Self-intersection of composite curves and surfaces

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Abstract

This paper provides computationally tractable conditions to determine whether a composite spline curve or patch self-intersects, according to a definition that includes the important limiting cases of cusps, singularities, and tangential intersections of adjacent components. These results follow upon our exposition of necessary and sufficient conditions to preclude such self-intersections. The paper includes a numerical example illustrating the results, and discusses an important application, namely, guaranteeing that a finite curvilinear simplicial complex in R^3, made up of properly-joined parametric patches, will retain its original topological form when its control points are perturbed.

1 Introduction

In this paper we give conditions permitting avoidance of self-intersections of composite spline curves and patches [1]. We also give limits on the size of control-point perturbations so that the perturbed curve or patch has no self-intersection. The results are formulated in terms of theorems for curves or patches having only one or two components. These theorems may be applied simultaneously to all parts of the curve or patch, and therefore apply directly to curves or patches composed of an arbitrary number of components. It is assumed that if a curve has two components, then they share only a single common endpoint, and if a patch has two components, they intersect in a single corner point or along a single common boundary curve. Further, the components are assumed to be polynomial Bézier curves or triangular polynomial Bézier patches, although we will also indicate where results generalize to the rational case.

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The main motivation for the results is found in the case of *patches*, in the context of representing the boundary of a solid object. Suppose that we are given a collection of properly-joined[3] single-component patches corresponding to the curvilinear boundary of a homogeneously three-dimensional solid object [3]. If the data defining the object (*i.e.,* the control points defining the patches) are manipulated using finite-precision arithmetic, then we may ask whether the perturbed data defines an approximate[4] well-formed three-dimensional object: this is (part of) the *robustness* problem [4, Ch.4]. Some *ad hoc* approaches to this problem place error zones around the original boundary faces, where the width of the error zone is on the order of the error in finite-precision arithmetic. While this is a useful heuristic, it does not permit proof of rigorous theorems, due to the possibility of extraneous intersections between neighboring faces (see Figure 1). Ideally, we would like to prove that the perturbed data defines the well-formed boundary of a nearby three-dimensional object. The results of this paper permit proof of such theorems, at least in simple cases, and therefore provide a first step towards a completely rigorous robustness theory. For example, suppose that a solid, defined by properly-joined patches forming its boundary, undergoes a rigid motion, *i.e.,* the control points for all patches are transformed by a rotation and a translation. Using standard facts about floating-point arithmetic [5], it is easy to bound the perturbations of the control points relative to an exact rigid motion. Since these perturbations will usually be very small, the results given here permit us in this case to ensure that there are no self-intersections introduced in any composite patch forming part of the boundary, and thus to conclude[5] that

[3]Properly-joined means that a single patch does not self-intersect, and two disjoint patches either coincide along a common boundary curve or in a common vertex [2, p. 202].

[4]As measured by the Hausdorff metric, for example.

[5]It is also necessary, in order to reach this conclusion, to use standard convex-hull results to check for intersection, after perturbation, of disjoint single-component patches. It is also possible to bound the Hausdorff distance between the boundaries of the approximate and exact versions of the object, and to show that they have the same topological form according to the criterion of [6, Part III].

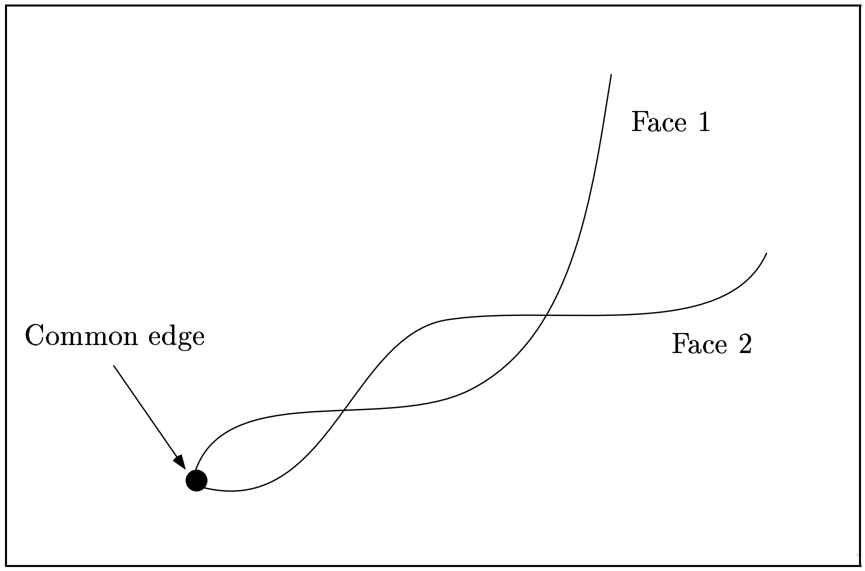

Figure 1: Cross-section of two neighboring faces

the computed result defines the properly-joined boundary of a three-dimensional solid object.

Similar comments apply to the modification, in other contexts, of the control points defining boundary patches. For example, in a system permitting interactive editing of curvilinear solids, the results provided here could be used to signal control-point changes that might lead to a self-intersecting boundary.

The results of this paper are more generally applicable in computer-aided geometric design, graphics and animation. Sufficient conditions for avoiding self-intersections are used in the construction of algorithms for patch-patch intersection for Boolean operations [7], and such conditions may also be applicable in other applications, such as rendering algorithms, to guarantee the so-called "single-sheet" property for texturing algorithms. More generally, as observed in [8], "...one of the basic problems in a [boundary-representation] system is the unexpected generation of self-intersecting solids...", and the property of non-self-intersection forms part of the STEP standard [9]. Also, the problem of identifying the range of perturbations, for which the topological form of a geometric object remains invariant, is of crucial importance in morphing for animation [10]. Furthermore, avoiding self-intersection is implicit in certain work [11] on the topologically reliable approximation of composite Bézier curves. Finally, avoidance of cusps and singularities is often important [12, 13, 14]; the conditions we give for preventing intersections preclude cusps and singularities as a special case.

Previous work on the problem of patch intersection has included conditions guaranteeing non-self-intersection of single-component patches and curves. While the method of [15], for curves, can be rigorously justified by theorem, [16] presents heuristic (but apparently effective) methods for patches. We note that difference quotients similar to those introduced below are classical [17], and have been used computationally to avoid points of self-intersection [18].

We now outline the organization of the paper. Section 2 deals with detecting self-intersections of curves, and Section 3 provides analogous results for patches. Most of the proofs are omitted, but they are given in [6]. The results of Section 3 can be used to guarantee that solid objects defined by perturbed data have retained their original topological form. This was mentioned above in relation to the robustness problem, but, as summarized in [19], maintenance of topological form is of interest in many applied fields, including tolerancing and metrology, solid modeling, engineering design, and computer graphics. In Section 4 we reformulate the previous results in terms of the maximum control-point perturbations that can be tolerated without risk of self-intersections. Section 4 also gives an example related to self-intersection of composite patches, and describes an implementation. Section 5 is the Conclusion.

2 Self-intersection of Bézier curves

2.1 Self-intersection of single-component curves

Let $\boldsymbol{Q} = \{\boldsymbol{R}_0, \boldsymbol{R}_1, \ldots, \boldsymbol{R}_n\}$ denote a sequence of vectors in R^3, which is the control polygon for an n-degree, parametric polynomial Bézier curve [1]. We introduce the operator $C(t)$ operating on the sequence $\boldsymbol{Q}$ and defined as follows:

$$C(t)\{\boldsymbol{R}_i\}_{i=0}^{n} = ((1-t)E_0 + tE_1)\{\boldsymbol{R}_i\}_{i=0}^{n} = \{(1-t)\boldsymbol{R}_i + t\boldsymbol{R}_{i+1}\}_{i=0}^{n-1}.$$

Here E_1 denotes the forward-shift operator, and E_0 the identity operator. Now, the n-degree Bézier curve having control polygon $\boldsymbol{Q}$ is given by

$$\boldsymbol{R}(t) = C^n(t)\boldsymbol{Q} = ((1-t)E_0 + tE_1)^n\boldsymbol{Q} = \left[\sum_{i=0}^{n}\binom{n}{i}(1-t)^{n-i}t^iE_i\right]\boldsymbol{Q}, \quad t \in [0,1],$$

where $E_i = E_1^i$; consequently

$$\boldsymbol{R}(t) = \sum_{i=0}^{n}\binom{n}{i}(1-t)^{n-i}t^i\boldsymbol{R}_i. \tag{1}$$

Self-intersection of a Bézier curve can be expressed in terms of a certain Bézier *patch*, and we therefore introduce the notation for a triangular Bézier patch here. Let $\boldsymbol{Q} = \{\boldsymbol{R}_{ijk}\}_{i,j,k\geq 0;\, i+j+k=n}$ denote a set of control-polygon vectors. The operators $E_{100}, E_{010}, E_{001}$ are shift operators in the i-, j- and k-directions:

$$E_{100}\{\boldsymbol{R}_{ijk}\}_{i,j,k\geq 0;\, i+j+k=n} = \{\boldsymbol{R}_{i+1,j,k}\}_{i,j,k\geq 0;\, i+j+k=n-1},$$

with E_{010} and E_{001} defined analogously. Introducing the operator $C(r,s,t) = rE_{100} + sE_{010} + tE_{001}$ we may represent the patch by $C^n(r,s,t)\boldsymbol{Q}$ or

$$\boldsymbol{R}(r,s,t) = \sum_{i+j+k=n}\frac{n!}{i!j!k!}r^is^jt^k\boldsymbol{R}_{ijk}.$$

Here, r,s,t denote barycentric coordinates with $r,s,t \geq 0$ and $r+s+t=1$. Also, this and similar summations are over non-negative values of the index variables.

Now, consider two points on the Bézier curve given by (1), denoted by $\boldsymbol{R}(1-s)$ and $\boldsymbol{R}(t)$ with $0 \leq t < 1-s \leq 1$, and form the difference quotient

$$\boldsymbol{S}(s,t) = \frac{1}{n}\frac{\boldsymbol{R}(1-s) - \boldsymbol{R}(t)}{(1-s)-t}. \tag{2}$$

Then, by the Remainder Theorem, $\boldsymbol{S}(s,t)$ is a vector-valued *polynomial*, well-defined for $1-s=t$ by $\boldsymbol{S}(1-t,t) = \frac{1}{n}\dot{\boldsymbol{R}}(t)$, where the dot denotes differentiation with respect to t. Thus, $\boldsymbol{S}(s,t)$ is defined over the triangle

$$\mathcal{T} = \{(s,t) : s+t \leq 1,\ s,t \geq 0\}. \tag{3}$$

We will say that $\boldsymbol{R}(t)$ is *self-intersecting* if for some $t_0, t_1 \in [0,1]$, with $t_0 < t_1$ we have $\boldsymbol{R}(t_0) = \boldsymbol{R}(t_1)$, or if for some $t \in [0,1]$ we have $\dot{\boldsymbol{R}}(t) = \boldsymbol{0}$. Points where $\dot{\boldsymbol{R}}(t) = \boldsymbol{0}$ are called critical points; they are limiting cases of genuinely self-intersecting curves, and are included among the self-intersecting curves. Let

$$d = \min\left\{|\boldsymbol{S}(s,t)| : (s,t) \in \mathcal{T}\right\}, \tag{4}$$

where $|\cdot|$ denotes the Euclidean norm on R^3. Then, we have:

Criterion 2.1 *A necessary and sufficient condition that the Bézier curve* $\boldsymbol{R}(t)$ *is not self-intersecting is that* $d > 0$.

It is clear that $\boldsymbol{S}(s,t)$ is an $(n-1)$-degree, triangular Bézier patch over the parameter triangle $\mathcal{T}$ given by (3), and in order to apply Criterion 2.1, *i.e.*, to determine whether $\boldsymbol{S}(s,t)$ takes the value $\boldsymbol{0}$, it is useful to know the vectors of its control polygon. Note that

$$\boldsymbol{R}(1-s) - \boldsymbol{R}(t) = [C^n(1-s) - C^n(t)]\boldsymbol{Q} \tag{5}$$

and that $C^n(1-s) - C^n(t) = [\sum_{i=0}^{n-1} C^{n-1-i}(1-s)C^i(t)][C(1-s) - C(t)]$. Since $C(1-s) - C(t) = (1-s-t)(E_1 - E_0)$, we have

$$\boldsymbol{S}(s,t) = \frac{1}{n}[C^{n-1}(1-s) + C^{n-2}(1-s)C(t) + \ldots + C^{n-1}(t)]\boldsymbol{q} \tag{6}$$

where we have introduced the sequence $\boldsymbol{q} = \{\boldsymbol{r}_i\}_{i=0}^{n-1}$ of differences $\boldsymbol{r}_i = \boldsymbol{R}_{i+1} - \boldsymbol{R}_i$, *i.e.*, $\boldsymbol{q} = (E_1 - E_0)\boldsymbol{Q}$. Thus, $d = d(\boldsymbol{q})$ depends on $\boldsymbol{q}$.

The following lemma, which motivated the introduction of the factor $\frac{1}{n}$ in (2), will be useful later. Here, $conv(\boldsymbol{q})$ denotes the convex hull of $\boldsymbol{q}$.

Lemma 2.1 *For the function $\boldsymbol{S}(s,t)$ we have $range(\boldsymbol{S}) \subseteq conv(\boldsymbol{q})$.*

Proof. For $0 \le t \le 1$, $C(t) = (1-t)E_0 + tE_1$ forms convex combinations of the vectors in the sequence on which it operates. It follows from (6) that for all $(s,t) \in \mathcal{T}$, $\boldsymbol{S}(s,t)$ is a convex combination of the vectors $\{\boldsymbol{r}_i\}_{i=0}^{n-1} = \boldsymbol{q}$. □

Now, using barycentric coordinates, we have

$$\boldsymbol{S}(s,t) = \frac{1}{n}\left[\sum_{i=0}^{n-1} C^{n-1-i}(r+t)C^i(t)\right]\boldsymbol{q} \tag{7}$$

where $C(r+t) = rE_1 + (sE_0 + tE_1)$ and $C(t) = rE_0 + (sE_0 + tE_1)$. The control polygon for the surface patch $\boldsymbol{S}(s,t)$ can now be obtained from

$$\frac{1}{n}\left[\sum_{i=0}^{n-1} C^{n-1-i}(r+t)C^i(t)\right]\boldsymbol{q} = \sum_{i+j+k=n-1} \frac{(n-1)!}{i!j!k!} r^i s^j t^k \frac{1}{i+1} E_k[E_0 + E_1 + \ldots + E_i]\boldsymbol{q}. \tag{8}$$

The proof [6, Part I] of this equality is a straightforward algebraic verification using the multinomial theorem. We conclude from (7) and (8) that the control-polygon vector associated with the indices (i,j,k) is (see Figure 2):

$$\boldsymbol{R}_{ijk} = \frac{1}{(i+1)}[\boldsymbol{r}_k + \boldsymbol{r}_{k+1} + \ldots + \boldsymbol{r}_{k+i}],\ i+j+k = n-1.$$

Criterion 2.1 gives a necessary and sufficient condition for self-intersection of a Bézier curve. However, in order to apply it, we must compute the parameter $d = d(\boldsymbol{q})$. This is not simple [20], and it is worthwhile to give less-sharp criteria that are easier to use. Let

$$d^* = d^*(\boldsymbol{q}) = dist(\boldsymbol{0}, conv(\boldsymbol{q})). \tag{9}$$

and, given a unit vector $\boldsymbol{n}$,

$$d^{**} = d^{**}(\boldsymbol{n}, \boldsymbol{q}) = \min\{\boldsymbol{r}\cdot\boldsymbol{n} : \boldsymbol{r} \in \boldsymbol{q}\}. \tag{10}$$

By Lemma 2.1 it follows that we have the following simpler criterion.

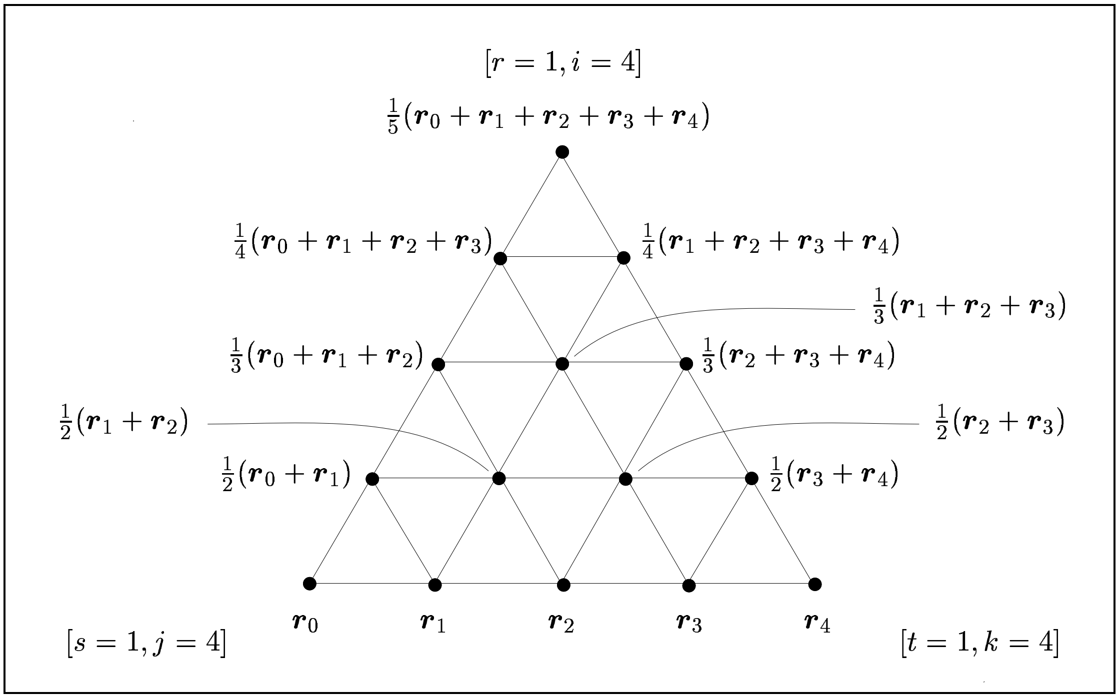

Figure 2: Control polygon for $n = 5$.

Criterion 2.1* *A sufficient condition for non-self-intersection of the Bézier curve* $\boldsymbol{R}(t)$ *is that* $d^*(\boldsymbol{q}) > 0$, *or, equivalently, that there exists a unit vector* $\boldsymbol{n}$ *such that* $d^{**}(\boldsymbol{n}, \boldsymbol{q}) > 0$.

Remark. Criterion 2.1* generalizes to the case of rational Bézier curves, by viewing such a curve as a central projection of a polynomial Bézier curve in R^4. The proof may be based on the Variation-Diminishing Principle.

2.2 Self-intersection of two-component curves

Let $\boldsymbol{R}^0(u) = \sum_{i=0}^n \binom{n}{i} u^i (1-u)^{n-i} \boldsymbol{R}_i^0$ and $\boldsymbol{R}^1(v) = \sum_{k=0}^n \binom{n}{k} v^k (1-v)^{n-k} \boldsymbol{R}_k^1$ be Bézier curves with control polygons $\boldsymbol{Q}^0 = \{\boldsymbol{R}_i^0\}_{i=0}^n$ and $\boldsymbol{Q}^1 = \{\boldsymbol{R}_k^1\}_{k=0}^n$, and assume that

$$\boldsymbol{R}^0(0) = \boldsymbol{R}^1(0) = \boldsymbol{R}_0^0 = \boldsymbol{R}_0^1.$$

We seek conditions which will guarantee that there are no other common points, *i.e.*, that $\boldsymbol{R}^0(u) \neq \boldsymbol{R}^1(v)$ if $u \in (0, 1]$ or $v \in (0, 1]$, and that the curves are non-tangential and non-critical at the initial points, *i.e.*, that

$$\dot{\boldsymbol{R}}^0(0) \times \dot{\boldsymbol{R}}^1(0) \neq \mathbf{0} \text{ or } \dot{\boldsymbol{R}}^0(0) \cdot \dot{\boldsymbol{R}}^1(0) < 0. \tag{11}$$

We therefore consider $\boldsymbol{R}(u,v) = \boldsymbol{R}^1(v) - \boldsymbol{R}^0(u)$, *i.e.*,

$$\boldsymbol{R}(u,v) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} v^k (1-v)^{n-k} \boldsymbol{R}_k^1 - \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} u^i (1-u)^{n-i} \boldsymbol{R}_i^0. \tag{12}$$

There are two cases: an extraneous intersection could occur *a)* with $v \leq u$, or *b)* with $u \leq v$. We will study the restriction of $\boldsymbol{R}$ to the subtriangles $T_a = \{(u,v) : 0 \leq v \leq u \leq 1\}$ and $T_b = \{(u,v) : 0 \leq u \leq v \leq 1\}$ of its domain.

For T_a we introduce barycentric coordinates (r,s,t) defined by $r = 1-u$, $s = u-v$ and $t = v$. Thus, $r+s+t = 1$, $(r,s,t) = \boldsymbol{\rho} = (1,0,0)$ corresponds to the point $(0,0) \in T_a$, $(r,s,t) = \boldsymbol{\sigma} = (0,1,0)$ to $(1,0) \in T_a$, and $(r,s,t) = \boldsymbol{\tau} = (0,0,1)$ to $(1,1) \in T_a$. As shown in [6, Part I], the restriction to T_a is then given by

$$\sum_{i+j+k=n} \frac{n!}{i!j!k!} r^i s^j t^k \boldsymbol{R}_{ijk}, \tag{13}$$

where the control polygon $\{\boldsymbol{R}_{ijk}\}_{i+j+k=n}$ is given by

$$\boldsymbol{R}_{ijk}^a = \boldsymbol{R}_k^1 - \boldsymbol{R}_{n-i}^0 = \boldsymbol{R}_k^1 - \boldsymbol{R}_{j+k}^0. \tag{14}$$

For the restriction of $\boldsymbol{R}$ to T_b we have (13) with $r = 1-v$, $s = v-u$, $t = u$ and

$$\boldsymbol{R}_{ijk}^b = -\boldsymbol{R}_k^0 + \boldsymbol{R}_{n-i}^1 = -\boldsymbol{R}_k^0 + \boldsymbol{R}_{j+k}^1. \tag{15}$$

Again considering the restriction to the triangle T_a, we write $\boldsymbol{P} = (r,s,t)$, using the barycentric coordinates introduced above; (13) may now be written as $\boldsymbol{R}(\boldsymbol{P})$. Let $\boldsymbol{P}_1 = (r_1, s_1, t_1)$ be a point in T_a, and let $\boldsymbol{P}_0 = (1,0,0)$ (see Figure 3). Since condition (11) fails to be satisfied if $\boldsymbol{R}(\boldsymbol{P})$ has some directional derivative at the point $\boldsymbol{P}_0$ equal to $\mathbf{0}$, the composite curve has a self-intersection if $\boldsymbol{R}(\boldsymbol{P}_1) = \mathbf{0}$ for some $\boldsymbol{P}_1 \neq \boldsymbol{P}_0$, or if this directional derivative at $\boldsymbol{P}_0$ is equal to $\mathbf{0}$. A similar conclusion is valid for the restriction of $\boldsymbol{R}$ to the triangle T_b.

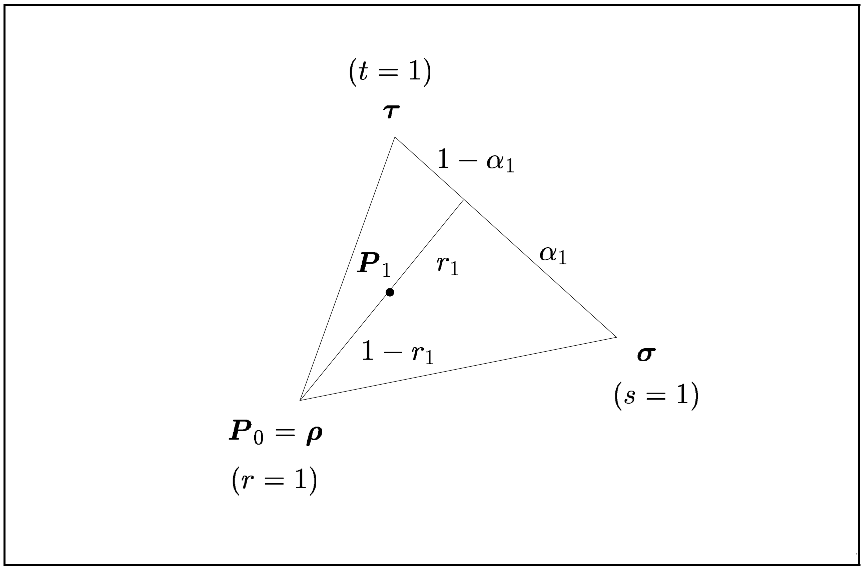

Figure 3: Triangle T_a

In direct analogy to the previous subsection, we can define a function that is equal to $\mathbf{0}$ if and only if the two components of a composite curve intersect:

$$\begin{aligned}\boldsymbol{S}_a(\boldsymbol{P}_0, \boldsymbol{P}_1) = \frac{1}{n}\,\frac{1}{1-r_1}\,[\boldsymbol{R}(\boldsymbol{P}_1) - \boldsymbol{R}(\boldsymbol{P}_0)] = \\ \frac{1}{n}\left[\sum_{k=0}^{n-1} C^{n-1-k}(r_1, s_1, t_1) C^k(1,0,0)\right] \\ [(1-\alpha_1)C(-1,1,0) + \alpha_1 C(-1,0,1)]\boldsymbol{Q}_a\end{aligned} \tag{16}$$

where $\alpha_1 = t_1/(1-r_1)$ and $\boldsymbol{Q}_a = \{\boldsymbol{R}^a_{ijk}\}_{i+j+k=n}$, as defined in (14).

The function $\boldsymbol{S}_a$ has domain $\{\boldsymbol{P}_1 \in T_a : \boldsymbol{P}_1 \neq \boldsymbol{P}_0\}$. If we keep α_1 fixed then

$$\lim_{\boldsymbol{P}_1 \to \boldsymbol{P}_0} \boldsymbol{S}_a(\boldsymbol{P}_0, \boldsymbol{P}_1) = C^{n-1}(\boldsymbol{P}_0)[(1-\alpha_1)C(-1,1,0) + \alpha_1 C(-1,0,1)]\boldsymbol{Q}_a, \tag{17}$$

a function of α_1. We now replace the variable $\boldsymbol{P}_1 = (r_1, s_1, t_1)$ by (α_1, r_1), and introduce $\tilde{\boldsymbol{S}}_a(\alpha_1, r_1)$, given by (16) if $\boldsymbol{P}_1 \neq \boldsymbol{P}_0$ and by (17) if $\boldsymbol{P}_1 = \boldsymbol{P}_0$. The case $\tilde{\boldsymbol{S}}_a(\alpha_1, 1) = \mathbf{0}$ corresponds to the case when $\boldsymbol{R}(u,v)$ has a directional derivative at $(u,v) = (0,0)$ equal to $\mathbf{0}$, *i.e.*, when (11) fails to be satisfied.

The factor $\frac{1}{1-r_1}$ in $\boldsymbol{S}_a$ is not arbitrary: this choice permits our assertions about the singular case, and it is crucial to the later perturbation theory, which uses Lemma 2.2 below. The factor $\frac{1}{n}$ is also not arbitrary: it is used in the proof of Lemma 2.2, just as the corresponding factor was used in Lemma 2.1.

Similarly, we define the corresponding functions $\boldsymbol{S}_b$ and $\tilde{\boldsymbol{S}}_b$ when we restrict to the triangle T_b; in this case, $\boldsymbol{S}_b$ is given by (16) with $\boldsymbol{Q}_a$ replaced by $\boldsymbol{Q}_b$, where $\boldsymbol{Q}_b = \{\boldsymbol{R}^b_{ijk}\}_{i+j+k=n}$, defined in (15). (Note that $\boldsymbol{Q}_a \cup \boldsymbol{Q}_b$ is just the set of differences $\boldsymbol{R}^1_{k_1} - \boldsymbol{R}^0_{k_0}$ for all combinations of k_0 and k_1.) Let

$$d_a = d_a(\boldsymbol{Q}^0, \boldsymbol{Q}^1) = \min\{|\tilde{\boldsymbol{S}}_a(\alpha_1, r_1)| : 0 \leq \alpha_1 \leq 1, 0 \leq r_1 \leq 1\},$$

with a corresponding definition for d_b, and

$$d = d(\boldsymbol{Q}^0, \boldsymbol{Q}^1) = \min\{d_a, d_b\}. \tag{18}$$

Criterion 2.2 *A necessary and sufficient condition, for non-intersection of the components $\boldsymbol{R}^0(u)$ and $\boldsymbol{R}^1(v)$ of a composite Bézier curve, is that $d(\boldsymbol{Q}^0, \boldsymbol{Q}^1) > 0$.*

Calculation of the differences in (16) gives $C(-1,1,0)\boldsymbol{Q}_a = \{-\boldsymbol{r}^0_{j+k}\}_{i+j+k=n-1}$, $C(-1,0,1)\boldsymbol{Q}_a = \{\boldsymbol{r}^1_k - \boldsymbol{r}^0_{j+k}\}_{i+j+k=n-1}$, and $C(-1,1,0)\boldsymbol{Q}_b = \{\boldsymbol{r}^1_{j+k}\}_{i+j+k=n-1}$, $C(-1,0,1)\boldsymbol{Q}_b = \{\boldsymbol{r}^1_{j+k} - \boldsymbol{r}^0_k\}_{i+j+k=n-1}$. Defining

$$\begin{aligned}\boldsymbol{q}_a &= (C(-1,1,0)\boldsymbol{Q}_a) \cup (C(-1,0,1)\boldsymbol{Q}_a)\\ \boldsymbol{q}_b &= (C(-1,1,0)\boldsymbol{Q}_b) \cup (C(-1,0,1)\boldsymbol{Q}_b),\end{aligned}$$

we have, in analogy with Lemma 2.1, the following lemma.

Lemma 2.2 *For the functions $\tilde{\boldsymbol{S}}_a$ and $\tilde{\boldsymbol{S}}_b$ we have*

$$\begin{aligned}range(\tilde{\boldsymbol{S}}_a) &\subseteq conv(\boldsymbol{q}_a)\\ range(\tilde{\boldsymbol{S}}_b) &\subseteq conv(\boldsymbol{q}_b)\end{aligned}$$

More tractable but less sharp criteria are obtained by excluding $\boldsymbol{R}_0^1 - \boldsymbol{R}_0^0$. We have $\boldsymbol{R}^1(v) - \boldsymbol{R}^0(u) = C^n(r,s,t)\boldsymbol{Q}_i$, for $(u,v) \in T_i$, $i = a$ or b. Let $\boldsymbol{Q}'_a = \{\boldsymbol{R}^a_{ijk}\}_{0 \le i < n}$ and $\boldsymbol{Q}'_b = \{\boldsymbol{R}^b_{ijk}\}_{0 \le i < n}$, and

$$\begin{aligned}
&d_a^*(\boldsymbol{Q}^0, \boldsymbol{Q}^1) = dist(\boldsymbol{0}, conv(\boldsymbol{Q}'_a)) \\
&d_b^*(\boldsymbol{Q}^0, \boldsymbol{Q}^1) = dist(\boldsymbol{0}, conv(\boldsymbol{Q}'_b)) \\
&d^* = d^*(\boldsymbol{Q}^0, \boldsymbol{Q}^1) = \min\{d_a^*(\boldsymbol{Q}^0, \boldsymbol{Q}^1), d_b^*(\boldsymbol{Q}^0, \boldsymbol{Q}^1)\}.
\end{aligned}$$

Also, given unit vectors $\boldsymbol{n}_a$ and $\boldsymbol{n}_b$, define

$$\begin{aligned}
&d_a^{**}(\boldsymbol{n}_a, \boldsymbol{Q}^0, \boldsymbol{Q}^1) = \min\{\boldsymbol{x} \cdot \boldsymbol{n}_a : \boldsymbol{x} \in \boldsymbol{Q}'_a\}, \\
&d_b^{**}(\boldsymbol{n}_b, \boldsymbol{Q}^0, \boldsymbol{Q}^1) = \min\{\boldsymbol{x} \cdot \boldsymbol{n}_b : \boldsymbol{x} \in \boldsymbol{Q}'_b\}, \\
&\boldsymbol{n} = (\boldsymbol{n}_a, \boldsymbol{n}_b), \\
&d^{**} = d^{**}(\boldsymbol{n}, \boldsymbol{Q}^0, \boldsymbol{Q}^1) = \min\{d_a^{**}, d_b^{**}\}.
\end{aligned}$$

Criterion 2.2* *A sufficient condition precluding intersection of two components of a composite Bézier curve is that* $d^*(\boldsymbol{Q}^0, \boldsymbol{Q}^1) > 0$, *or, equivalently, that there exist unit vectors* $\boldsymbol{n}_a$ *and* $\boldsymbol{n}_b$ *such that* $d^{**}(\boldsymbol{n}, \boldsymbol{Q}^0, \boldsymbol{Q}^1) > 0$.

Proof. If $(u,v) \in T_a, (u,v) \neq (0,0)$ then $\boldsymbol{R}^1(v) - \boldsymbol{R}^0(u) = C^n(r,s,t)\boldsymbol{Q}_a$ with $(r,s,t) \neq (1,0,0)$. This implies that $C^n(r,s,t)\boldsymbol{Q}_a$ is a convex combination of vectors from $\boldsymbol{Q}_a$ not consisting only of $\boldsymbol{R}_0^1 - \boldsymbol{R}_0^0 = \boldsymbol{0}$. Then, $\boldsymbol{0} \notin conv(\boldsymbol{Q}'_a)$ implies that $C^n(r,s,t)\boldsymbol{Q}_a \neq \boldsymbol{0}$, *i.e.*, that $\boldsymbol{R}^1(v) - \boldsymbol{R}^0(u) \neq \boldsymbol{0}$ if $(u,v) \in T_a, (u,v) \neq (0,0)$. Similarly, $\boldsymbol{R}^1(v) - \boldsymbol{R}^0(u) \neq \boldsymbol{0}$ if $(u,v) \in T_b, (u,v) \neq (0,0)$. It remains to show that $\boldsymbol{R}^1(v)$ and $\boldsymbol{R}^0(u)$ do not have parallel and identically directed tangent vectors at the initial point, and that they are non-critical there. Assume the contrary. Since $d^* > 0$ we must have $\boldsymbol{R}_1^1 - \boldsymbol{R}_0^1 \neq \boldsymbol{0}$ and $\boldsymbol{R}_1^0 - \boldsymbol{R}_0^0 \neq \boldsymbol{0}$ and without loss of generality $\boldsymbol{R}_1^1 - \boldsymbol{R}_0^1 = \alpha(\boldsymbol{R}_1^0 - \boldsymbol{R}_0^0)$ with $\alpha \geq 1$. Then, from the definition (14), $\boldsymbol{R}_0^1 - \boldsymbol{R}_1^0 \in \boldsymbol{Q}_a$ and $\boldsymbol{R}_1^1 - \boldsymbol{R}_1^0 \in \boldsymbol{Q}_a$. However $\boldsymbol{R}_1^1 - \boldsymbol{R}_1^0 = \boldsymbol{R}_0^1 + \alpha(\boldsymbol{R}_1^0 - \boldsymbol{R}_0^0) - \boldsymbol{R}_1^0 = (1-\alpha)(\boldsymbol{R}_0^1 - \boldsymbol{R}_1^0)$. Since $\boldsymbol{R}_0^1 - \boldsymbol{R}_1^0 \in \boldsymbol{Q}_a$ and $(1-\alpha)(\boldsymbol{R}_0^1 - \boldsymbol{R}_1^0) \in \boldsymbol{Q}_a$ with $1 - \alpha \leq 0$ it follows that $\boldsymbol{0} \in conv(\boldsymbol{Q}'_a)$ which is a contradiction. □

3 Self-intersection of Bézier patches

In this section we present analogous results for patches. The proofs [6, Part II] are similar to those for curves.

3.1 Self-intersection of single triangular Bézier patches

Let $\boldsymbol{Q} = \{\boldsymbol{R}_{ijk}\}_{i,j,k\geq 0;\, i+j+k=n}$ be the control polygon of an n-degree triangular Bézier patch, $C(r,s,t) = rE_{100} + sE_{010} + tE_{001}$, and $\mathcal{U}$ the triangle $\{\boldsymbol{P} = r\boldsymbol{\rho} + s\boldsymbol{\sigma} + t\boldsymbol{\tau} : r+s+t = 1,\ r,s,t \geq 0\}$, where $\boldsymbol{\rho}, \boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are vectors in general position. The triangular Bézier patch is then

$$\boldsymbol{R}(r,s,t) = \sum_{i+j+k=n} \frac{n!}{i!j!k!} r^i s^j t^k \boldsymbol{R}_{ijk},$$

defined on $\mathcal{U}$. Also, $\boldsymbol{R}(r,s,t) = C^n(r,s,t)\boldsymbol{Q} = C^n(\boldsymbol{P})\boldsymbol{Q}$ where

$$C^n(r,s,t) = C^n(\boldsymbol{P}) = \sum_{i+j+k=n} \frac{n!}{i!j!k!} r^i s^j t^k E_{ijk}$$

with $E_{ijk} = E_{100}^i E_{010}^j E_{001}^k$.

In analogy to Section 2, we say that the patch is self-intersecting if $\boldsymbol{R}(\boldsymbol{P}_0) = \boldsymbol{R}(\boldsymbol{P}_1)$ for two distinct points $\boldsymbol{P}_0$ and $\boldsymbol{P}_1$ in $\mathcal{U}$ or if some directional derivative of $\boldsymbol{R}(\boldsymbol{P})$ at a point $\boldsymbol{P}_0$ equals $\boldsymbol{0}$, and we derive a function that is $\boldsymbol{0}$ if and only if the patch is self-intersecting.

Assume that $\boldsymbol{P}_0 \neq \boldsymbol{P}_1$ and $\boldsymbol{R}(\boldsymbol{P}_0) = \boldsymbol{R}(\boldsymbol{P}_1)$, and let L be the line passing through $\boldsymbol{P}_0$ and $\boldsymbol{P}_1$. Then, there is a parallel line L' passing through

a. the corner point $\boldsymbol{\rho}$ and intersecting the opposite edge $\boldsymbol{\sigma\tau}$; or,

b. the corner point $\boldsymbol{\sigma}$ and intersecting the opposite edge $\boldsymbol{\tau\rho}$; or,

c. the corner point $\boldsymbol{\tau}$ and intersecting the opposite edge $\boldsymbol{\rho\sigma}$.

Similarly, the directional derivative along some line L through $\boldsymbol{P}_0$ might be equal to the null vector. In this event, we have the same possibilities for L'.

Consider Case a, with $\boldsymbol{P}_0 \neq \boldsymbol{P}_1$. The function $\boldsymbol{S}_a$ is defined by

$$\begin{aligned} \boldsymbol{S}_a(\boldsymbol{P}_0, \boldsymbol{P}_1) &= \tfrac{1}{n} \tfrac{1}{(r_0 - r_1)} \left[\boldsymbol{R}(\boldsymbol{P}_1) - \boldsymbol{R}(\boldsymbol{P}_0)\right] \\ &= \tfrac{1}{n} \left[\textstyle\sum_{k=0}^{n-1} C^{n-1-k}(\boldsymbol{P}_1) C^k(\boldsymbol{P}_0)\right] [(1-\alpha)C(-1,1,0) + \alpha C(-1,0,1)]\boldsymbol{Q}, \end{aligned} \tag{19}$$

where $\alpha = (t_1 - t_0)/(r_0 - r_1)$. This function has domain

$$\{(\boldsymbol{P}_0, \boldsymbol{P}_1) \in \mathcal{U} \times \mathcal{U} : \boldsymbol{P}_0 \neq \boldsymbol{P}_1, \boldsymbol{P}_0 \text{ and } \boldsymbol{P}_1 \text{ positioned as in case a}\}.$$

Note that the parameter α in (19) depends only on the direction of $\boldsymbol{P}_1 - \boldsymbol{P}_0$. If we think of α as a fixed parameter, then $\boldsymbol{S}_a(\boldsymbol{P}_0, \boldsymbol{P}_1)$ is an $(n-1)$-degree polynomial in the two 2-D variables $\boldsymbol{P}_1$ and $\boldsymbol{P}_0$. As in Section 2, the factors $\frac{1}{n}$ and $\frac{1}{(r_0 - r_1)}$ in (19) are not arbitrary, but play a crucial role in the proofs.

Now, if we keep α fixed and let $\boldsymbol{P}_1 \to \boldsymbol{P}_0$ in (19), then

$$\lim_{\boldsymbol{P}_1 \to \boldsymbol{P}_0} \boldsymbol{S}_a(\boldsymbol{P}_0, \boldsymbol{P}_1) = C^{n-1}(\boldsymbol{P}_0)[(1-\alpha)C(-1,1,0) + \alpha C(-1,0,1)]\boldsymbol{Q}, \tag{20}$$

which depends on $\boldsymbol{P}_0$ and α. We now replace $\boldsymbol{P}_0$ and $\boldsymbol{P}_1$ by the variables $(\alpha_0, r_0, \alpha, r_1)$, where $\alpha_0 = t_0/(1 - r_0)$ if $r_0 \neq 1$ (α_0 is arbitrary if $r_0 = 1$), and introduce $\tilde{\boldsymbol{S}}_a(\alpha_0, r_0, \alpha, r_1)$ given by (19) if $\boldsymbol{P}_0 \neq \boldsymbol{P}_1$, *i.e.*, if $r_0 > r_1$, and by (20) if $\boldsymbol{P}_0 = \boldsymbol{P}_1$, *i.e.*, if $r_0 = r_1$. The domain of $\tilde{\boldsymbol{S}}_a$ is

$$D_a = \{(\alpha_0, r_0, \alpha, r_1) : 0 \leq \alpha_0, \alpha \leq 1, 0 \leq r_1 \leq r_0 \leq 1\} \subset R^4.$$

Similarly, we may define functions $\boldsymbol{S}_b$, $\boldsymbol{S}_c$, $\tilde{\boldsymbol{S}}_b$ and $\tilde{\boldsymbol{S}}_c$ for Cases b and c. Expressions for these functions are obtained from those for $\boldsymbol{S}_a$ and $\tilde{\boldsymbol{S}}_a$ by cyclic permutation of the variables (r, s, t) and of the variables in the operator $C(\cdot, \cdot, \cdot)$.

Clearly, if $\boldsymbol{P}_0 \neq \boldsymbol{P}_1$ with $\boldsymbol{P}_0$ and $\boldsymbol{P}_1$ positioned as in Case a, then $\boldsymbol{R}(\boldsymbol{P}_0) = \boldsymbol{R}(\boldsymbol{P}_1)$ if and only if $\boldsymbol{S}_a(\boldsymbol{P}_0, \boldsymbol{P}_1) = \tilde{\boldsymbol{S}}_a(\alpha_0, r_0, \alpha, r_1) = \boldsymbol{0}$ (where $r_0 > r_1$). On the other hand, $\boldsymbol{R}(\boldsymbol{P})$ is singular at $\boldsymbol{P}_0$, with α defining the singular direction, if and only if $\tilde{\boldsymbol{S}}_a(\alpha_0, r_0, \alpha, r_0) = \boldsymbol{0}$ (here $r_1 = r_0$).

Let

$$\begin{aligned} \boldsymbol{q}_a &= (C(-1,1,0)\boldsymbol{Q}) \cup (C(-1,0,1)\boldsymbol{Q}) \\ \boldsymbol{q}_b &= (C(0,-1,1)\boldsymbol{Q}) \cup (C(1,-1,0)\boldsymbol{Q}) \\ \boldsymbol{q}_c &= (C(1,0,-1)\boldsymbol{Q}) \cup (C(0,1,-1)\boldsymbol{Q}) \\ \boldsymbol{q} &= (\boldsymbol{q}_a, \boldsymbol{q}_b, \boldsymbol{q}_c) \end{aligned}$$

and

$$d_a = d_a(\boldsymbol{q_a}) = \min\{|\tilde{\boldsymbol{S}}_a(\alpha_0, r_0, \alpha, r_1)| : (\alpha_0, r_0, \alpha, r_1) \in D_a\},$$

with corresponding definitions for d_b and d_c. Also, let

$$d = d(\boldsymbol{q}_a, \boldsymbol{q}_b, \boldsymbol{q}_c) = d(\boldsymbol{q}) = \min\{d_a, d_b, d_c\}. \tag{21}$$

We now have the following criterion for non-self-intersection.

Criterion 3.1 *A necessary and sufficient condition, that the Bézier patch $\boldsymbol{R}(\boldsymbol{P})$ has no self-intersection, is that $d > 0$.*

In parallel to Section 2, we now formulate more tractable criteria.

Lemma 3.1 *For the functions $\tilde{\boldsymbol{S}}_a$, $\tilde{\boldsymbol{S}}_b$ and $\tilde{\boldsymbol{S}}_c$ we have*

$$range(\tilde{\boldsymbol{S}}_i) \subseteq conv\{\boldsymbol{q}_i\}, \quad i = a, b, c. \tag{22}$$

We define

$$d_a^*(\boldsymbol{q}_i) = dist(\boldsymbol{0}, conv(\boldsymbol{q}_i)) \quad i = a, b, c$$
$$d^* = d^*(\boldsymbol{q}) = \min\{d_a^*, d_b^*, d_c^*\}.$$

Also, letting $\boldsymbol{n} = (\boldsymbol{n}_a, \boldsymbol{n}_b, \boldsymbol{n}_c)$, define d_a^{**}, d_b^{**}, d_c^{**} and d^{**} by

$$d_i^{**} = d_i^{**}(\boldsymbol{n}_i, \boldsymbol{q}_i) = \min\{\boldsymbol{r} \cdot \boldsymbol{n}_i : \boldsymbol{r} \in \boldsymbol{q}_i\}, \quad i = a, b, c,$$

$$d^{**} = d^{**}(\boldsymbol{n}, \boldsymbol{q}) = \min\{d_a^{**}, d_b^{**}, d_c^{**}\}.$$

Then, by Lemma 3.1 and Criterion 3.1, we have the following criterion.

Criterion 3.1* *A sufficient condition for non-self-intersection of the Bézier patch $\boldsymbol{R}(\boldsymbol{P})$ is that $d^*(\boldsymbol{q}) > 0$, or, equivalently, that there exist unit vectors $\boldsymbol{n}_a$, $\boldsymbol{n}_b$ and $\boldsymbol{n}_c$ such that $d^{**}(\boldsymbol{n}, \boldsymbol{q}) > 0$.*

Remark. Similarly to Criterion 2.1*, Criterion 3.1* generalizes to rational Bézier patches, by considering the properties of central projections in R^4.

3.2 Self-intersection of two-component patches

We again begin with necessary and sufficient conditions. The two cases, that the component patches coincide along a common boundary curve or in a common vertex, are treated separately.

First, note that when we say that two patches coincide along a boundary curve, we mean that the corresponding points of their control polygons coincide, so that the boundary curves are to be identical as mappings from a parameter interval. Two patches coinciding along a common boundary curve (or, respectively, at a common vertex), forming a composite patch, have a self-intersection

1. if they have a common point outside the common boundary curve (outside the common vertex); or

2. if for some point $\boldsymbol{P}$ on the common boundary curve (or for the common vertex), the adjacent patches are tangential. (This is the limiting case of the first condition, where there is a genuine self-intersection.)

3.2.1 Two-component patches with common boundary curve

Suppose given two n-degree Bézier patches $\boldsymbol{R}^0(\boldsymbol{P}_0) = \sum_{i+j+k=n} \frac{n!}{i!j!k!} r_0^i s_0^j t_0^k \boldsymbol{R}_{ijk}^0$ and $\boldsymbol{R}^1(\boldsymbol{P}_1) = \sum_{i+j+k=n} \frac{n!}{i!j!k!} r_1^i s_1^j t_1^k \boldsymbol{R}_{ijk}^1$, with $\boldsymbol{Q}^0 = \{\boldsymbol{R}_{ijk}^0\}_{i+j+k=n}$ and $\boldsymbol{Q}^1 = \{\boldsymbol{R}_{ijk}^1\}_{i+j+k=n}$ as control polygons. Assume that the patches coincide along the curve defined by $r_0 = r_1 = 0$, *i.e.*,

$$\boldsymbol{R}_{0jk}^0 = \boldsymbol{R}_{0jk}^1, \qquad j + k = n. \tag{23}$$

Now suppose that we have an intersection corresponding to Case 1, above, so that for either $r_0 > 0$ or $r_1 > 0$ we have

$$C^n(\boldsymbol{P}_0)\boldsymbol{Q}^0 = C^n(\boldsymbol{P}_1)\boldsymbol{Q}^1.$$

In analogy with Section 2, we investigate $\boldsymbol{R}(\boldsymbol{P}_0, \boldsymbol{P}_1) = \boldsymbol{R}^1(\boldsymbol{P}_1) - \boldsymbol{R}^0(\boldsymbol{P}_0)$; this is an n-degree polynomial mapping from $\mathcal{U} \times \mathcal{U}$ into R^3. We now decompose

$\mathcal{U} \times \mathcal{U}$ into subsimplices in the following way:

$$T_a = \{(\boldsymbol{P}_0, \boldsymbol{P}_1) \in \mathcal{U} \times \mathcal{U} : 0 \leq r_0 \leq r_1, 0 \leq t_0 \leq t_1, r_1 + t_1 \leq 1\}$$

$$T_b = \{(\boldsymbol{P}_0, \boldsymbol{P}_1) \in \mathcal{U} \times \mathcal{U} : 0 \leq s_1 \leq s_0, 0 \leq t_1 \leq t_0, s_0 + t_0 \leq 1\}$$

$$T_c = \{(\boldsymbol{P}_0, \boldsymbol{P}_1) \in \mathcal{U} \times \mathcal{U} : 0 \leq r_0 \leq r_1, 0 \leq s_0 \leq s_1, r_1 + s_1 \leq 1\}.$$

The sets T_d, T_e and T_f are defined from T_a, T_b and T_c, respectively, by interchanging the indices 0 and 1 on the variables r, s and t. We note that each of the sets $T_a, \ldots, T_f$ is defined by five independent linear inequalities and is therefore a 4-D simplex (with five barycentric coordinates). Further, $T_a, \ldots, T_f$ have disjoint interiors and their union is $\mathcal{U} \times \mathcal{U}$. In Figure 4 we have indicated the regions

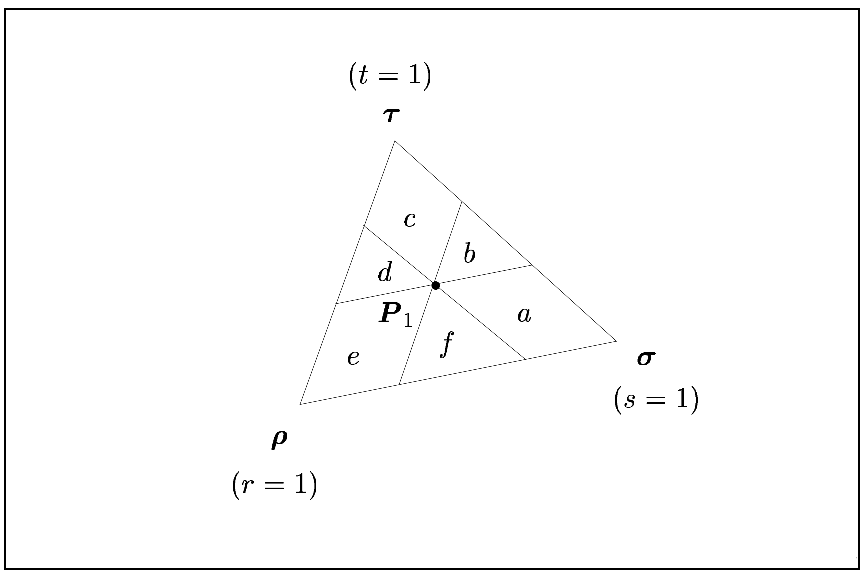

Figure 4: Positions corresponding to $T_a, \ldots, T_f$

defining the range of positions for $\boldsymbol{P}_0$ when $(\boldsymbol{P}_0, \boldsymbol{P}_1)$ is in each of $T_a, \ldots, T_f$.

Now, the restriction of the mapping $\boldsymbol{R}(\boldsymbol{P}_0, \boldsymbol{P}_1)$ to T_a is a 4-D, n-degree, Bézier mapping with a control polygon that we denote by

$$\boldsymbol{Q}_a = \{\boldsymbol{R}^a_{ijklm}\}_{i+j+k+l+m=n},$$

so that

$$\boldsymbol{R}(v,w,x,y,z) = \sum_{i+j+k+l+m=n} \frac{n!}{i!j!k!l!m!} v^i w^j x^k y^l z^m \boldsymbol{R}^a_{ijklm},$$

where v, w, x, y, z are barycentric coordinates for T_a, with $v, w, x, y, z \geq 0$ and $v+w+x+y+z = 1$: $v = r_0$, $w = r_1 - r_0$, $x = t_0$ $y = t_1 - t_0$ and $z = 1 - r_1 - t_1 = s_1$. (The complexity here is illusory: it will be seen in Subsection 4.4 that the d^* and d^{**} criteria, derived below, are very easily implemented.) The control polygons $\boldsymbol{Q}_b, \ldots, \boldsymbol{Q}_f$ for the other subsimplices are defined similarly. Then [6, Part II]

$$\begin{aligned}
\boldsymbol{R}^a_{ijklm} &= \boldsymbol{R}^1_{i+j,m,k+l} - \boldsymbol{R}^0_{i,j+l+m,k}, \\
\boldsymbol{R}^b_{ijklm} &= \boldsymbol{R}^1_{j+l+m,i,k} - \boldsymbol{R}^0_{m,i+j,k+l}, \\
\boldsymbol{R}^c_{ijklm} &= \boldsymbol{R}^1_{i+j,k+l,m} - \boldsymbol{R}^0_{i,k,j+l+m};
\end{aligned} \tag{24}$$

and $\boldsymbol{R}^d$, $\boldsymbol{R}^e$ and $\boldsymbol{R}^f$ may be obtained from $\boldsymbol{R}^a$, $\boldsymbol{R}^b$ and $\boldsymbol{R}^c$ respectively by exchanging indices 0 and 1, and the sign, in the expressions in (24). Thus,

$$\begin{aligned}
\boldsymbol{R}^d_{ijklm} &= \boldsymbol{R}^1_{i,j+l+m,k} - \boldsymbol{R}^0_{i+j,m,k+l} \\
\boldsymbol{R}^e_{ijklm} &= \boldsymbol{R}^1_{m,i+j,k+l} - \boldsymbol{R}^0_{j+l+m,i,k} \\
\boldsymbol{R}^f_{ijklm} &= \boldsymbol{R}^1_{i,k,j+l+m} - \boldsymbol{R}^0_{i+j,k+l,m}.
\end{aligned} \tag{25}$$

From conditions (23)-(25), we also conclude that for all indices

$$\boldsymbol{R}^a_{00k0m} = \boldsymbol{R}^d_{00k0m} = \boldsymbol{R}^b_{i0k00} = \boldsymbol{R}^e_{i0k00} = \boldsymbol{R}^c_{00k0m} = \boldsymbol{R}^f_{00k0m} = \boldsymbol{0}. \tag{26}$$

A point $(\boldsymbol{P}_0, \boldsymbol{P}_1) = \boldsymbol{P}$ in one of the subsimplices $T_a, \ldots, T_f$ will be expressed in terms of the barycentric coordinates (v, w, x, y, z) introduced above. For T_a,

$$\boldsymbol{R}(\boldsymbol{P}_0, \boldsymbol{P}_1) = C^n(v,w,x,y,z)\boldsymbol{Q}_a.$$

It can then be shown [6, Part II] that

$$\begin{aligned}
\boldsymbol{S}_a(\boldsymbol{P}_0, \boldsymbol{P}_1) = \frac{1}{n}\left[\frac{1}{r_1 + (t_1 - t_0)}\right] \boldsymbol{R}(\boldsymbol{P}_0, \boldsymbol{P}_1) = \\
\frac{1}{n}\left[\sum_{k=0}^{n-1} C^{n-1-k}(v,w,x,y,z) C^k(0,0,x,0,1-x)\right] \\
[\overline{v}C(1,0,0,0,-1) + \overline{w}C(0,1,0,0,-1) + \overline{y}C(0,0,0,1,-1)]\, \boldsymbol{Q}_a,
\end{aligned} \tag{27}$$

where $\overline{v} = v/(v+w+y)$, $\overline{w} = w/(v+w+y)$, and $\overline{y} = y/(v+w+y)$.

Similarly, for T_b, with $r_1 > 0$, $v = s_1$, $w = s_0 - s_1$, $x = t_1$, $y = t_0 - t_1$, $z = r_0$, and $\overline{w} = w/(w+y+z)$, $\overline{y} = y/(w+y+z)$, $\overline{z} = z/(w+y+z)$, we obtain

$$\boldsymbol{S}_b(\boldsymbol{P}_0, \boldsymbol{P}_1) = \frac{1}{nr_1}\boldsymbol{R}(\boldsymbol{P}_0, \boldsymbol{P}_1) = \frac{1}{n}\left[\sum_{k=0}^{n-1} C^{n-1-k}(v,w,x,y,z)C^k(1-x,0,x,0,0)\right] \left[\overline{w}C(-1,1,0,0,0) + \overline{y}C(-1,0,0,1,0) + \overline{z}C(-1,0,0,0,1)\right]\boldsymbol{Q}_b. \quad (28)$$

Then, expressions for $\boldsymbol{S}_c$, $\boldsymbol{S}_d$, $\boldsymbol{S}_e$ and $\boldsymbol{S}_f$ corresponding to T_c, T_d, T_e and T_f are obtained from $\boldsymbol{S}_a$ and $\boldsymbol{S}_b$ by permutation of variables and indices.

In analogy with previous results, we define

$$\begin{aligned}\boldsymbol{q}_a &= (C(1,0,0,0,-1)\boldsymbol{Q}_a) \cup (C(0,1,0,0,-1)\boldsymbol{Q}_a) \cup (C(0,0,0,1,-1)\boldsymbol{Q}_a),\\ \boldsymbol{q}_b &= (C(-1,1,0,0,0)\boldsymbol{Q}_b) \cup (C(-1,0,0,1,0)\boldsymbol{Q}_b) \cup (C(-1,0,0,0,1)\boldsymbol{Q}_b).\end{aligned}$$

Similarly, we may find expressions for $\boldsymbol{q}_c$, $\boldsymbol{q}_d$, $\boldsymbol{q}_e$ and $\boldsymbol{q}_f$ corresponding to the functions $\boldsymbol{S}_c$, $\boldsymbol{S}_d$, $\boldsymbol{S}_e$ and $\boldsymbol{S}_f$. Let $\boldsymbol{q} = (\boldsymbol{q}_a, \boldsymbol{q}_b, \boldsymbol{q}_c, \boldsymbol{q}_d, \boldsymbol{q}_e, \boldsymbol{q}_f)$ and

$$d_i = d_i(\boldsymbol{q}_i) = \inf\{|\boldsymbol{S}_i(\boldsymbol{P}_0, \boldsymbol{P}_1)| : r_0 > 0, r_1 > 0, (\boldsymbol{P}_0, \boldsymbol{P}_1) \in T_i\}, \quad i = a,b,c,d,e,f,$$

$$d = d(\boldsymbol{q}) = \min_{i=a,b,c,d,e,f} d_i(\boldsymbol{q}_i). \quad (29)$$

Criterion 3.2.1 *A necessary and sufficient condition for non-intersection of the component patches* $\boldsymbol{R}^0(\boldsymbol{P}_0)$ *and* $\boldsymbol{R}^1(\boldsymbol{P}_1)$ *outside their common boundary curve is that* $d(\boldsymbol{q}) > 0$.

We also get the following lemma, which is used later.

Lemma 3.2.1 *For the functions* $\boldsymbol{S}_i$, $i = a,b,c,d,e$ *and* f, *we have*

$$range(\boldsymbol{S}_i) \subseteq conv(\boldsymbol{q}_i).$$

To establish simpler criteria we consider the reduced control polygons

$$\boldsymbol{Q}'_\nu = \{\boldsymbol{R}^\nu_{ijklm} \in \boldsymbol{Q}_\nu : k+m < n\}, \quad \nu = a, c, d, f, \tag{30}$$

$$\boldsymbol{Q}'_\nu = \{\boldsymbol{R}^\nu_{ijklm} \in \boldsymbol{Q}_\nu : i+k < n\}, \quad \nu = b, e, \tag{31}$$

and define

$$\begin{aligned} d^*_\nu &= d^*_\nu(\boldsymbol{Q}^0, \boldsymbol{Q}^1) = dist(\boldsymbol{0}, conv(\boldsymbol{Q}'_\nu)), \\ d^* &= d^*(\boldsymbol{Q}^0, \boldsymbol{Q}^1) = \min\{d^*_\nu : \nu = a, b, c, d, e, f\}. \end{aligned} \tag{32}$$

Also, for given unit vectors $\boldsymbol{n}_\nu$ we define, for $\nu = a, b, c, d, e, f$,

$$\begin{aligned} &d^{**}_\nu(\boldsymbol{n}_\nu, \boldsymbol{Q}^0, \boldsymbol{Q}^1) = \min\{\boldsymbol{x} \cdot \boldsymbol{n}_\nu : \boldsymbol{x} \in \boldsymbol{Q}'_\nu\}, \\ &\boldsymbol{n} = (\boldsymbol{n}_a, \boldsymbol{n}_b, \boldsymbol{n}_c, \boldsymbol{n}_d, \boldsymbol{n}_e, \boldsymbol{n}_f) \\ &d^{**} = d^{**}(\boldsymbol{n}, \boldsymbol{Q}^0, \boldsymbol{Q}^1) = \min\{d^{**}_a, d^{**}_b, d^{**}_c, d^{**}_d, d^{**}_e, d^{**}_f\}. \end{aligned} \tag{33}$$

Criterion 3.2.1* *A sufficient condition precluding intersection of component patches outside their common boundary curve is that* $d^*(\boldsymbol{Q}^0, \boldsymbol{Q}^1) > 0$*, or, equivalently, that there exist unit vectors* $\boldsymbol{n}_a, \dots, \boldsymbol{n}_f$ *such that* $d^{**}(\boldsymbol{n}, \boldsymbol{Q}^0, \boldsymbol{Q}^1) > 0$.

3.2.2 Two-component patches with a common vertex

Assume given two n-degree patches with control polygons $\boldsymbol{Q}^0 = \{\boldsymbol{R}^0_{ijk}\}_{i+j+k=n}$ and $\boldsymbol{Q}^1 = \{\boldsymbol{R}^1_{ijk}\}_{i+j+k=n}$ which intersect in the vertex defined by $r = 1$, *i.e.*,

$$\boldsymbol{R}^0_{n00} = \boldsymbol{R}^1_{n00}. \tag{34}$$

Suppose that we have an intersection, so that $\boldsymbol{R}^1(\boldsymbol{P}_1) = C^n(\boldsymbol{P}_1)\boldsymbol{Q}^1 = \boldsymbol{R}^0(\boldsymbol{P}_0) = C^n(\boldsymbol{P}_0)\boldsymbol{Q}^0$ with $r_0 < 1$ or $r_1 < 1$, corresponding to Case 1 (mentioned at the beginning of this section). We now decompose $\mathcal{U} \times \mathcal{U}$ as before into the subsimplices $T_a, T_b, \dots, T_f$. By (24), (25) and (34) we get

$$\boldsymbol{R}^a_{n0000} = \boldsymbol{R}^b_{0000n} = \boldsymbol{R}^c_{n0000} = \boldsymbol{R}^d_{n0000} = \boldsymbol{R}^e_{0000n} = \boldsymbol{R}^f_{n0000} = \boldsymbol{0}.$$

First we consider $\boldsymbol{R}(\boldsymbol{P}_0, \boldsymbol{P}_1) = C^n(v, w, x, y, z)\boldsymbol{Q}_a$, where $\boldsymbol{R}(\boldsymbol{P}_0, \boldsymbol{P}_1) = \boldsymbol{R}^1(\boldsymbol{P}_1) - \boldsymbol{R}^0(\boldsymbol{P}_0)$. By (34) we have $C^n(1,0,0,0,0)\boldsymbol{Q}_a = \boldsymbol{0}$. Therefore we conclude that

$$\boldsymbol{R}(\boldsymbol{P}_0, \boldsymbol{P}_1) = [C^n(v, w, x, y, z) - C^n(1,0,0,0,0)]\boldsymbol{Q}_a.$$

Introducing $\overline{w} = w/(w+x+y+z)$, $\overline{x} = x/(w+x+y+z)$, $\overline{y} = y/(w+x+y+z)$, $\overline{z} = z/(w+x+y+z)$, we have, with a development analogous to that for components sharing a common curve:

$$
\begin{aligned}
\boldsymbol{S}_a(\boldsymbol{P}_0, \boldsymbol{P}_1) &= \frac{1}{n(1-r_0)}\boldsymbol{R}(\boldsymbol{P}_0, \boldsymbol{P}_1) = \\
&\frac{1}{n}\left[\sum_{k=0}^{n-1} C^{n-1-k}(v,w,x,y,z)C^k(1,0,0,0,0)\right] \\
&[\overline{w}C(-1,1,0,0,0) + \overline{x}C(-1,0,1,0,0) + \overline{y}C(-1,0,0,1,0) + \\
&\qquad \overline{z}C(-1,0,0,0,1)]\boldsymbol{Q}_a. \qquad (35)
\end{aligned}
$$

Similarly we define the functions

$$\boldsymbol{S}_i(\boldsymbol{P}_0, \boldsymbol{P}_1) = \tfrac{1}{n(1-r_0)}\boldsymbol{R}(\boldsymbol{P}_0, \boldsymbol{P}_1) \qquad (36)$$

on T_i, $i = b, c, d, e, f$. We define $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$, respectively, by

$$
\begin{aligned}
&C(-1,1,0,0,0)\boldsymbol{Q}_a \cup C(-1,0,1,0,0)\boldsymbol{Q}_a \cup C(-1,0,0,1,0)\boldsymbol{Q}_a \cup C(-1,0,0,0,1)\boldsymbol{Q}_a \\
&C(1,0,0,0,-1)\boldsymbol{Q}_b \cup C(0,1,0,0,-1)\boldsymbol{Q}_b \cup C(0,0,1,0,-1)\boldsymbol{Q}_b \cup C(0,0,0,1,-1)\boldsymbol{Q}_b
\end{aligned}
$$

with similar notation for $\boldsymbol{q}_c, \boldsymbol{q}_d, \boldsymbol{q}_e$ and $\boldsymbol{q}_f$. Also, for $i = a, b, c, d, e, f$, define

$$
\begin{aligned}
d_i &= d_i(\boldsymbol{q}_i) = \inf\{|\boldsymbol{S}_i(\boldsymbol{P}_0, \boldsymbol{P}_1)| : r_0 < 1 \text{ or } r_1 < 1, (\boldsymbol{P}_0, \boldsymbol{P}_1) \in T_i\}, \\
d &= d(\boldsymbol{q}) = \min\nolimits_{i=a,b,c,d,e,f} d_i(\boldsymbol{q}_i).
\end{aligned} \qquad (37)
$$

Criterion 3.2.2 *A necessary and sufficient condition for non-intersection of the component patches $\boldsymbol{R}^0(\boldsymbol{P}_0)$ and $\boldsymbol{R}^1(\boldsymbol{P}_1)$ outside their common vertex is that $d(\boldsymbol{q}) > 0$.*

Lemma 3.2.2 *For the functions $\boldsymbol{S}_i$, $i = a, b, c, d, e, f$, $range(\boldsymbol{S}_i) \subseteq conv(\boldsymbol{q}_i)$.*

We again define reduced control polygons

$$\boldsymbol{Q}'_\nu = \{\boldsymbol{R}^\nu_{ijklm} \in \boldsymbol{Q}_\nu : i < n\}, \quad \nu = a, c, d, f,$$

$$\boldsymbol{Q}'_\nu = \{\boldsymbol{R}^\nu_{ijklm} \in \boldsymbol{Q}_\nu : m < n\}, \quad \nu = b, e.$$

We also take

$$d^*_\nu = d_\nu(\boldsymbol{Q}^0, \boldsymbol{Q}^1) = dist(\boldsymbol{0}, conv(\boldsymbol{Q}'_\nu)),$$

$$d^* = d^*(\boldsymbol{Q}^0, \boldsymbol{Q}^1) = \min\{d^*_\nu : \nu = a, b, c, d, e, f\},$$

and, for a given unit vector $\boldsymbol{n}_\nu$, $\nu = a, b, c, d, e, f$,

$$\begin{aligned} &d^{**}_\nu(\boldsymbol{n}_\nu, \boldsymbol{Q}^0, \boldsymbol{Q}^1) = \min\{\boldsymbol{x} \cdot \boldsymbol{n}_\nu : \boldsymbol{x} \in \boldsymbol{Q}'_\nu\} \\ &\boldsymbol{n} = (\boldsymbol{n}_a, \boldsymbol{n}_b, \boldsymbol{n}_c, \boldsymbol{n}_d, \boldsymbol{n}_e, \boldsymbol{n}_f) \\ &d^{**} = d^{**}(\boldsymbol{n}, \boldsymbol{Q}^0, \boldsymbol{Q}^1) = \min\{d^{**}_a, d^{**}_b, d^{**}_c, d^{**}_d, d^{**}_e, d^{**}_f\}. \end{aligned}$$

Criterion 3.2.2 * *A sufficient condition precluding intersection of component patches outside their common vertex is that* $d^*(\boldsymbol{Q}^0, \boldsymbol{Q}^1) > 0$*, or, equivalently, that there exist* $\boldsymbol{n}_a, \ldots, \boldsymbol{n}_f$ *such that* $d^{**}(\boldsymbol{n}, \boldsymbol{Q}^0, \boldsymbol{Q}^1) > 0$.

4 Perturbation analyses

As observed in Section 1, one of the main motivations for the results of this paper is to ensure preservation of the well-formed nature of objects defined in terms of Bézier curves or patches. Given that there are no self-intersections of composite curves or patches, the criteria in Sections 2 and 3 can be transformed, by means of the triangle inequality, into perturbation results, *i.e.*, we can give the maximum control-point perturbations that can be tolerated without risk of introducing self-intersections.

The theorem numbers in this section are determined by the corresponding criterion. We give a proof only for the first theorem; the others are similar [6].

4.1 Avoiding self-intersections when perturbing curves

Let an n-degree Bézier curve $\boldsymbol{R}(t)$, as in (1), be given. Assume that $\boldsymbol{R}$ is non-self-intersecting, *i.e.*, that $d(\boldsymbol{q})$, given by (4), is positive. If we perturb $\boldsymbol{Q}$, replacing it by $\boldsymbol{Q} + \boldsymbol{\delta Q} = \{\boldsymbol{R}_i + \boldsymbol{\delta R}_i\}_{i=0}^n$, then we may ask whether the perturbed curve remains non-self-intersecting. Let $\boldsymbol{q} + \boldsymbol{\delta q} = \{\boldsymbol{r}_i + \boldsymbol{\delta r}_i\}_{i=0}^{n-1}$ be the corresponding perturbed sequence of forward differences, and $|\boldsymbol{q}| = \max\{|\boldsymbol{r}_i| : \boldsymbol{r}_i \in \boldsymbol{q}\}$.

Theorem 2.1 *If* $d(\boldsymbol{q}) > 0$ *and* $|\boldsymbol{\delta q}| < d(\boldsymbol{q})$ *then the perturbed curve remains non-self-intersecting.*

Theorem 2.1* *If* $d^*(\boldsymbol{q}) > 0$ *and if* $|\boldsymbol{\delta q}| < d^*(\boldsymbol{q})$*, then the perturbed curve remains non-self-intersecting. Also, if there exists* $\boldsymbol{n}$ *such that* $d^{**}(\boldsymbol{n}, \boldsymbol{q}) > 0$*, and if* $d^{**}(\boldsymbol{n}, \boldsymbol{\delta q}) + d^{**}(\boldsymbol{n}, \boldsymbol{q}) > 0$*, then the perturbed curve remains non-self-intersecting.*

Proof of Theorem 2.1. Let $\boldsymbol{\delta S}(s,t)$ be defined by (6) with $\boldsymbol{q}$ replaced by $\boldsymbol{\delta q}$. By definition we have $|\boldsymbol{S}(s,t)| \geq d(\boldsymbol{q}) > 0$. Further, by Lemma 2.1, which applies also to $\boldsymbol{\delta q}$ and $\boldsymbol{\delta S}$, $range(\boldsymbol{\delta S}) \subseteq conv(\boldsymbol{\delta q})$, which implies that $|\boldsymbol{\delta S}(s,t)| \leq |\boldsymbol{\delta q}|$. Now, by the triangle inequality,

$$|\boldsymbol{S}(s,t) + \boldsymbol{\delta S}(s,t)| \geq |\boldsymbol{S}(s,t)| - |\boldsymbol{\delta S}(s,t)| \geq d(\boldsymbol{q}) - |\boldsymbol{\delta q}| > 0,$$

$$i.e., \quad d(\boldsymbol{q} + \boldsymbol{\delta q}) \geq d(\boldsymbol{q}) - |\boldsymbol{\delta q}| > 0,$$

and, by Criterion 2.1, the proof is complete. □

Remark. Simililary to Criterion 2.1*, Theorem 2.1* generalizes to the case of rational Bézier curves.

We assume now that two n-degree Bézier curves with control polygons $\boldsymbol{Q}^0$ and $\boldsymbol{Q}^1$ are given and that they are non-intersecting outside their common point, *i.e.*, that $d(\boldsymbol{Q}^0, \boldsymbol{Q}^1)$ defined in (18) is positive. We perturb $\boldsymbol{Q}^0$ and $\boldsymbol{Q}^1$ by $\boldsymbol{\delta Q}^0$ and $\boldsymbol{\delta Q}^1$ respectively, retaining the condition of a common initial point.

Theorem 2.2 *If* $|\boldsymbol{\delta q}_i| < d_i(\boldsymbol{Q}^0, \boldsymbol{Q}^1)$ *for* $i = a$ *and* b*, then the perturbed component curves remain non-intersecting outside the initial point.*

Theorem 2.2* *If* $d_i^*(\boldsymbol{Q}^0, \boldsymbol{Q}^1) > 0$ *and* $|\boldsymbol{\delta Q}'_i| < d_i^*(\boldsymbol{Q}^0, \boldsymbol{Q}^1)$ *for* $i = a$ *and* b*, then the perturbed component curves remain non-intersecting outside the initial point. Also, if* $\boldsymbol{n}_a$ *and* $\boldsymbol{n}_b$ *are unit vectors such that* $d_i^{**}(\boldsymbol{n}_i, \boldsymbol{Q}^0, \boldsymbol{Q}^1) > 0$ *and* $d_i^{**}(\boldsymbol{n}_i, \boldsymbol{\delta Q}^0, \boldsymbol{\delta Q}^1) + d_i^{**}(\boldsymbol{n}_i, \boldsymbol{Q}^0, \boldsymbol{Q}^1) > 0$ *for* $i = a$ *and* b*, then the same is true.*

4.2 Avoiding self-intersections when perturbing patches

We now state, for patches, theorems analogous to those in Subsection 4.1.

Assume that an n-degree triangular Bézier patch is given, with control polygon $\boldsymbol{Q}$, and that it is non-self-intersecting, *i.e.*, that $d(\boldsymbol{q})$ is positive. For a perturbed patch with control polygon $\boldsymbol{Q} + \boldsymbol{\delta Q}$ we have the following results concerning the preservation of non-self-intersection.

Theorem 3.1 *If $d(\boldsymbol{q}) > 0$ and if $|\boldsymbol{\delta q}_a| < d_a(\boldsymbol{q}_a)$, $|\boldsymbol{\delta q}_b| < d_b(\boldsymbol{q}_b)$, $|\boldsymbol{\delta q}_c| < d_c(\boldsymbol{q}_c)$, then the perturbed patch remains non-self-intersecting.*

Theorem 3.1* *If $d^*(\boldsymbol{q}) > 0$ and if $|\boldsymbol{\delta q}_a| < d^*_a(\boldsymbol{q}_a)$, $|\boldsymbol{\delta q}_b| < d^*_b(\boldsymbol{q}_b)$, $|\boldsymbol{\delta q}_c| < d^*_c(\boldsymbol{q}_c)$, then the perturbed patch remains non-self-intersecting. Further, if $d^{**}(\boldsymbol{n}, \boldsymbol{q}) > 0$ and if $d^{**}(\boldsymbol{n}_a, \boldsymbol{\delta q}_a) + d^{**}(\boldsymbol{n}_a, \boldsymbol{q}_a) > 0$, $d^{**}(\boldsymbol{n}_b, \boldsymbol{\delta q}_b) + d^{**}(\boldsymbol{n}_b, \boldsymbol{q}_b) > 0$, $d^{**}(\boldsymbol{n}_c, \boldsymbol{\delta q}_c) + d^{**}(\boldsymbol{n}_c, \boldsymbol{q}_c) > 0$, then the perturbed patch remains non-self-intersecting.*

Remark. Similary to Theorem 2.1*, Theorem 3.1* generalizes to the case of rational Bézier surfaces.

Now, assume that two n-degree patches with control polygons $\boldsymbol{Q}^0$ and $\boldsymbol{Q}^1$ are given as in Subsection 3.2, without any extraneous intersection. Here, Theorem 3.2 corresponds to both Criterion 3.2.1 and Criterion 3.2.2, and Theorem 3.2* to both Criterion 3.2.1* and Criterion 3.2.2*.

Theorem 3.2 *If $|\boldsymbol{\delta q}_i| < d_i(\boldsymbol{q}_i)$ for $i = a, b, c, d, e, f$, then the component patches remain non-intersecting outside their common boundary curve or vertex.*

Theorem 3.2* *If $d^*_\nu(\boldsymbol{Q}^0, \boldsymbol{Q}^1) > 0$ for $\nu = a, b, c, d, e, f$ and $|\boldsymbol{\delta Q}'_\nu| < d^*_\nu(\boldsymbol{Q}^0, \boldsymbol{Q}^1)$ then the component patches remain non-intersecting outside their common boundary curve or vertex. Further, if $\boldsymbol{n}_\nu$ are unit vectors such that $d^{**}_\nu(\boldsymbol{n}_\nu, \boldsymbol{Q}^0, \boldsymbol{Q}^1) > 0$ and if also $d^{**}_\nu(\boldsymbol{n}_\nu, \boldsymbol{\delta Q}^0, \boldsymbol{\delta Q}^1) + d^{**}_\nu(\boldsymbol{n}_\nu, \boldsymbol{Q}^0, \boldsymbol{Q}^1) > 0$, $\nu = a, b, c, d, e, f$, then the same is true.*

4.3 Preservation of the topological form of solids

As mentioned in the Introduction, it is of interest in many areas (including solid modeling) to give conditions under which a finite curvilinear simplicial complex, made up of properly-joined parametric patches defined in terms of control points, will retain its original topological form when the control points are perturbed. These applications are described in more detail in [19].

Sufficient conditions are given in [6, Part III] to guarantee that topological form is preserved, where this is interpreted to mean that there is a homeomorphism, defined on the whole ambient space R^3, mapping the object onto the perturbed object. It is shown in [6, Part III] that the main conditions to be satisfied are that the original object should be continuously perturbed in a way that introduces no self-intersection of one- or two-component composite patches, and no intersections of disjoint patches. Thus, the perturbation theory above can be applied directly, along with classical results ensuring that disjoint patches do not intersect after perturbation, to guarantee preservation of topological form.

4.4 Numerical example

We present an example illustrating Criterion 3.2.1*, a sufficient condition precluding self-intersection of a two-component patch, and the corresponding Perturbation Theorem (Theorem 3.2*). The two components coincide along the boundary curve $r = 0$. We have written a computer program[6] implementing the necessary calculations, as well as a program corresponding to Criterion 3.2.2* (self-intersections of composite patches formed by patches coinciding at the corner point $r = 1$).

The two degree-3 patches used here have been chosen to provide a transparent example. We define $\boldsymbol{\rho} = (1, 0, 0)$, $\boldsymbol{\sigma} = (0, 1, 0)$ and $\boldsymbol{\tau} = (0, 0, 0)$, so that the parametric domain is the triangle $\{(r, s) : 0 \leq s \leq 1 - r \leq 1\}$, with $t = 1 - r - s$. The control vectors $\boldsymbol{R}^0_{i,j,3-i-j}$ for $0 \leq j \leq 3 - i \leq 3$, are shown in Figure 5. The

[6] Available by anonymous ftp at ftp.iro.umontreal.ca, directory pub/vision/softs/stewart.

control vectors $\boldsymbol{R}^1_{i,j,3-i-j}$ are identical, except in the z-component, which is .5 for $i = 1$, .75 for $i = 2$, and 1 for $i = 3$. The two patches coincide along the segment $\{(0, y, 0) : 0 \le y \le 3\}$ in R^3.

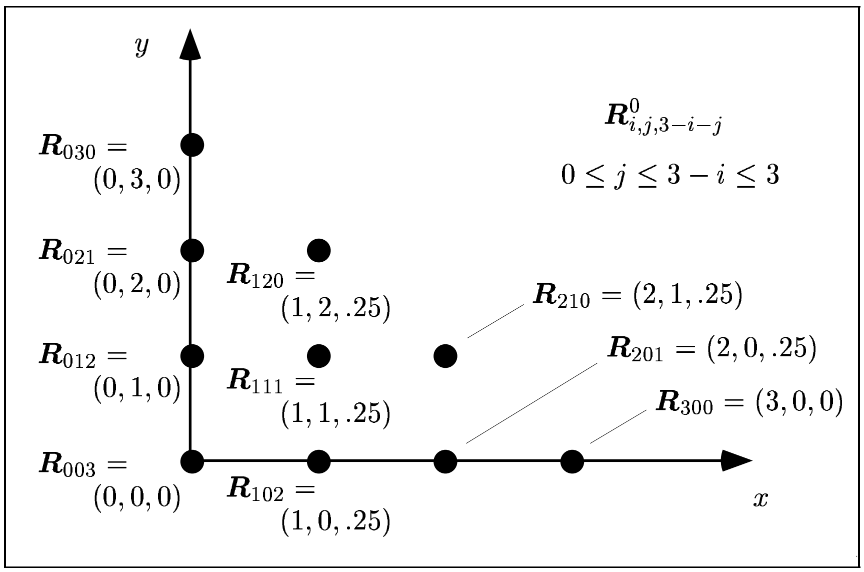

Figure 5: Control vectors for Patch 0

As is easily verified, each of the arrays $\boldsymbol{Q}'_a$, $\boldsymbol{Q}'_b$, $\boldsymbol{Q}'_c$, $\boldsymbol{Q}'_d$, $\boldsymbol{Q}'_e$, and $\boldsymbol{Q}'_f$ defined by (30) and (31) contains exactly 31 vectors. Thus, the vectors in each array can be stored as a linear sequence of 31 vectors and their values calculated inside four nested **for**-loops with indices

$$\begin{array}{ccccc} 0 & \le & i & \le & 3 \\ 0 & \le & j & \le & 3 - i \\ 0 & \le & l & \le & 3 - i - j \\ 0 & \le & m & \le & 3 - i - j - l. \end{array}$$

Using (24) we find for example that all 31 vectors in $\boldsymbol{Q}'_a$ are in the orthant $\{(x, y, z) : x \ge 0, y \le 0, z \ge 0\}$, and that $d^*_a = dist(\boldsymbol{0}, conv(\boldsymbol{Q}'_a)) \cong 0.2425$. Note that it is not necessary to compute an explicit representation of $conv(\boldsymbol{Q}'_a)$ to calculate the distance from $\mathbf{0} \in R^3$ to the convex hull.

The results are similar for the other arrays $\boldsymbol{Q}'_b$, $\boldsymbol{Q}'_c$, $\boldsymbol{Q}'_d$, $\boldsymbol{Q}'_e$, and $\boldsymbol{Q}'_f$. The minimal distances in each case are

$$d_a^* = d_c^* \cong 0.243, \quad d_b^* = 0.25, \quad d_d^* \cong 0.237, \quad d_e^* \cong 0.223, \quad d_f^* \cong 0.218$$

and consequently $d^* \cong 0.218$. It follows by Criterion 3.2.1* that the composite patch defined by $\boldsymbol{Q}^0 = \{\boldsymbol{R}^0_{ijk}\}$ and $\boldsymbol{Q}^1 = \{\boldsymbol{R}^1_{ijk}\}$ has no self-intersection. Furthermore, by Theorem 3.2*, the vectors in, say, $\boldsymbol{Q}^1 \setminus \{\boldsymbol{R}_{0,j,3-j} : j = 0,1,2,3\}$ can be perturbed by 0.218 without risk of a self-intersection. This shows that for this example, the bound provided by Theorem 3.2* is quite sharp. For example, the edges corresponding to $s = 0$, of the two patches, are Bézier curves with parameter $r \in [0,1]$ having tangent $\boldsymbol{R}^1_{102} - \boldsymbol{R}^1_{003}$ and $\boldsymbol{R}^0_{102} - \boldsymbol{R}^0_{003}$ respectively at $r = 0$. Consequently, if the perturbation $(0,0,-0.25)$ is added to $\boldsymbol{R}^1_{102}$, there is a self-intersection (a tangential intersection at $r = 0$), since $\boldsymbol{R}^1_{102} + (0,0,-0.25) - \boldsymbol{R}^1_{102} = \boldsymbol{R}^0_{102} - \boldsymbol{R}^0_{003}$.

The computations necessary for the application of Criterion 3.2.2* are similar: in this case, each of the sets $\boldsymbol{Q}'_a, \boldsymbol{Q}'_b, \ldots, \boldsymbol{Q}'_f$ contains 34 vectors.

5 Conclusion

We have given conditions precluding self-intersection of composite Bézier curves and composite Bézier patches. In each case we first gave necessary and sufficient conditions, followed by more tractable sufficient conditions. These were followed in turn by a perturbation analysis. These non-intersection results included, as a special case, detection of cusps, singularities, and tangential intersections. We also described how these results can be used to guarantee that the topological form of a curvilinear simplicial complex will be preserved under perturbation of its defining data, and finally, we presented a numerical example.

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