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Preserving computational topology by subdivision of quadratic and cubic Bézier curves

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Abstract

Non-self-intersection is both a topological and a geometric property. It is known that non-self-intersecting regular Bézier curves have non-self-intersecting control polygons, *after* sufficiently many uniform subdivisions. Here a sufficient condition is given within $\mathbb{R}^3$ for a non-self-intersecting, regular C^2 cubic Bézier curve to be ambient isotopic to its control polygon formed after sufficiently many subdivisions. The benefit of using the control polygon as an approximant for scientific visualization is presented in this paper.

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1. Introduction

Three results from the literature motivate the new theorem presented here. Repeated subdivision permits arbitrarily close approximation of any spline curve, as is used widely in graphics and other applications [6]. Sufficient conditions are known for a non-self-intersecting Bézier curve to eventually have a non-self-intersecting control polygon under repeated subdivision [15]. For many parametric curves, sufficient conditions are known to construct a piecewise linear (PL) interpolation that is ambient isotopic to the curve [12].

For any spline curve $\mathbf{c}$ of degree n, with control points $\{P_0, \dots, P_n\}$, its *control polygon* is the piecewise linear (PL) curve formed by connecting the points $\{P_0, \dots, P_n\}$, in order, by line segments, such that P_0 is the initial point and P_n is the final point. If $P_0 \neq P_n$ then the control polygon is an open curve, otherwise it is a closed curve. Let $CP(\mathbf{c})$ denote the control polygon of $\mathbf{c}$. For non-self-intersecting curves, it follows easily [15] that there will eventually exist a subdivision whose control polygon is homeomorphic to the original curve, but the stronger topological equivalence under isotopy is the focus of this paper.

The preference [3] for isotopy appears since an isotopy preserves the embedding of the curve, whereas a homeomorphism may not. For instance, any two knots are homeomorphic but different knots [2] are distinguished by their embeddings within $\mathbb{R}^3$. The definition of an ambient isotopy within $\mathbb{R}^n$ follows [9].

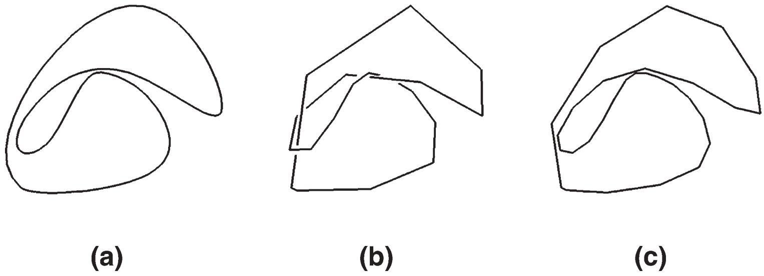

Fig. 1a. The unknot. (**b**) The figure-8 knot. (**c**) Good approximation

Definition 1.1: *Let X and Y be two subspaces of* $\mathbb{R}^n$. *Then there is an* **ambient isotopy**, *H between X and Y if there exists a continuous function* $H : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ *with the following conditions:*

1. $H(\cdot, 0)$ *is the identity,*
2. $H(X, 1) = Y$, *and*
3. $\forall t \in [0, 1]$, $H(\cdot, t)$ *is a homeomorphism from* $\mathbb{R}^n$ *onto* $\mathbb{R}^n$.

The sets X and Y are then said to be **ambient isotopic**.

Summarizing previous work [3], the space curve in Fig. 1a is equivalent to the unknot. An undesirable approximation, shown in Fig. 1b, is the figure-8-knot. This crude approximation would not be acceptable for most visualization applications, but the more refined ambient isotopic approximation of Fig. 1c would. Using this as a motivating example, the main theorem presented in Sect. 3 gives sufficient conditions to obtain an ambient isotopic approximation by repeated subdivision.

2. Background and motivation

A previous result [12] constructed an ambient isotopic PL approximation to a spline curve via interpolant points. The extension here is to show that uniform subdivision will generate a control polygon that is also isotopic to the curve. This may facilitate integration with graphics and visualization applications. That previous result is given, followed by definitions that are used in the rest of the development[1].

Theorem 2.1: *Let* **c** *be a regular non-self-intersecting* C^2 *Bézier curve in* $\mathbb{R}^3$. *Then there exists a PL interpolation of* **c** *that is ambient isotopic to* **c**. *(Note that* **c** *may be open or closed.)*

[1] The use of C^2 here is in the classical mathematical sense [9], referring to *all* points of *c*, as distinguished from the emphasis sometimes understood in more applied contexts of focusing attention *only* upon the junction points between spline pieces.

Definition 2.1: *A function f, from the space X to X has* **compact support** *if there is a compact set $K \subseteq X$ such that for any $x \in X - K$, $f(x) = x$. Then, K is called a* **set of compact support** *for f.*

Pipe surfaces have been studied since the 19th century [13], but the presentation here follows a contemporary source [12].

Definition 2.2: *The* **pipe surface** *of radius r of a parameterized curve $\mathbf{c}(t)$, where $t \in [0, 1]$ is given by*

$$\mathbf{p}(t, \theta) = \mathbf{c}(t) + r[cos(\theta)\mathbf{n}(t) + sin(\theta)\mathbf{b}(t)],$$

where $\theta \in [0, 2\pi]$ and $\mathbf{n}(t)$ and $\mathbf{b}(t)$ are, respectively, the normal and bi-normal vectors at the point $\mathbf{c}(t)$, as given by the Frenet-Serret trihedron.

Then the previous ambient isotopic PL approximation proof follows two key steps:

(1) Determine a radius r such that $\mathbf{p}(t, \theta)$ is non-self-intersecting.
(2) Select points $\{q_0, q_1, \ldots, q_n\}$ on $\mathbf{c}$ such that each line segment $[q_i, q_{i+1}]$ lies within r of $\mathbf{c}$. Connect them as a polyline ambient isotopic to $\mathbf{c}$.

Considering Fig. 2, let $\mathbf{c}_i$ be the segment of $\mathbf{c}$ between q_i and q_{i+1}, let D_i be the closed disc of radius r normal to $\mathbf{c}$ at q_i and let $\mathbf{p}_i$ be the section of the pipe surface between D_i and D_{i+1}. Then, the above proof constructs local ambient isotopies from $[q_i, q_{i+1}]$ to $\mathbf{c}_i$ for each i, such that each local isotopy has a set of compact support bounded by

$$D_i \cup \mathbf{p}_i \cup D_{i+1}.$$

Each D_i remains fixed under the local isotopies, so this proof [12] is completed by invoking the following "folk lemma", whose proof is elementary.

Lemma 2.1: *For $n \geq 0$, let F be an ambient isotopy defined on $\mathbb{R}^n \times [0, 1]$ onto $\mathbb{R}^n$ so that subsets A and B of $\mathbb{R}^n$ are ambient isotopic under F. Similarly, let G be an*

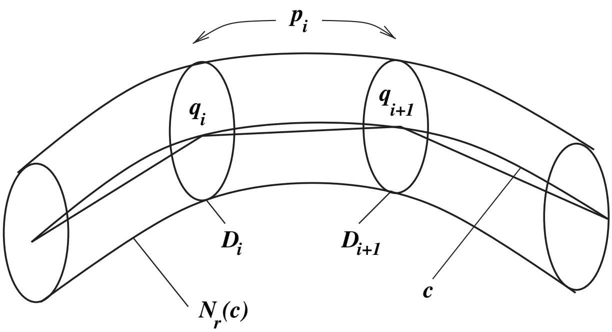

Fig. 2. A curve with its pipe surface and PL approximant.

ambient isotopy defined on $\mathbb{R}^n \times [0, 1]$ *onto* $\mathbb{R}^n$ *so that subsets* C *and* D *of* $\mathbb{R}^n$ *are ambient isotopic under* G. *Furthermore, suppose that* F *has compact support* $CS(F)$ *and* G *has compact support* $CS(G)$. *If for each point* $x \in CS(F) \cap CS(G)$, *it is true that* $F(x, t) = G(x, t) = x$, *for all* $t \in [0, 1]$, *then the function*

$$H : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n,$$

defined by

$$H(x, t) = \begin{cases} F(x, t), & x \in CS(F), \\ G(x, t), & x \in CS(G), \\ x, & otherwise, \end{cases}$$

for $t \in [0, 1]$ *is an ambient isotopy with compact support* $CS(F) \cup CS(G)$ *such that* $A \cup C$ *is ambient isotopic to* $B \cup D$ *under* H.

3. Ambient isotopy via subdivision

This section gives the main result of this paper, Theorem 3.1. The curves considered are all assumed to be non-self-intersecting Bézier space curves, as the case of non-self-intersecting planar curves follows easily from combining recent [15] and classical results [7]. The main result relies upon recognizing when a PL knot is the unknot, a problem of considerable continuing theoretical interest [8], [10], but which is fully understood for small PL knots[2].

Lemma 3.1: *Any PL knot with less than six segments must be the unknot.*

The proof of the main theorem also relies on the following lemma showing that the control points of the curve between D_i and D_{i+1} can be assumed to lie within $\mathbf{p}_i$.

Lemma 3.2: *Let* w *be a point where a regular cubic Bézier curve* c *is subdivided. Let* Π *be the plane normal to* c *at* w. *There exists a subdivision of* c *such that the control polygon of the single segment Bézier sub-curve ending at* w *and the control polygon of the single segment Bézier sub-curve beginning at* w *intersect* Π *only in the single point* w.

Proof: Choose a plane Λ that is perpendicular to Π, with $w \in \Lambda$ and consider the projection $\psi : \mathbb{R}^3 \to \Lambda$. Then $\psi(c)$ is a cubic Bézier curve and $\psi(\Pi)$ is a line, denoted as ℓ. The proof focuses upon $\psi(c)$ since a control polygon of $\psi(c)$ will cross ℓ if and only if the corresponding control polygon of c crosses Π. Denote the control points of the sub-curve of $\psi(c)$ beginning at w as b_0, b_1, b_2, b_3, with $b_0 = w$. The point b_1 must lie to one side of ℓ, due to $b_0 b_1$ being collinear with the tangent at b_0. Since $\psi(c)$ is a cubic curve, it can be assumed, without loss of generality, that

[2] These PL knots are also known as 'stick knots', with each stick being one of the linear segments.

sufficiently many subdivisions have been taken so that b_3 lies on the same side of ℓ as b_1. Since this sub-curve is cubic, it can have at most 2 inflection points, so it can be further assumed that additional subdivisions have been taken so that no inflection point can occur except perhaps at $b_0 = w$ and so that this sub-curve is convex. Using arguments similar to those presented elsewhere [18], for the *particular* case of a planar cubic Bézier curve, its convexity implies that its control polygon is also convex. But, if b_2 were to be separated from b_1 and b_3 by ℓ, then ℓ would have non-empty intersection with each of the segments b_0b_1, b_1b_2 and b_2b_3, contradicting the convexity of the control polygon. In the special case where $b_2 \in \ell$, then one more subdivision would cause that relation to fail. Hence, the three points b_1, b_2, b_3 can all be assumed to lie to one side of ℓ, implying that the control points of c all lie to one side of Π. A similar argument shows that the part of c ending at w can also be subdivided so that its control points all lie to the opposite side of Π. □

Theorem 3.1: *Let* $\mathbf{c}$ *be a non-self-intersecting* C^2 *Bézier curve of degree less than 4, having regular Bézier parameterization in* $\mathbb{R}^3$. *Then there exists a positive integer* M*, such that for* $m \geq M$*, the control polygon of the* m*-th subdivision of* $\mathbf{c}$ *is ambient isotopic to* $\mathbf{c}$. (*Note that* $\mathbf{c}$ *may be open or closed.*)

Proof: Since the curve is of degree less than 4, for each subdivision, then the closed polyline formed by $CP(\mathbf{c}_i) \cup [q_i, q_{i+1}]$, has at most 4 linear segments and is unknotted by Lemma 3.1. This unknottedness is a necessary condition for the rest of the proof.

For a *closed* curve $\mathbf{c}$, choose r as defined in Sect. 2 so that its pipe surface of radius r will not have any self-intersections. Then, take sufficiently many subdivisions such that

- the control polygon is contained in $\mathbf{N}_r(\mathbf{c})$ and
- the control polygon is non-self-intersecting [15].

It can be assumed that each control polygon $\mathbf{P}_i \subset \mathbf{p}_i$, with $\mathbf{p}_i$ defined as in Fig. 2. The cases of degree 1 and 2 are trivial and the case of degree 3 is given by Lemma 3.2.

For each $i = 1, \ldots, n$, let V_i be the vertices of $\mathbf{P}_i$ and let U_i be defined by set subtraction as

$$U_i = V_i - \{q_i, q_{i+1}\}.$$

Then pick the point $u \in U_i$ that is farthest away from $\mathbf{c}$ (a tie can be broken by arbitrarily choosing either point). Following previous work on ambient isotopies for polyhedra [4], there exists an isotopy within $\mathbf{p}_i$ that "pushes" u to form a triangle, which leaves vertices q_i and q_{i+1} as fixed points. Then, another "push" within $\mathbf{p}_i$ will produce an isotopy between $[q_i, q_{i+1}]$ and $\mathbf{c}_i$ which also leaves q_i and q_{i+1} as fixed points.

The completion of the proof relies upon Lemma 2.1 to show the existence of an ambient isotopy between the subdivided control polygon for the whole curve $\mathbf{c}$

and the polyline formed from joining the vertices $\{q_0, q_1, \ldots q_{n-1}, q_n\}$, for some appropriate value of n (where explicit *a priori* bounds on n are known [11]), with $q_n = q_0$. However, this union of line segments has already been shown to be ambient isotopic to **c** [12]. Since ambient isotopy defines an equivalence relation, this polyline is also ambient isotopic to **c**. (The minor modifications required to complete the proof for open curves are left as an exercise for the reader.) □

A primary motivation for this work is to perform ambient isotopic approximations in support of scientific visualization of molecular simulations. A design requirement for this visualization is that the graphics have provably the same topology as the spline models undergoing simulation. The theory presented here is responsive to that requirement. Theorem 3.1 gives sufficient conditions for a PL isotopic approximation of a single static curve. This static approximation result is being coupled with a published result on constraints on the perturbation of spline control points to preserve isotopy class [5] into prototype code for scientific visualization [14] with a representative animation available for viewing [16].

This visualization role also validates the focus on approximation by control points. Certainly, interpolant points could also be perturbed for dynamic graphics, but creation of the newly perturbed spline curve would then require the use of an interpolation algorithm, with no known guarantees that the interpolant algorithm will produce a curve that is ambient isotopic to the original curve. The approach presented here specifically avoids that dilemma since the same control points are perturbed simultaneously on the spline representation and on the PL graphical approximation.

4. Conclusion

Sufficient conditions are given for a subdivided control polygon of quadratic or cubic, regular C^2 non-self-intersecting Bézier curves to be *topologically equivalent* to the curve. The equivalence criterion uses ambient isotopy, since ambient isotopy will both yield a homeomorphic approximation and preserve the curve's embedding. Applications to scientific visualization are discussed, which focus upon the desirability of using control points for the approximation, as opposed to approximation by interpolant points.

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