Polyhedral perturbations that preserve topological form

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October 23, 1995

Abstract

The idea, that we are willing to accept variation in an object but that we insist it should retain its original topological form, has powerful intuitive appeal, and the concept appears in many applied fields. Some of the most important of these are tolerancing and metrology, solid modeling, engineering design, finite element analysis, surface reconstruction, computer graphics, path planning in robotics, fairing procedures, image analysis, and medical imaging. In this paper we focus on the field of tolerancing and metrology. The requirement that two objects or sets should have the same topological form requires a precise definition. We specify "same topological form" to mean that there exists a "space homeomorphism" from R^3 onto R^3 that carries a nominal object S onto another design object. In general, establishing the existence of such space homeomorphisms can be considerably more difficult than demonstrating classical topological equivalence by a homeomorphism. In the special case when the boundary of S is a polyhedral two-sphere in R^3, one of the authors has previously given a simple sufficient condition for the existence of a space homeomorphism mapping S onto another design object. This paper presents an analogous sufficient condition for the case when S is a finite polyhedron in R^3. The result relies upon a triangulation of the boundary and upon a dependent parameter that specifies the maximum size of permissible perturbations of the vertices of the polyhedron.

Suggested keywords: solid modeling, tolerancing and metrology, topological equivalence, topological form, simplicial complexes, computer graphics

[1]The first author was supported by the Swedish Natural Science Research Council under contract F-FU-8448-309.

[2]The second author acknowledges, with appreciation, partial funding for this work received under National Aeronautics and Space Administration (NASA) Grant Number NGT 40037. The views expressed herein are of this author, not of NASA.

[3]The third author acknowledges, with appreciation, partial funding for this work received under National Aeronautics and Space Administration (NASA) Grant Number NGT 40037 and under National Science Foundation Grant Number MII-9308346. The views expressed herein are of this author, not of these funding agencies.

[4]The research of the fourth author was supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada.

1 Introduction

In many applied fields, including tolerancing and metrology [1, 2, 3, 4, 5, 6, 7, 8, 9], solid modeling [10, p.110], engineering design [11], finite element analysis [12], surface reconstruction [13], computer graphics [14], path planning in robotics [15, p. 91], fairing procedures [16], image analysis [17, p. 69], and medical imaging [18, 19], it is natural to require that a perturbed object should have the same topological form as a given original object S.

The primary focus of this paper is the first-mentioned area, tolerancing and metrology, although the result proved here could be applied equally well in other fields. Rigorous specification of the requirement that toleranced objects should have the correct topological form is part of the more general problem of establishing precise mathematical criteria on which to base evaluation of the quality of manufactured objects. The "metrology crisis" described by Voelcker [20, 21, 22] illustrates the possible consequences of failure to define such criteria. Indeed, it was observed in [21] that the American tolerancing standard [23] is "... dangerously inadequate because it is cast in prose and special-case examples, and does not provide the precise mathematical definitions that are required to write control programs for inspection machines". These shortcomings led in particular to the GIDEP alert of 1988 [20]. As Voelcker states

> For several weeks the flow of mainly military hardware worth hundreds of millions (possibly billions) of dollars simply stopped, because work could not proceed until the goods could be dimensionally qualified. *Ad hoc* agreements between DoD authorities and standards organizations got the flow restarted, but the underlying problem(s) remain.

Similarly, Besl has observed [24] that the problem of checking for correct topological form is a serious difficulty in current algorithms in the field of metrology.

The idea, that we are willing to accept variation in an object but that we insist it should retain its original topological form, has powerful intuitive appeal. On the other hand, various definitions have been used to define this concept, and the idea is often used in an ill-defined and *merely* intuitive way. In this paper we will use the precise mathematical condition that two objects have the same topological form provided that there is a homeomorphism from R^3 onto R^3 which carries one object onto the other [7, 8, 9]. Intuitively we might imagine that the object is made of red putty, and that the rest of R^3 is filled with black putty; if the object *and* the surrounding space are elastically deformed, with no ripping or cutting of the boundary of the object, and without introducing new self-intersections of the boundary, then the modified object (the deformed red putty) has the same topological form as the original. Thus, a cube might be transformed into a solid spherical ball, but two disjoint rings cannot become interlinked [14, Figure 1], and a torus cannot become a torus with a knot[5] in it. In many application areas, including tolerancing and metrology, this seems like an almost obvious minimal requirement to impose.

A simple example of two objects with the same topological form is illustrated in Figure 1. This example will be useful later.

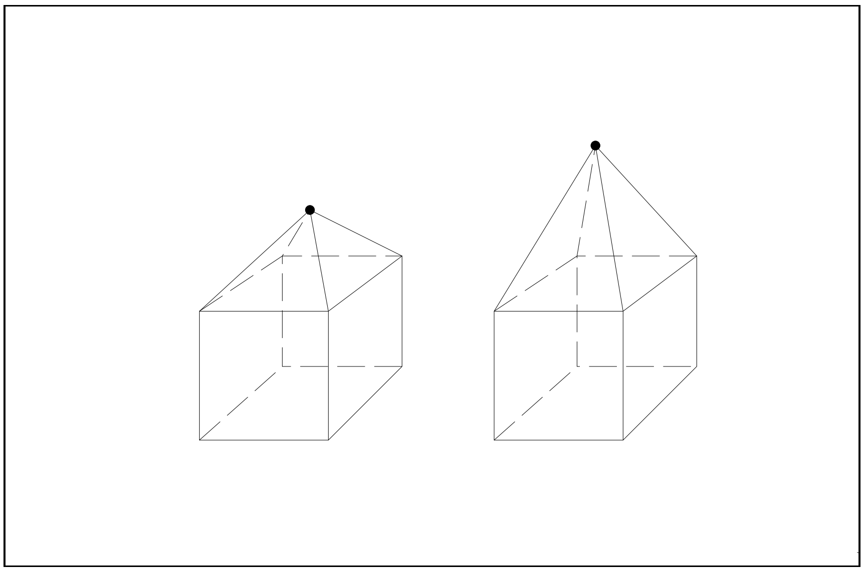

Figure 1: House-shaped objects with roofs of different pitch

Having defined what we mean by the statement that two objects should have the same topological form, it is also natural to ask how this requirement might be verified. Such verification may not be straightforward, either in the field of tolerancing and metrology, or in other fields.

Consider first the field of computer graphics. It will typically be easy to tell whether a simple object, displayed on the screen and rotated, has the correct topological form. On the other hand, as pointed out in [13, Sec. 4.1], if an algorithm is producing large numbers of surface elements in a complicated way, then the verification of correctness of topological form may not be simple[6].

Similarly, in tolerancing and metrology, it will typically be easy to visually inspect simple objects to determine whether they have the correct topological form: it is unlikely that a competent technician would fail to notice a design

[5] We mean, of course, a knot that cannot be untied.

[6] For example, the authors of [13] state that " ... constraining [their optimization process] to always produce an embedding appears to be difficult." See also [13, Sec. 3 and Figure 7(d)].

suggestion that led to, say, an object with an unwanted hole in it. On the other hand, if we envisage the possibility of a fully-integrated computer-aided design/analysis loop [27], then we begin to feel less secure. As observed by Requicha, the semi-autonomous or autonomous systems of the future will not be able to depend on human intervention to resolve their internal anomalies. It is at least plausible that a competent human might fail to notice that a cutter-path, proposed by a semi-autonomous module, will lead to an unwanted hole in an object. If we wanted to check for this, how exactly should we proceed? It is even more plausible that a completely autonomous computer-module, which is supposed to verify designs and cutter-paths, might fail to notice a proposal that would lead to an unwanted hole. If we go one step further, and consider the possibility that the designs themselves are being generated by expert systems and other artificial-intelligence modules, then it is quite possible that designs having incorrect topological form might (in the absence of tests to detect this possibility) sometimes be produced.

Of course, other tests of the quality of proposed designs will be necessary; testing for correctness of topological form is only a high-level screening process. On the other hand, the test described by our main theorem is correspondingly simple.

For example, when developing reliable solid modeling systems, these topological concerns merit significant attention during software development. Even though it is unlikely that a competent human designer would create a topologically invalid object, such pathological models are frequently generated as the output of Boolean operations within solid modeling. These Boolean operators are standard tools within CAGD modeling systems. Much of the seminal work of the role of regular closed sets within CAGD was motivated by a need to develop a rigorous theoretical basis to avoid such pathologies being generated by the underlying geometrical/topological algorithms of solid modeling systems. The reliability of these algorithms continues to be an active area of research pursuit, and mathematical extensions [25, 26] of this theory are useful for refining these software tools.

2 Mathematical preliminaries and notation

The criterion for sameness of topological form, mentioned in the introduction, was introduced in the context of mechanical tolerancing in [7, 8, 9]. Specifically, an object A is an acceptable approximation to the nominal solid S if there exists a *space homeomorphism*[7] that carries the nominal solid S onto A.

[7] A space homeomorphism is a homeomorphism from R^3 onto R^3.

The class of objects considered here is the class of *finite polyhedra*[8], by which we mean sets that are a finite union of properly-joined simplexes[9] within R^3, where these simplexes are understood to have rectilinear faces. We will use the words "object" and "solid" interchangeably to refer to members[10] of this class, denoted $\mathcal{A}$. As a comparison, the class of r-sets[11], a *de facto* standard for many applications, specifically excludes mixed-dimensional subsets, since they do not satisfy the regular-closed criterion. Since such mixed-dimensional subsets are allowed within $\mathcal{A}$, it is clear that $\mathcal{A}$ includes the class of non-empty rectilinear polyhedral r-sets. Recently, attention has been given [32, 25] to an even broader class of mixed-dimensional objects, which are more convenient for modeling in many applications. The main theorem presented in the paper applies to the mixed-dimensional class $\mathcal{A}$, and this generality is obtained with essentially no complication of the proof.

The restriction of the class $\mathcal{A}$ to *rectilinear* polyhedra (that is, the objects must have flat faces) is a serious one, and it would be desirable to give analogous theorems in the case of free-form objects. The result given here should be viewed as a fundamental first step towards a complete solution to the problem. On the other hand, objects with flat faces are not wholly without interest [23, Sec. 4.4.1, Sec. 6.4].

While our theorem is currently restricted to the subclass of polyhedra just mentioned, this is consistent with the historical development of many CAGD tools. The underlying algorithms for these tools internally represent design objects by using polyhedra for abstract data types. Frequently, these internal polyhedral abstract data types persist within these tools, even after they have matured to model free-form design objects, whereupon these polyhedra serve as desirable computational approximants to the free-form models [33].

Following [7, 8, 9], suppose that the nominal solid S is in $\mathcal{A}$, and let $\mathcal{M}$ be a subset of $\mathcal{A}$ defining a variational class: $\mathcal{M}$ is interpreted as the class of objects "within specification". Thus, the pair $(S, \mathcal{M})$ describes a tolerance specification: S is an ideal or nominal solid, and $\mathcal{M}$ is the class of all solids that are permissible approximations to S. Note that roman characters (S, A, ...) denote subsets of R^3, while script characters ($\mathcal{M}$, $\mathcal{H}$, ...) denote classes of subsets of R^3.

The class of objects $A \in \mathcal{A}$ for which there is a space homeomorphism that carries S onto A is denoted $\mathcal{H}(S)$, and the requirement that A should have the same topological form as S may then be written $\mathcal{M} \subseteq \mathcal{H}(S)$. In [9], a simple condition for membership in $\mathcal{H}(S)$ was given when S was a polyhedral solid with

[8] While our survey of the literature revealed varying terminology for such sets, our usage here is consistent with one of our principal references [28, p. 34].

[9] Two simplexes are properly joined [29] if they are either disjoint or intersect in a subsimplex common to both.

[10] We exclude from $\mathcal{A}$ the trivial case of the empty set.

[11] An r-set is a compact, regular-closed, semi-analytic subset of R^3 [30, 31].

boundary homeomorphic to a two-sphere [28]. This condition stated that the perturbation of the vertices (zero-dimensional simplices) should be smaller than one-half of the minimal distance between disjoint parts of the boundary. In the sequel, we will define these concepts precisely, and we will show that the theorem generalizes to all members S of $\mathcal{A}$, at least in the case when the boundary of S has been completely triangulated.

Denote the convex hull of $n+1$ affinely independent points $v_0, ..., v_n$ by $[v_0, ..., v_n]$, $n = 0, 1, 2$ or 3. For $n = 0, 1, 2$ or 3 such a convex hull is designated, respectively, as a vertex, an edge, a triangle, or a tetrahedron.

Let the *separation* between two compact sets X and Y be defined by

$$dist(X, Y) = \min d(x, y),$$

where the minimum is taken over $x \in X$ and $y \in Y$, and where $d(\ \cdot\ ,\ \cdot)$ denotes the Euclidean metric. (Thus, $d(x, y) = \|x - y\|$, where $\|\cdot\|$ denotes the Euclidean norm.)

3 Triangulations and discussion of hypotheses

An important distinction between the sufficient condition presented in this paper and that of its antecedent [9] is that the proof of the main theorem of this paper is significantly dependent upon the presence of a triangulation of the boundary of the object. The earlier work [9], while applicable to a much smaller class of objects, did not stipulate any requirement for the triangulation of the boundary. The role of the triangulation affords additional insight into the commonly expressed tolerance condition of *perfect form* [1, 2, 3, 8]. Specifically, triangulating the boundary ensures that *any* vertex perturbation results in flat-faced polyhedron, even while the resultant polyhedron may have a different number of faces than the original polyhedron. Thus, it is clear that the preservation of perfect form is an additional constraint-satisfaction problem that needs to be addressed separately. The issues of preserving topological form and perfect form are not equivalent.

If τ is a triangulation of ∂S, where ∂S denotes the boundary of S, let $\mathcal{P}(\tau)$ denote the set of vertices, edges and triangles of τ; thus, the faces (in the sense of solid modeling) of the solid S have been decomposed into triangles[12]. The notation $\mathcal{P}$ is intended to suggest the "parts" of the triangulation of ∂S. Let

$$\nu(\tau) = \min\{dist(X, Y)\},$$

[12] By definition, S is a finite polyhedron and can be represented as a finite union of properly joined simplexes. We note that such a representation need not be unique. A cube, for instance, may obviously be decomposed in many ways into properly joined simplexes. Further, the triangulation τ of ∂S is not necessarily induced by such an underlying representation but may be chosen based upon conditions pertinent only to the boundary.

where the minimum is taken over $X, Y \in \mathcal{P}(\tau)$ and $X \cap Y = \emptyset$. Note that $\nu(\tau)$ is a property of a *particular triangulation* τ of the *boundary* of S. In this paper, we will show that constraining the maximum size of perturbations of the vertices of the boundary of S to be less than $\nu(\tau)/2$ leads to a perturbed object that is in $\mathcal{H}(S)$. This dependency of the parameter upon the triangulation raises the issue of the choice of optimal triangulations [34, 35].

In the sequel, the explicit dependence of $\mathcal{P}$ and ν upon the triangulation τ will not be shown. (We have deliberately chosen the notation ν for the minimal separation between disjoint parts of the triangulation, in order to distinguish it from the quantity μ, defined in [9], which is the minimal separation between disjoint parts of the untriangulated boundary; clearly $\nu \leq \mu$.) The triangles making up the boundary of S will be denoted T_S^k, where $k \in K$, for some finite set K used to index all such triangles. The boundary parts (elements of $\mathcal{P}$) are closed subsets of R^3. A typical *vertex* will be denoted v_j, a typical *edge* will be denoted $[v_j, v_{j'}]$, and a typical *triangle* will be denoted $[v_j, v_{j'}, v_{j''}]$. (We will always ensure, sometimes by hypothesis, that two vertices defining an edge, and that three vertices defining a triangle, are affinely independent.) The *logical* or *symbolic* information [10, p. 110] associated with the representation of S specifies the set of *logical vertices* j, the set of *logical edges* (j, j'), and the set of *logical triangles* (j, j', j'') corresponding to the vertices, edges and triangles referred to above.

Let $v_1, ..., v_n$ be the vertices of τ, taken in any order. For each j, let δv_j be a vector defining the perturbation of v_j, and let the perturbed vertices be designated as $v_j + \delta v_j$.

Lemma 3.1: Let S be a finite polyhedron within R^3. If there is a space homeomorphism satisfying the following properties:

- for each logical vertex j, v_j is carried onto $v_j + \delta v_j$;
- for each logical edge (j, j'), $[v_j, v_{j'}]$ is carried onto the (non-degenerate) line-segment $[v_j + \delta v_j, v_{j'} + \delta v_{j'}]$;
- for each logical triangle (j, j', j''), $[v_j, v_{j'}, v_{j''}]$ is carried onto the (non-degenerate) triangle $T^k \equiv [v_j + \delta v_j, v_{j'} + \delta v_{j'}, v_{j''} + \delta v_{j''}]$;

then the image of S under the space homeomorphism is a finite polyhedron within R^3, and this image is also in $\mathcal{H}(S)$.

Proof. Let the image of triangle T_S^k be represented by T^k (the perturbed boundary triangle). Because of the displayed conditions, the collection of such triangles T^k provides a rectilinear triangulation of $\cup_k T^k$, where $\cup_k T^k$ is a compact set. The hypothesis of the existence of a space homeomorphism ensures that the image of S is embedded in R^3 and its boundary is $\cup_k T^k$. Hence, this image of S is a member of $\mathcal{A}$ and of $\mathcal{H}(S)$. □

We are now in a position to describe, in general terms, the method of proof that will be used below. In the proof of the main theorem we will show that there is a space homeomorphism g^n linking the original polyhedron S, and a perturbed polyhedron S' for which v_i has been replaced, for each i, by $v_i + \delta v_i$. We will do this by defining, for each i, $1 \leq i \leq n$, a homeomorphism g^i as the composition of i homeomorphisms $h^j, j = 1, ..., i$:

$$g^i = h^i \circ h^{i-1} \circ ... \circ h^1,$$

where the effect of h^j, for each j, is to move the vertex v_j to $v_j + \delta v_j$. Thus, the vertices are moved one at a time: g^1 results in a finite polyhedron having vertices $v_1 + \delta v_1, v_2, ..., v_n$, $g^2 = h^2 \circ h^1$ results in a polyhedron having vertices $v_1 + \delta v_1, v_2 + \delta v_2, v_3, ..., v_n$, and so on; g^n results in a polyhedron having vertices $v_1 + \delta v_1, ..., v_n + \delta v_n$.

The homeomorphism h^i moving v_i to $v_i + \delta v_i$ will be defined using pushes [28]. In addition to moving the vertex, it will move certain adjacent parts of the boundary of the image (under g^{i-1}) of S, as well as certain neighboring points of R^3 that are not in the boundary of this set. It will also be shown that h^i does not disturb any other (*i.e.* non-adjacent) parts of the boundary. Figure 2 shows a cross-section illustrating the push necessary to transform the first object

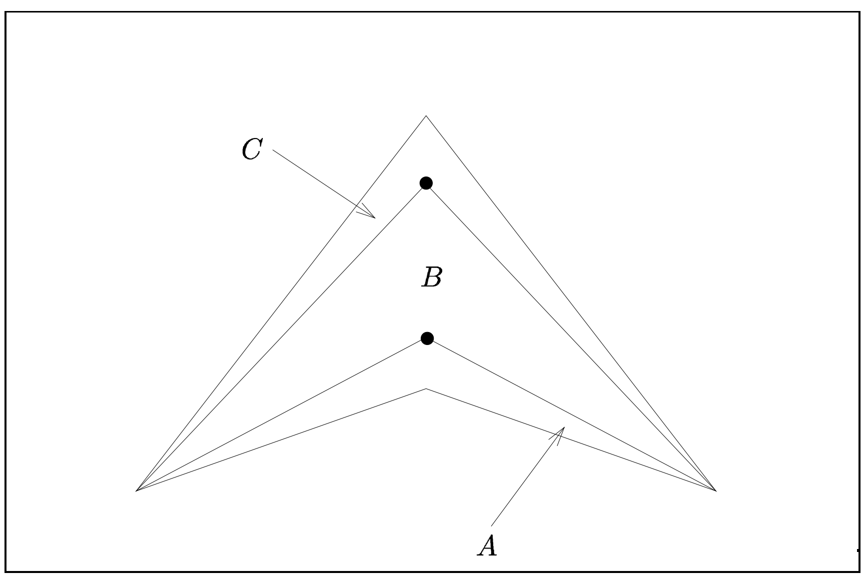

Figure 2: Cross-section illustrating push for Figure 1

in Figure 1 into the second object in Figure 1. The points shown by large dots in Figure 2 correspond to the two positions of the peak of the roofs of the house-shaped objects. The regions A and C are thin enough that they do not interfere with non-adjacent parts of the boundary (of the current *or future* transformed versions of the object). Using again the metaphor of Section 1, the red putty in region A is stretched to fill regions A and B, while the black putty in regions B and C is compressed into region C; the remainder of R^3 is left unaltered by h^i.

The main technical difficulty in the proof comes from the fact that for a more complicated object, there may have been another piece of the object (red putty) directly above the peak of the roof, or there may be a cavity in the object, below the peak of the roof; moreover, the resulting other pieces of the boundary of the object may later be moved, or may already have been moved, by some other homeomorphism $h^j, j \neq i$, from their original positions, towards the roof. It is crucial to avoid conflicts between these other parts of the boundary and the region modified by h^i.

A more realistic illustration of a push is given in Figure 3, where v_i and

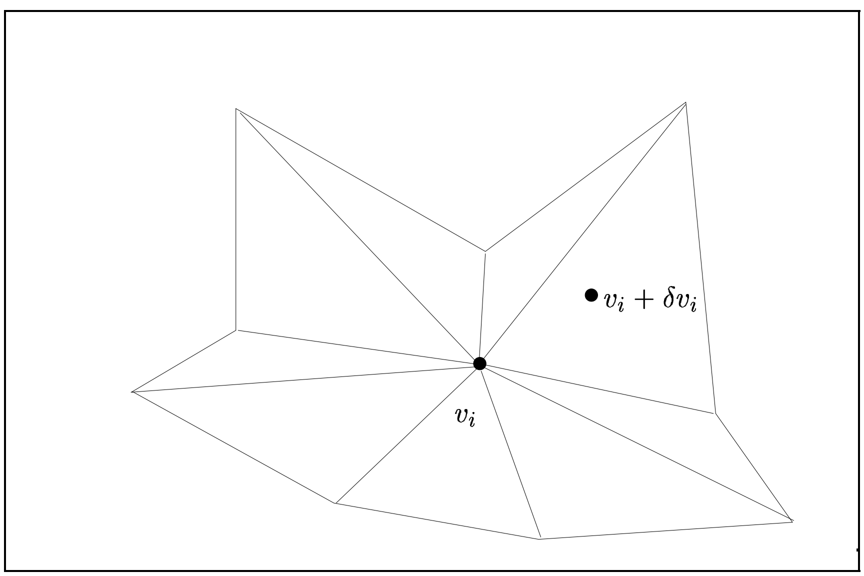

Figure 3: Star-shaped region containing v_i and $v_i + \delta v_i$

$v_i + \delta v_i$ are shown in a star-shaped region that does not interfere with previous or impending vertex displacements. (In our application, the star-shaped region is three-dimensional.) Changing the metaphor, we may imagine that v_i is connected to the points of the star shaped region by rubber bands, and that v_i is moved to $v_i + \delta v_i$ with the rubber bands shrinking or stretching appropriately. We must ensure that such a star-shaped region exists, and that a situation like that illustrated by the dashed lines in Figure 4 cannot occur.

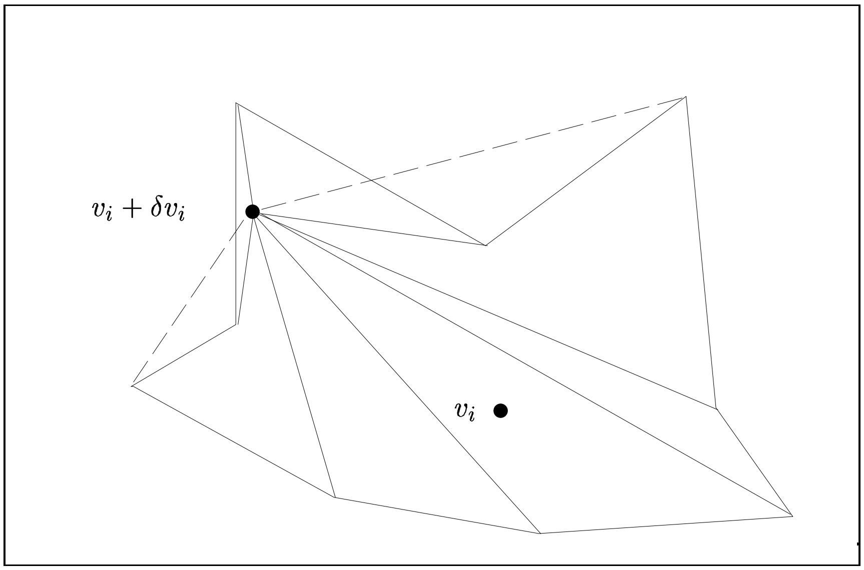

Figure 4: A displacement that is not a push

4 Inductive hypothesis

In this section, we establish the inductive framework corresponding to the successive displacements of vertices. We will show that moving one vertex at a time, by a sufficiently small amount, is consistent with the type of space homeomorphisms required in the hypotheses of Lemma 3.1. The finite composition of the homeomorphisms associated with each vertex displacement will then be the space homeomorphism used to execute all perturbations of the finitely many vertices. As has already been stated, we will exhibit a finite sequence of space homeomorphisms

$$h^i, i = 1, ..., n, \text{ where each } h^i \text{ maps } v_i \text{ into } v_i + \delta v_i.$$

Indeed, for $i = 1, ..., n$, if g^i is the composition $h^i \circ h^{i-1} \circ ... \circ h^1$ of the first i of these homeomorphisms, then the exhibited g^i will satisfy the following conditions:

1. $g^i(v_j) = v_j + \delta v_j, j = 1, ..., i$ and $g^i(v_j) = v_j, j = i + 1, ..., n;$

2. the image under g^i of any edge $[v_j, v_{j'}]$ is the non-degenerate edge $[g^i(v_j), g^i(v_{j'})]$;

3. the image under g^i of any triangle $[v_j, v_{j'}, v_{j''}]$ is the non-degenerate triangle $[g^i(v_j), g^i(v_{j'}), g^i(v_{j''})]$.

Define g^0 to be the identity mapping. The proof in the next section proceeds by induction. The conditions just displayed, with i replaced by $i-1$, constitute the *inductive hypothesis*, and it will be shown that if the conditions are satisfied by g^{i-1} for some value of $i-1$, $0 \le i-1 \le n-1$, then we can define h^i so that $g^i = h^i \circ g^{i-1}$ satisfies the conditions as displayed.

5 Perturbing boundaries via space homeomorphisms

The main result is given in this section, with several of the supporting lemmas being consolidated in the Appendix, Section 8.

Theorem 5.1: Let S be a finite polyhedron within R^3 and let the boundary ∂S have a triangulation τ with corresponding parameter $\nu = \nu(\tau)$. If for each j, the vertex perturbations are such that $\|\delta v_j\| < \nu/2$, then there exists a space homeomorphism satisfying the properties in the statement of Lemma 3.1.

Proof. We will proceed by induction. Suppose that the inductive hypothesis is satisfied by g^{i-1} for some value of $i-1$, $0 \le i-1 \le n-1$. We want to define h^i so that $g^i = h^i \circ g^{i-1}$ satisfies the properties displayed at the end of Section 4.

Let $B_{\nu/2}(v_i)$ be the ball of radius $\nu/2$ centered at v_i. Since $\|\delta v_j\| < \nu/2$, it follows by condition 1 of the inductive hypothesis that $g^{i-1}(v_j) \notin B_{\nu/2}(v_i)$ for all $j \neq i$. It then follows by condition 2 of the inductive hypothesis that

$$[g^{i-1}(v_{j'}), g^{i-1}(v_{j''})] \cap B_{\nu/2}(v_i) = \emptyset$$

for all logical edges (j', j'') such that $i \neq j', j''$: otherwise, the edge $[v_{j'}, v_{j''}]$ must have been separated from v_i by less than ν, which is a contradiction.

Let $\mathcal{F}_{v_i}$ be the set of subsimplexes that have v_i as a vertex. We will explicitly treat only the case when $\mathcal{F}_{v_i}$ consists exclusively of triangles (i.e. 2-simplexes); the cases of lower-dimensional subsimplexes can be treated using a similar proof. A typical triangle of $\mathcal{F}_{v_i}$ is $[v_i, g^{i-1}(v_{j'}), g^{i-1}(v_{j''})]$. All such triangles are disjoint, except possibly along an edge $[v_i, g^{i-1}(v_{j'})]$ or $[v_i, g^{i-1}(v_{j''})]$ or at the vertex v_i; this follows from the inductive hypothesis.

Define $T_\theta \equiv [v_i + \theta\delta v_i, g^{i-1}(v_{j'}), g^{i-1}(v_{j''})], 0 \le \theta \le 1$, so that, in particular, a typical triangle $[v_i, g^{i-1}(v_{j'}), g^{i-1}(v_{j''})]$ is denoted T_0. The homeomorphism h^i will be defined below as a push [28, pp. 13, 162] that maps each $T_0 \in \mathcal{F}_{v_i}$ into its corresponding T_1, leaving the edge $[g^{i-1}(v_{j'}), g^{i-1}(v_{j''})]$ fixed. Assuming for the moment that it is possible to exhibit such a push, we would have, for each i, j', j'' that,

- vertex $g^{i-1}(v_i) = v_i$ is carried into $v_i + \delta v_i = g^i(v_i)$;
- edge $[v_i, g^{i-1}(v_{j'})]$ is carried into $[v_i + \delta v_i, g^{i-1}(v_{j'})] = [g^i(v_i), g^i(v_{j'})]$;
- edge $[v_i, g^{i-1}(v_{j''})]$ is carried into $[v_i + \delta v_i, g^{i-1}(v_{j''})] = [g^i(v_i), g^i(v_{j''})]$;
- triangle $[v_i, g^{i-1}(v_{j'}), g^{i-1}(v_{j''})]$ is carried into

$$[v_i + \delta v_i, g^{i-1}(v_{j'}), g^{i-1}(v_{j''})] = [g^i(v_i), g^i(v_{j'}), g^i(v_{j''})].$$

None of these edges or triangles degenerates, since $g^{i-1}(v_{j'}), g^{i-1}(v_{j''}) \notin B_{\nu/2}(v_i)$, and $[g^{i-1}(v_{j'}), g^{i-1}(v_{j''})] \cap B_{\nu/2}(v_i) = \emptyset$. Consequently, provided that h^i is a space homeomorphism that leaves other triangles $[g^{i-1}(v_j), g^{i-1}(v_{j'}), g^{i-1}(v_{j''})]$ (*i.e.* those *not* in $\mathcal{F}_{v_i}$) undisturbed, it follows that conditions 1, 2 and 3, displayed at the end of Section 4, are satisfied by g^i, and the inductive step is complete.

It remains to define the push h^i. Consider a typical triangle of $\mathcal{F}_{v_i}$ of the form $[v_i, g^{i-1}(v_{j'}), g^{i-1}(v_{j''})]$. In keeping with the spirit of Edelsbrunner and Mücke's 'Simulation of Simplicity' (SoS) [14, Sec. 6.2], [36], we will present here only the case when the four points

$$v_i,\ v_i + \delta v_i,\ g^{i-1}(v_{j'}),\ g^{i-1}(v_{j''})$$

are affinely independent. (The degenerate case, when these points are not affinely independent, is presented in the Appendix, Proposition 8.6.)

Let $\nu^* = 2\max_{1 \le i \le n} \|\delta v_i\|$, so that $\nu^*/2 < \nu/2$, and let ϵ be chosen such that $0 < \epsilon < \nu/2 - \nu^*/2$. By Lemma 8.1, there exists a tetrahedral neighborhood $N_{(i,j',j'')}$ whose interior contains $(T_\theta - [g^{i-1}(v_{j'}), g^{i-1}(v_{j''})])$ for each $\theta \in [0,1]$ and such that $dist(x, [v_i, v_i + \delta v_i, g^{i-1}(v_{j'}), g^{i-1}(v_{j''})]) < \epsilon$, for all $x \in N_{(i,j',j'')}$. Then, let $U_i = \cup N_{(i,j',j'')}$, where the union is taken over all (j', j'') such that $[v_i, g^{i-1}(v_{j'}), g^{i-1}(v_{j''})] \in \mathcal{F}_{v_i}$. All of the vertices of the form $g^{i-1}(v_m)$ (except v_i) of triangles in $\mathcal{F}_{v_i}$ are in the boundary of U_i. (This is proved in the Appendix, Proposition 8.5.)

Since the boundary of U_i is homeomorphic to a piecewise linear two-sphere (Lemma 8.2), we may triangulate [28, p. 23] the *boundary* of U_i using *all* of these vertex-neighbors $g^{i-1}(v_m)$ of v_i; let $\tau_0(\partial U_i)$ denote such a triangulation. Furthermore, since U_i is star-convex (Definition 8.3) with respect to both of its interior points v_i and $v_i + \delta v_i$ (Lemma 8.4), joining each vertex of the boundary triangulation $\tau_0(\partial U_i)$ to v_i produces a triangulation $\tau_0(U_i)$ of U_i, and joining the vertices of the boundary triangulation $\tau_0(\partial U_i)$ to $v_i + \delta v_i$ produces another triangulation of U_i, denoted $\tau_1(U_i)$. The star of v_i [28, p. 13] in $\tau_0(U_i)$ and the star of $v_i + \delta v_i$ in $\tau_1(U_i)$ are both equal to U_i, and we define h^i by means of the corresponding push [28, pp. 13, 162]: that is, h^i is the homeomorphism from R^3 onto itself that is fixed outside of U_i, takes v_i to $v_i + \delta v_i$, and takes the simplexes of $\tau_0(U_i)$ in the star of v_i linearly onto simplexes of $\tau_1(U_i)$ that are in the star of $v_i + \delta v_i$. This push carries each T_0 onto the corresponding T_1, leaving $[g^{i-1}(v_{j'}), g^{i-1}(v_{j''})]$ fixed for $j', j'' \neq i$, as required above.

Finally, also as required above, each such push leaves triangles not in $\mathcal{F}_{v_i}$ undisturbed. To prove this, it is sufficient to show that if $N_{(i,j',j'')}$ corresponds to $[v_i, g^{i-1}(v_{j'}), g^{i-1}(v_{j''})] \in \mathcal{F}_{v_i}$ then the intersection of $N_{(i,j',j'')}$ and a triangle

$$T^* = [g^{i-1}(v_{j_0}), g^{i-1}(v_{j_1}), g^{i-1}(v_{j_2})] \notin \mathcal{F}_{v_i}$$

is either empty or equal to exactly one of the sets

$$[g^{i-1}(v_{j'}), g^{i-1}(v_{j''})] \text{ or } \{g^{i-1}(v_{j'})\} \text{ or } \{g^{i-1}(v_{j''})\}.$$

This can be shown by a straightforward but tedious argument.

This completes the proof. □

For these finite polyhedra, the retention of perfect form (flat faces) would require that points that were initially co-planar remain co-planar after the vertex perturbations. The above proof should make it clear that although each perturbed boundary triangle necessarily remains planar, the perfect-form constraint need not be fulfilled. Investigating the additional theoretical implications of satisfying this perfect-form constraint remains of interest. However, it is likely that this issue would be resolved by quite different, yet complementary, theoretical techniques.

6 Conclusions and future work

In accordance with previously postulated conditions for preservation of topological form under tolerance variations [9], we have given upper bounds on the size of perturbations of polyhedral vertices so as to maintain the appropriate topological form. It would be desirable to have more general theorems giving bounds on the perturbations in terms of the minimal separation between parts of the untriangulated boundary, as in [9], and bounds for permissible perturbations for non-polyhedral objects. However, so far, we have only been able to prove the theorem in the special case. One of our principal objectives for the future is to generalize this tolerance theory to the case of objects with free-form boundaries. There are difficult problems involved in such a generalization. However, the result presented here is a fundamental, probably essential, first step. In particular, historically, internal polyhedral abstract data types have often been used as the basis for transition of the mathematical theory of free-form geometric models to their practical application in algorithms having computationally tractable performance, where the polyhedral approximations are often done ‘opportunistically’ to yield ‘small polygons’ only in regions where they are essential.

Ultimately, these tolerance bounds would be included as constraints within software systems incorporating automatic design optimization via interaction, under program control, between design and analysis modules.

7 Acknowledgements

We express our appreciation to V. Shapiro, J. A. Roulier, M. D. Rice, S. Seidman, W. W. Comfort, W. Abikoff, and D. A. Lear, for helpful comments and discussions. While we gratefully acknowledge the help of these colleagues, any remaining errors or omissions are solely the responsibility of the authors. The authors thank the anonymous reviewer for helpful suggestions.

The results of this paper were developed concurrently and collaboratively with the doctoral research [37] of one of the authors; the present paper generalizes some of the results presented there.

8 Appendix

We collect here several geometrical definitions and tools needed in support of our main proof.

Lemma 8.1: In R^3, for any[13] four affinely independent points, r, s, t, u, and any $\epsilon > 0$, there exists a tetrahedron $[p, q, t, u]$ such that

$$[r, s, t, u] - [t, u] \subset int_{3D}([p, q, t, u])$$

and for all $x \in [p, q, t, u], dist(x, [r, s, t, u]) < \epsilon$. □

Lemma 8.2: If A is a finite union of tetrahedra, each of which has q as an interior point, then A is homeomorphic to a piecewise linear 3-ball. □

Definition 8.3: A non-empty set A is star convex with respect to a point $q \in A$, if for each $x \in A$, with $q \neq x$, the line segment $[q, x]$ is contained in A. □

Lemma 8.4 [38, p. 132]**:** If A is a union of sets, each of which is star convex with respect to the same point q, then A is star convex with respect to q. □

Proposition 8.5: Vertices of the form $g^{i-1}(v_m)$ satisfying the conditions in the proof of Theorem 5.1 are in ∂U_i.

Proof. Suppose $g^{i-1}(v_m)$, where $m \neq i$, is a vertex of a triangle in $\mathcal{F}_{v_i}$. This vertex may share the edge $[v_i, g^{i-1}(v_m)]$ with other faces in $\mathcal{F}_{v_i}$, and this vertex will be in the boundary of the union of the neighborhoods $N_{(i,j',j'')}$corresponding to these triangles. It cannot however be in the neighborhood $N_{(i,j',j'')}$ corresponding to any other triangle, since in this case it would be at distance less than $(\nu - \nu^*)/2 + \nu^*/2$ from the triangle $[v_i, v_{j'}, v_{j''}]$, and since v_m is at distance less than $\nu/2$ from $g^{i-1}(v_m)$, we would have by the triangle inequality that v_m is at distance less than ν from a disjoint part of the triangulation (*viz* the triangle

[13]While this lemma is presented in its full generality, it should be clear that the intended use within this paper applies to the affinely independent points v_i, $v_i + \delta v_i$, $g^{i-1}(v_{j'})$, $g^{i-1}(v_{j''})$, with a fixed edge of $[g^{i-1}(v_{j'}), g^{i-1}(v_{j''})]$.

$[v_i, v_{j'}, v_{j''}]$), a possibility which has been excluded by hypothesis. Since each $N_{(i,j',j'')}$ is closed, it follows that $g^{i-1}(v_m)$ is in the boundary of U_i. □

Proposition 8.6: *The Degenerate Case.*
When the points

$$v_i,\ v_i + \delta v_i,\ g^{i-1}(v_{j'}),\ g^{i-1}(v_{j''})$$

are not affinely independent, the space homeomorphism h^i mapping v_i into $v_i + \delta v_i$ can be given as a composition of two pushes.

Proof. For all triangles of the form $[v_i, g^{i-1}(v_{j'}), g^{i-1}(v_{j''})]$ consider also their perturbed images $[v_i + \delta v_i, g^{i-1}(v_{j'}), g^{i-1}(v_{j''})]$. Since there are only finitely many such triangles, let $\{P_1, ..., P_k\}$ be the finite collection of planes containing all the above triangles. Let γ be chosen such that $0 < \gamma < (\nu - \nu*)/2$, and let B be the closed ball of radius γ centered at $v_i + \delta v_i$. It is clear that the set

$$C = B - \bigcup_{1 \le m \le k} P_m$$

is nonempty and does not contain $v_i + \delta v_i$. Choose any point of C and denote it w_i. Then, the points of the form $v_i,\ g^{i-1}(v_{j'}),\ g^{i-1}(v_{j''}),\ w_i$ are affinely independent, as are the points of the form $v_i + \delta v_i,\ g^{i-1}(v_{j'}),\ g^{i-1}(v_{j''}),\ w_i$. Hence, h^i can be given as the composition of the two pushes, v_i onto w_i and w_i onto $v_i + \delta v_i$, where each such push is constructed similarly to the general case given in the proof of Theorem 5.1. □

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