \pm 15,\text{m/s}$</td>
<td></td>
</tr>
<tr>
<td>Resolution Cell</td>
<td>$N_e = 9600$</td>
<td>$N_e = 3280$</td>
</tr>
<tr>
<td></td>
<td>$\Delta\beta = 6^\circ$, $\sigma_{\beta}^2 = 3,\text{degrees}^2$</td>
<td>$\Delta\beta = 3^\circ$, $\sigma_{\beta}^2 = 1.5,\text{degrees}^2$</td>
</tr>
<tr>
<td></td>
<td>$\Delta r = 75,\text{m}$, $\sigma_r^2 = 469,\text{m}^2$</td>
<td>$\Delta r = 375,\text{m}$, $\sigma_r^2 = 11718,\text{m}^2$</td>
</tr>
<tr>
<td></td>
<td>$\Delta \dot{r} = 0.75,\text{m/s}$, $\sigma_{\dot{r}}^2 = 0.0469,(\text{m/s})^2$</td>
<td></td>
</tr>
<tr>
<td>CFAR Detector</td>
<td>16 cell averaging</td>
<td>8 cell averaging</td>
</tr>
</tbody>
</table>

real-time target validation procedures as the detection and tracking parameters that influence the statistics of the LLR surface are varied.

A. Scenario Description

The two scenarios considered in this paper represent different active sonar problems. In the first scenario (called the 3D scenario, signifying the dimensionality of the measurement space), an underwater autonomous vehicle conducts an active sonar search using a short to medium range sonar system. Detections are given by an amplitude at a given bearing, range, and range rate. In the second scenario (called the 2D scenario), a stationary sensor conducts an active sonar search using a medium to long range sonar system. Detections are given by an amplitude at a given bearing and range.

The detection signal processing model is similar for both scenarios. Measurement amplitude for each resolution cell is modeled as a Rayleigh-distributed random variable. A Rayleigh distributed amplitude corresponds to a Swerling-I fluctuation model and is considered appropriate for active sonar applications [1], [22]. Amplitude data is thresholded using a cell averaging Constant False Alarm Rate (CFAR) detector to obtain a set of detections at a given single frame $P_{fa}$. Table II summarizes key parameters that define each scenario.

The set of thresholded measurements is then passed to the ML-PDA tracking algorithm which computes the track estimate. The method used to find the LLR global maximum was the subspace-based procedure.
TABLE III

SIMULATION CASES FOR THE 3D SCENARIO

<table>
<thead>
<tr>
<th>Case Number</th>
<th>SNR</th>
<th>$P_{fa}$</th>
<th>$P_{d}$</th>
<th>$N_W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6 dB</td>
<td>1%</td>
<td>35%</td>
<td>5 frames</td>
</tr>
<tr>
<td>2</td>
<td>10 dB</td>
<td>1%</td>
<td>62%</td>
<td>5 frames</td>
</tr>
<tr>
<td>3</td>
<td>6 dB</td>
<td>2%</td>
<td>42%</td>
<td>5 frames</td>
</tr>
<tr>
<td>4</td>
<td>6 dB</td>
<td>1%</td>
<td>35%</td>
<td>10 frames</td>
</tr>
</tbody>
</table>

TABLE IV

SIMULATION CASES FOR THE 2D SCENARIO

<table>
<thead>
<tr>
<th>Case Number</th>
<th>SNR</th>
<th>$P_{fa}$</th>
<th>$P_{d}$</th>
<th>$N_W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7 dB</td>
<td>1%</td>
<td>38%</td>
<td>5 frames</td>
</tr>
<tr>
<td>2</td>
<td>12 dB</td>
<td>1%</td>
<td>69%</td>
<td>5 frames</td>
</tr>
<tr>
<td>3</td>
<td>7 dB</td>
<td>2%</td>
<td>45%</td>
<td>5 frames</td>
</tr>
<tr>
<td>4</td>
<td>7 dB</td>
<td>1%</td>
<td>38%</td>
<td>10 frames</td>
</tr>
</tbody>
</table>

from [3], [4]. The LLR global maximum was then compared to a threshold using either the off-line or real-time target validation procedure. The real-time procedure uses $K = 4$ exponential components in the mixture density, $S = 10^4$ samples of the LLR surface, and $P = 100$ discrete values of $\alpha$.

Simulations were conducted under $H_0$ with the target absent and under $H_1$ with the target present. Target validation thresholding from the $H_0$ simulations yields $P_{FT}$ by finding the percent of track estimates that were (wrongly) accepted. Target validation thresholding from the $H_1$ simulations yields $P_{DT}$ by finding the percent of track estimates that were (correctly) accepted.

Because the LLR is a function of assumed target SNR, detector $P_{fa}$ and $N_W$, four cases within each scenario are simulated that vary one of these variables from its baseline value. Tables III and IV show the values of these variables for each case. Note that SNR=6 dB in a resolution cell has a very low $P_d$ even for the $P_{fa} = 2\%$, and is considered to be a VLO target.
Fig. 3. LLR global maximum exceedance probabilities for the 3D scenario cases (under $H_0$) fit to Gumbel and Gaussian distributions.

**B. Distribution of the LLR Global Maximum under $H_0$**

We first consider the distribution of the LLR global maximum under $H_0$. For each scenario and case, 5000 simulations were performed under $H_0$ and the LLR global maximum found. Using the off-line target validation procedure, the Gumbel parameters that describe the distribution of the LLR global maximum were then estimated. As a comparison, the same data set of LLR global maxima was also used to fit a Gaussian pdf by obtaining maximum likelihood estimates of the data set mean and variance.

Results are presented in Figs. 3 and 4 and are plotted as exceedance probabilities, defined as the complementary cdf, on a semilog scale. For a given target validation threshold, the exceedance probability corresponds to $P_{FT}$. These figures show that the empirical pdf of the LLR global maximum matches closely the Gumbel distribution predicted by the EVT approach taken in this paper, and is much more accurate than the Gaussian distribution. In particular, for values of $P_{FT}$ that tracking systems would typically use (below about 5%), the Gaussian approximation errs in an optimistic way—it predicts a significantly lower $P_{FT}$ than would actually be experienced.

For the 2D scenario cases, there is some deviation between the empirical result and the Gumbel
approximation at exceedance probabilities below about $10^{-2}$ and also errs in an optimistic way (although its optimism is less than that if the Gaussian approximation were to be used). The cause of this deviation has not been identified although further analysis of the approximations and assumptions used in ascribing a Gumbel distribution to this process (see end of Section III-A) may provide some insight.

C. Target Validation Performance—System Operating Characteristic (SOC)

We next examine the ability of the off-line and real-time target validation procedures to reject false targets and accept true targets. The off-line target validation thresholds were computed using the simulations from the previous section and the procedure described in section IV-A. The real-time target validation thresholds were computed using the procedure described in section IV-B. For each scenario and case 7500 simulations were performed under $H_0$ and $H_1$.

Figs. 5 and 6 show the performance of the off-line and real-time target validation procedures compared to empirical results on a SOC. The $P_{FT}$ plotted is the actual $P_{FT}$ achieved for a specific target validation threshold. Because it uses a fixed threshold value, the off-line target validation results will lie on the curve defined by the empirical result.
Fig. 5. System Operating Characteristic for the 3D scenario cases comparing off-line and real-time target validation results.

Fig. 6. System Operating Characteristic for the 2D scenario cases comparing off-line and real-time target validation results.
The real-time target validation procedure is sub-optimal in that for a given $P_{FT}$, the achieved $P_{DT}$ is less than that given by the empirical results. In both scenarios, cases 1 and 3 represent tracking situations near the edge of the performance envelope of the ML-PDA algorithm where SNR is approaching the lowest limit at which the ML-PDA algorithm can track a target. In terms of the LLR surface, this translates to the average value of the LLR global maximum under $H_1$ approaching the average value under $H_0$. Threshold selection in this case becomes more critical in terms of maximizing $P_{DT}$ for a given $P_{FT}$.

The performance penalty incurred in using the real-time target validation threshold is therefore increased near the edge of the ML-PDA tracking envelope. Cases 2 and 4 represent situations where ML-PDA in general performs well, and a sub-optimal choice of target validation threshold is not penalized as much in terms of reducing $P_{DT}$.

In applying either target validation technique to an actual tracking problem, a track validation threshold is selected to achieve a maximum allowed $P_{FT}$. Based on the approximations and errors associated with a particular method, the actual $P_{FT}$ may deviate from the maximum allowed value. For example, if a validation method achieved a $P_{FT}$ lower than the allowable maximum, then the procedure is suboptimal from the respect that by achieving a higher $P_{FT}$ (closer to the maximum), a higher actual $P_{DT}$ will result.

Fig. 7 compares the specified maximum $P_{FT}$ to that achieved over the set of 7500 simulations for each scenario and case for each target validation procedure. The magnitude of the errors in actual vs. allowable $P_{FT}$ are generally smaller for the off-line target validation procedure. However the off-line procedure tends to err in the optimistic direction in that the maximum allowable $P_{FT}$ will be exceeded. The real-time procedure is less prone to exceeding the maximum allowable $P_{FT}$, but suffers from larger errors.

Overall the results from this section show that the off-line target validation procedure does provide the best target validation performance of the two methods presented. However this method may not be practical due to the extensive off-line simulations that are required to obtain target validation thresholds for all operating conditions of the tracking system.

The real-time target validation threshold does not require the extensive off-line simulations and is able to produce a target validation threshold in real-time (defined as within the interval between successive frames of data) using the data set from which the track estimate was extracted. However this procedure incurs performance penalties at the limits of the ML-PDA tracker performance operating envelope. Despite this performance penalty, use of the ML-PDA algorithm combined with the real-time target validation procedure can be effective for tracking VLO targets.
VI. CONCLUSIONS

In Track-Before-Detect systems, such as the ML-PDA tracker, target detection and track estimation functions are performed simultaneously. As a consequence the track estimates obtained must be tested for the existence of a target. In this paper we have presented a target validation procedure based on the Neyman-Pearson lemma for the ML-PDA tracker. We have shown that the pdf of the LLR global maximum under the “no target” hypothesis used by previous researchers (a Gaussian distribution) is incorrect and by using Extreme Value Theory have shown that a Gumbel distribution better approximates this distribution.

Two procedures for obtaining a target validation threshold were next presented—an off-line procedure and a real-time procedure. Through simulations, the performance of a detection/tracking system using the off-line target validation procedure was shown to closely match empirical results. However the off-line method suffers from the requirement to perform extensive simulations to obtain a target validation threshold for each operating point of the detection/tracking system.

The real-time target validation procedure achieves a lower true track acceptance probability for a given
limiting false target/track acceptance probability than either the off-line method or the empirical results. This performance penalty increases near the limits of effective operation of the detection/tracking system. Despite this performance penalty, a detection/tracking system using the real-time target validation method remains a feasible method of validating targets in the ML-PDA tracker.

**APPENDIX A**

**THE NUMBER OF TWO-MEASUREMENT ASSOCIABLE CELLS**

Under certain conditions, a combination of two measurements in different frames will generate a single LLR maximum. To estimate the number of such two-measurement associations, assume that

1) only the two measurements of interest (labeled $Z_1$ and $Z_2$) contribute to the LLR.
2) the measurements are of equal amplitude $a$ corresponding to likelihood amplitude ratio $\rho$.
3) the measurement amplitude used is $a = E[a_{ij} | a_{ij} > \tau]$.

Such a pair of measurements can be associated to form a single LLR maximum if there exists a target motion parameter $x^*$ within the allowable parameter space that, when applied to $Z_2$ to project it to the data frame containing $Z_1$, the shifted measurement $Z'_2$ falls within a hyper-ellipsoid of a given size (based on $\delta$ whose value is determined later) centered at $Z_1$. The hyper-ellipsoid is defined by

$$Z'_2 \in \{ Z : (Z - Z_1)^T \Sigma^{-1} (Z - Z_1) \leq \delta^2 \} \quad (26)$$

where superscript $T$ represents the transpose operator.

To determine $\delta^2$, one must find the maximum Mahalanobis distance between $Z_1$ and $Z'_2$ where a single LLR maximum can be formed. When a single LLR maximum is formed, by symmetry it will be located at the midpoint ($Z^*$) along a line connecting $Z_1$ and $Z'_2$ (in the measurement space), where the parameter $x^*$ is set such that $Z'_2(x^*)$ maximizes the value of this LLR peak. From (9) the value of this maximum is

$$\Lambda_{max}(Z_1, Z'_2, a|x^*) = 2 \ln \{ 1 + c \exp[-(\delta/2)^2/2] \} \quad (27)$$

where

$$c = \frac{P_d \rho}{\lambda(1 - P_d)(2\pi)^{1.5} \sigma_\beta \sigma_r \sigma_f} \quad (28)$$

Let $x_i(Z_i)$ be a one to many mapping of $Z_i$ to $x_i$. As a simplification, assume that when two distinct LLR maxima form from $Z_1$ and $Z_2$, the value of the maxima is not influenced by the second measurement and the maxima will form at parameters $x_1(Z_1)$ and $x_2(Z_2)$ at a value of

$$\Lambda_{max}(Z_i, a|x_i) = \ln(1 + c) \quad (29)$$
The maximum distance at which a single LLR maximum will form occurs when \( \Lambda_{max}(Z_1, Z'_2, a|x^*) \) is equal to \( \Lambda_{max}(Z_i, a|x_i) \) or,

\[
2 \ln\{1 + c \exp[-(\delta/2)^2/2]\} = \ln(1 + c)
\]

and by solving for \( \delta^2 \) we obtain

\[
\delta^2 = -8 \ln \left( \frac{-1 + \sqrt{1 + c}}{c} \right)
\]

In general, \( c \gg 1 \) and so (31) can be approximated as

\[
\delta^2 \simeq 4 \ln c
\]

Measurements are provided by the detector as the center of resolution cells in the measurement space. For a given \( \delta^2 \), one can compute the number of resolution cell centers from the second data frame that can be associated with \( Z_1 \) to form a single LLR maximum by testing all of the resolution cells in the second frame using (26). Call this quantity the number of associable cells, \( N_{AC}(l) \) which is a function of the frame difference, \( l \), between the two measurements. This quantity is then used in (11) to compute the number of maxima on the LLR surface formed by two-measurement associations.

It should be noted that in the case where a 2D measurement space of bearing and range is used, a unique velocity will fit exactly through any two measurements in different frames and the Mahalanobis distance between \( Z_1 \) and \( Z'_2 \) will be zero. In this case, the two measurements will form a single LLR maximum provided this velocity lies inside the allowable parameter space (typically less than a maximum velocity).

### Appendix B

**Real-Time Target Validation EM Algorithm Implementation**

In this appendix, we outline the EM algorithm implementation used to estimate the pdf of the local LLR maxima, \( f_y(y) \), from a set of samples of the LLR surface, \( q_s \). First approximate \( f_y(y) \) as a mixture of \( K \) exponential distributions

\[
f_y(y) = \sum_{k=1}^{K} \pi_k \lambda_k \exp[-\lambda_k(y - y_{min})]
\]

where \( \pi_k \) are the weights such that \( \pi_k \geq 0 \forall i, \sum_{k=1}^{K} \pi_k = 1 \) and \( \lambda_k > 0 \) are the exponential distribution parameters.

Recall that a specific sample, \( q_s \) is related to a specific LLR local maximum \( y_m \) through the relation \( q_s = \alpha_s y_m \). The smallest value any \( y_m \) can take occurs as a result of a single-measurement peak at an
amplitude equal to the detector threshold and from (9) is given by

\[ y_{\text{min}} = \ln \left( 1 + \frac{P_d}{\lambda (1 - P_d)} \rho_{\text{min}} \right) \]  
(34)

and the minimum likelihood amplitude ratio, \( \rho_{\text{min}} \), is given by

\[ \rho_{\text{min}} = \frac{P_{fa}}{P_d (1 + \text{SNR})} \exp \left( \frac{\tau \text{SNR}}{1 + \text{SNR}} \right) \]  
(35)

where \( \tau \) is the detector threshold.

In order to apply the EM algorithm, \( f_\alpha(\alpha) \) is approximated by \( P \) values (ordered smallest to largest) representing partitions of the probability space, \( \alpha \in \{\alpha_1, \ldots, \alpha_P\} \). The discrete values taken by \( \alpha \) represent regions of equal probability such that

\[ \frac{1}{P} = \int_{(\alpha_{i+1}-\alpha_i)/2}^{(\alpha_i-\alpha_{i-1})/2} f_\alpha(\alpha) \, d\alpha \]  
(36)

for all values of \( i \). For \( i = 1 \) the lower integration limit is zero and for \( i = P \) the upper integration limit is 1.

From this discretization \( \epsilon \) (the smallest value of an LLR sample used in the EM algorithm) is set to

\[ \epsilon = \alpha_1 y_{\text{min}} \]  
(37)

Using a standard EM algorithm [7], weights are defined for each measurement \( q_s, s = 1, \ldots, S \), every mode \( k = 1, \ldots, K \), and every partition of \( \alpha, p = 1, \ldots, P \) and are given by

\[ w_{k,p,s} = \frac{\pi_k \lambda_k \exp[-\lambda_k (q_s/\alpha_p - y_{\text{min}})]}{\sum_{k=1}^{K} \sum_{p=1}^{P} \pi_k \lambda_k \exp[-\lambda_k (q_s/\alpha_p - y_{\text{min}})]} \]  
(38)

From a set of initial values of \( \pi_k \) and \( \lambda_k \), the following equations are used iteratively, along with (38) to obtain the estimate of \( f_y(y) \) from the exponential mixture:

\[ \lambda_k = \frac{\sum_{p=1}^{P} \sum_{s=1}^{S} w_{k,p,s}}{\sum_{p=1}^{P} \sum_{s=1}^{S} (q_s/\alpha_p - y_{\text{min}}) w_{k,p,s}} \]  
(39)

\[ \pi_k = N^{-1} \sum_{p=1}^{P} \sum_{s=1}^{S} w_{k,p,s} \]  
(40)

REFERENCES


