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Perturbing-Non-Bézier Curves

Modifications

1 Self-intersection of Bézier curves

1.1 Self-intersection of single-component curves

Let $\boldsymbol{Q} = \{\boldsymbol{R}_0, \boldsymbol{R}_1, \ldots, \boldsymbol{R}_n\}$ denote a sequence of vectors in R^3, which is the control polygon for an n-degree, parametric polynomial Bézier curve [Farin93]. We introduce the operator $C(t)$ operating on the sequence $\boldsymbol{Q}$ and defined as follows:

$$C(t)\{\boldsymbol{R}_i\}_{i=0}^{n} = ((1-t)E_0 + tE_1)\{\boldsymbol{R}_i\}_{i=0}^{n} = \{(1-t)\boldsymbol{R}_i + t\boldsymbol{R}_{i+1}\}_{i=0}^{n-1}.$$

Here E_1 denotes the forward-shift operator, and E_0 the identity operator. Now, the n-degree Bézier curve having control polygon $\boldsymbol{Q}$ is given by

$$\boldsymbol{R}(t) = C^n(t)\boldsymbol{Q} = ((1-t)E_0 + tE_1)^n\boldsymbol{Q} = \left[\sum_{i=0}^{n}\binom{n}{i}(1-t)^{n-i}t^iE_i\right]\boldsymbol{Q}, \;\; t \in [0,1],$$

where $E_i = E_1^i$; consequently

$$\boldsymbol{R}(t) = \sum_{i=0}^{n}\binom{n}{i}(1-t)^{n-i}t^i\boldsymbol{R}_i. \tag{1}$$

Now, consider two points on the Bézier curve given by (1), denoted by $\boldsymbol{R}(1-s)$

and $\boldsymbol{R}(t)$ with $0 \leq t < 1 - s \leq 1$, and form the difference quotient

$$\boldsymbol{S}(s,t) = \frac{1}{n}\frac{\boldsymbol{R}(1-s) - \boldsymbol{R}(t)}{(1-s) - t}. \tag{2}$$

Then, by the Remainder Theorem, $\boldsymbol{S}(s,t)$ is a vector-valued *polynomial*, well-defined for $1 - s = t$ by $\boldsymbol{S}(1-t,t) = \frac{1}{n}\dot{\boldsymbol{R}}(t)$, where the dot denotes differentiation with respect to t.

TJP: Maybe Remainder Theorem is of no use in non-Bézier case. In general, indicated quotient won't be a polynomial.

Thus, $\boldsymbol{S}(s,t)$ is defined over the triangle

$$\mathcal{T} = \{(s,t) : s + t \leq 1,\ s,t \geq 0\}. \tag{3}$$

We will say that $\boldsymbol{R}(t)$ is *self-intersecting* if for some $t_0, t_1 \in [0,1]$, with $t_0 < t_1$ we have $\boldsymbol{R}(t_0) = \boldsymbol{R}(t_1)$, or if for some $t \in [0,1]$ we have $\dot{\boldsymbol{R}}(t) = \mathbf{0}$. Points where $\dot{\boldsymbol{R}}(t) = \mathbf{0}$ are called critical points; they are limiting cases of genuinely self-intersecting curves, and are included among the self-intersecting curves. Let

$$d = \min\left\{|\boldsymbol{S}(s,t)| : (s,t) \in \mathcal{T}\right\}, \tag{4}$$

where $|\cdot|$ denotes the Euclidean norm on R^3. Then, we have:

/bf Criterion 1 A necessary and sufficient condition that the Bézier curve $\boldsymbol{R}(t)$ is not self-intersecting is that $d > 0$.

TJP: May no longer represent a Bézier patch, so we eliminate all the material about its development.

It is clear that $\boldsymbol{S}(s,t)$ is an $(n-1)$-degree, triangular Bézier .... *now gone!!*.

Note that

$$\boldsymbol{R}(1-s) - \boldsymbol{R}(t) = [C^n(1-s) - C^n(t)]\boldsymbol{Q} \tag{5}$$

and that $C^n(1-s) - C^n(t) = [\sum_{i=0}^{n-1} C^{n-1-i}(1-s)C^i(t)][C(1-s) - C(t)]$. Since $C(1-s) - C(t) = (1-s-t)(E_1 - E_0)$, we have

$$\boldsymbol{S}(s,t) = \frac{1}{n}[C^{n-1}(1-s) + C^{n-2}(1-s)C(t) + \ldots + C^{n-1}(t)]\boldsymbol{q} \tag{6}$$

where we have introduced the sequence $\boldsymbol{q} = \{\boldsymbol{r}_i\}_{i=0}^{n-1}$ of differences $\boldsymbol{r}_i = \boldsymbol{R}_{i+1} - \boldsymbol{R}_i$, *i.e.*, $\boldsymbol{q} = (E_1 - E_0)\boldsymbol{Q}$. Thus, $d = d(\boldsymbol{q})$ depends on $\boldsymbol{q}$.

The following lemma, which motivated the introduction of the factor $\frac{1}{n}$ in (2), will be useful later. Here, $conv(\boldsymbol{q})$ denotes the convex hull of $\boldsymbol{q}$.

TJP: Why does this motivate $1/n$?

Lemma 2 For the function $\boldsymbol{S}(s,t)$ we have $range(\boldsymbol{S}) \subseteq conv(\boldsymbol{q})$.

Proof For $0 \leq t \leq 1$, $C(t) = (1-t)E_0 + tE_1$ forms convex combinations of the vectors in the sequence on which it operates. It follows from (6) that for all $(s,t) \in \mathcal{T}$, $\boldsymbol{S}(s,t)$ is a convex combination of the vectors $\{\boldsymbol{r}_i\}_{i=0}^{n-1} = \boldsymbol{q}$.

TJP: Kill all patch development.

Let

$$d^* = d^*(\boldsymbol{q}) = dist(\boldsymbol{0}, conv(\boldsymbol{q})). \tag{7}$$

By Lemma 2 it follows that we have the following simpler criterion.

Criterion 2* A sufficient condition for non-self-intersection of the Bézier curve $\boldsymbol{R}(t)$ is that $d^*(\boldsymbol{q}) > 0$.

Remark. Criterion 2 generalizes to the case of rational Bézier curves, by viewing such a curve as a central projection of a polynomial Bézier curve in R^4. The proof may be based on the Variation Diminishing Property.

TJP: Could we also substitute deBoor algorithm instead of deCastlejau? This would follow aspects mentioned in Farin, I, p. 234, 240, 243, noting on p. 234 that positive weights ensures convex hull property. Also, see p. 157 comparing deBoor and deCasteljau. Do curves generated in G-K satisfy this weight condition? In any case, these are curves in parameter space (from G-K). How do topological conditions in parameter space map to conditions in model space?