Explicit Ambient Isotopic Approximations of 2-Manifolds

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Abstract

We present an efficient, constructive algorithm for adaptive, piecewise linear, ambient isotopic approximation of a parametric surface. In contrast to previous algorithms, our approach gives an explicit method for computing the sample set and does so within comparable size guarantees. Our algorithm has a further feature of interest: It relies solely on evaluation of the shape operator at chosen points in parameter space and an a priori promise on maximum curvature. The algorithm is thus quite independent of the underlying representation of the surface.

1 Introduction

In this article, we describe an explicit adaptive surface approximation algorithm for $C^2$ parametric surfaces, $S : [0, 1]^2 \to \mathbb{R}^3$, embedded as a compact manifold with boundary. Our main results on approximation extend work of Clarkson and Leibon and Letscher [9, 18]. Despite a recent preprint outlining problems with their approach, the results in dimension 2 remain valid [15, pg 45, discussion A.2]. These articles give precise, quantitative connections between curvature, approximation quality, and sample set size, but do not yield an explicit construction of the desired sample set. While these other results provide a priori bounds on the number of approximating piecewise-linear (PL) surface elements needed to approximate $S$, as a function of root Gauss curvature $\kappa$, we give an explicit construction to achieve these goals. Given $\eta > 0$ satisfying mild constraints in terms of an upper bound $\beta$ on maximal curvature of $S$ and a bound $\Lambda$ on the directional derivative of curvature and the first derivative of $S$, this PL approximant will be everywhere within distance $\eta$ of $S$ (as measured by Hausdorff distance) and have $O(\eta^{-1} (\ln \eta^{-1})^2 \int_S \kappa)$ surface elements; where $\kappa$ denotes

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root Gauss curvature. Our asymptotic dependence upon curvature is tight; our dependence on an approximation error tolerance $\eta$ is tight to within a $\log^2$ factor.

Combining with results of Dai et al. [11] we obtain a strong explicit algorithm for producing ambient isotopic approximations of parametric surfaces. By adjusting $\eta$ to meet bounds in terms of the local feature size; i.e., the distance from the surface to the medial axis, the output of our algorithm can be provably assured to be isotopic (see Theorem 5.7.) Even though computation of the medial axis is well-known to be numerically unstable [10], there are recent advances [24] which give methods for handling the instability. However, we note that while our bound involves the distance to the medial axis, the computation of the approximant does not depend on computation of the medial axis itself, hence it is free from the associated numerical instability.

We remark that once the parameters $\beta$ and $\Lambda$ are provided, the algorithm evaluates the surface $S$ and its derivatives at a polynomially-bounded number of points (in terms of $\eta^{-1}$ and maximum curvature) in order to ultimately construct the sample set. We remark that sufficient estimates for $\beta$ and $\Lambda$ can be computed efficiently for the large class of spline surfaces, see [20, 22] for more details.

To determine the sample points for our approximation, we stratify the surface, and corresponding parameter space, into ‘level sets’ with respect to the shape operator. Specifically, Lemma 5.2 grants us a sufficiently dense finite set of linear operators, each of which defines a level set of the surface for which the shape operators at the points in the level set are close in operator norm to the chosen operator in the dense family, i.e., regions of roughly constant curvature. We use the roughly constant curvature value on each level set to build a low discrepancy set for the whole of parameter space. Our algorithm then selects specific points out of each of these low discrepancy sets to build our sample set. Lemma 5.3 shows that this selection meets the sample criteria required to obtain our distance bounds. This algorithm does not address the quality of the triangles produced, leaving these important questions as an open topic of investigation.

2 Related Work

Many methods for ambient isotopic approximation of manifolds without boundary [3, 4, 5, 8] have arisen from the seminal paper [2]. These sampling criteria depend upon the medial axis, which has unfortunate computational implications [10]. Two recent methods for ambient isotopic approximations of manifolds with boundary [1, 12] also build upon these foundations [2], but the work [1] has the advantage of using an auxiliary geometric construction to avoid some of the difficulties inherent in computation of the medial axis. By integrating topological criteria [11] into approximation methods that are directly relevant to our work [9, 18]. As noted in the introduction, these methods are subtle in larger dimensions, but for dimension 2 work as advertised [15, pg 45, discussion A.2]. The combined result is an approximation method which depends only on the local feature size but it does not depend directly on computing any part of the medial axis itself.

Discrepancy theory [6, 19, 21] has been the basis of many construction problems in computational geometry. The monograph [7] by Chazelle is a good overview of classic applications of discrepancy in computational geometry, specifically towards reducing computation time of computing Delaunay triangulations and $\varepsilon$-nets. Surprisingly, this approach has not been developed in the arbitrary parametric settings that we consider, nor are any explicit constructions given which have the adaptive quality of the Leibon-Letscher approximation. We utilize discrepancy theory to build our sample sets so that they have sufficiently strong enough pseudorandom properties to
appropriately saturate level sets of $S$ according to the shape operator.

To establish an approximation by a Delaunay triangulation, Clarkson [9] used $\varepsilon$-nets of the manifold extending previous work [13, 18]. For our approximation, like Clarkson, we use triangles which are piecewise-linear elements in the Euclidean space in which the surface is embedded. However, it is unclear from the methods in [9] how to efficiently construct Clarkson’s $\varepsilon$-net or any approximation to the net sufficient to preserve the size bounds. While exhaustive methods can be shown produce $\varepsilon$-nets in this setting [14], they rely on potentially extravagant algorithmic primitives, such as computing a point of maximum distance from a given family of points. In our setting, computing even a single distance (in the curvature metric) may computationally be quite expensive. Of course, for practical applications it is critical to have an efficient construction for sample sets. Our constructive method yields a sample set with size proportional to $\eta^{-1}(\ln(\eta^{-1}))^2$ whereas the previous (nonconstructive) methods [9] achieve (optimal) size proportional to $\eta^{-1}$, where $\eta$ is the tolerance on Hausdorff distance.

3 Notation and Preliminaries

We first describe our notation and setting. We concentrate on approximation of embedded surfaces given by a map $S : [0,1]^2 \to \mathbb{R}^3$. Here by a surface we mean a smooth parameterized non-self-intersecting orientable 2-manifold which is embedded in $\mathbb{R}^3$. We measure curvature by using the Gauss map, $N : S([0,1]^2) \to \Sigma^2$ where $\Sigma^2$ denotes the unit sphere and the map $N$ takes a point to its unit normal vector.

For each point $p \in [0,1]^2$, the differential of $N$ at $S(p)$ is a self-adjoint linear map on the tangent space of $S$ at $S(p)$. By the spectral theorem [23], we can choose a local basis which orthogonally diagonalizes this operator.

Definition 3.1. Let $p \in [0,1]^2$ and $\{\kappa_1, \kappa_2\}$ be an orthonormal basis that diagonalizes the differential of $N$ at $S(p)$. The vectors $\kappa_1, \kappa_2$ are called the directions of principal curvature at $S(p)$ and the eigenvalues $\kappa_1, \kappa_2$ the principal curvatures at $S(p)$. The shape operator at $p \in [0,1]^2$ is the the linear map $dN(S(p))$. We denote it by $h_p$ and its operator norm is called the principal curvature of $S$ at $S(p)$. In particular, the operator norm of $h_p$ is $\max\{|\kappa_1|, |\kappa_2|\}$. The value $\kappa(p) = |\kappa_1\kappa_2|$ is called the Gaussian curvature of $S$ at $S(p)$.

The three constructions [9, 11, 18] follow the same basic steps, taking as input a set of sufficient sample density of the manifold and then building a triangulation from it. Our new approach focuses upon building an explicit sample set (see Section ??) which generates the same simplicial triangulation $\tau$ as [9], which is formed by taking each geodesic triangle and replacing it with the unique Euclidean triangle having the same vertices. Both works [9, 11] present analyses for determining the quality of this approximation. As such, our algorithm focuses on how to explicitly build the sample set.

The primary result we utilize is an estimate on the Hausdorff distance of the approximation in terms of its “density.” To be precise, here we recall notation previously used [9]. Consider the second fundamental form

$$q_{\text{II}}(v, S(p)) = v \cdot h_p v$$

where $v$ is a tangent vector to $S$ at $S(p)$ and $h_p$ is the shape operator at $p$. Associated to this form is a metric

$$d_{\text{II}}(p, p') = \inf_{\sigma} \int_{a}^{b} \sqrt{q_{\text{II}}(\sigma'(t), \sigma(t))} dt$$

IsoGeometric Analysis (IGA). Fundamentally, this is a new approach to finite element analysis (SIAM_News, ~1 year ago) for stress, strain, fluid and other engineering calculations. Instead of doing the analysis on a PL approximation of a surface, values for the engineering items of interest are calculated at each of the sample points from the parametric domain (Tom_Hughes_UTexas, web site and multiple recent publications). I would, of course, then stress that our explication of the sample set permits IGA, whereas the Clarkson method would not.
where the infimum is taken over all smooth paths \( \sigma : [a, b] \to S \) from \( S(p) \) to \( S(p') \).

**Definition 3.2.** Given \( \epsilon > 0 \), a set \( Y \subset S \) is an \( \epsilon \)-cover provided for each point \( S(p) \in S \), the set \( Y \) non-trivially intersects
\[
B^H_\epsilon(S(p)) := \{ S(q) \mid d_H(S(p), S(q)) \leq \epsilon \}.
\]

### 4 Sample Sets

In this section we summarize our algorithm and describe more precisely what pseudorandom properties we require of our sample sets, the algorithm used to build our sample sets, and how the sample sets compare to the \( \epsilon \)-covering of \([9]\).

The algorithm takes as input a smooth parametric surface \( S : [0, 1]^2 \to \mathbb{R}^3 \), a desired tolerance \( \eta > 0 \) which serves as the user-defined bound on Hausdorff distance between \( S \) and the approximant, as well as an upper bound \( \beta \) on the curvature of the surface \( (\beta \geq \max\{\|h_p\|_{\text{op}}, 1\}) \) for each \( p \in \mathbb{R}^2 \), and an upper bound \( \Lambda \) on the directional derivative of curvature and on the first derivative of the manifold \( (\Lambda \geq \max\{\|\partial h/\partial u\|_{\text{op}}, \|\partial S/\partial u\|_2\}) \). Here \( \| \cdot \|_{\text{op}} \) denotes operator norm (equal to the absolute value of the largest eigenvalue, in this case), \( \| \cdot \|_2 \) denotes the normal Euclidean norm on vectors. Theorem 5.7 gives precise conditions on \( \eta \) when ambient isotopy is achieved.

Central to our algorithm is the formal definition of our sample set, which we call a hitting set (see Definition 4.2.) In this formal definition, we utilize the shape operators. However, to compare different operators at different points we realize each of these as operators on the ambient Euclidean space which contains our parameter space. Some care will be needed to consider areas of extremely low curvature or with degenerate shape operators and as such we define a parameter \( \alpha \) which later will be set to \( 1/\sqrt{\ln \eta^{-1}} \).

**Remark 4.1.** For each \( p \) in the parameter space, let \( h_p \) denote the self-adjoint shape operator of the surface \( S \) at \( p \). This is naturally an operator on the tangent space \( T_{S(p)} \) which has a basis diagonalizing \( h_p \). Since our parameterization is an embedding, it is also an immersion and so the differential \( T_p \to T_{S(p)} \) is injective. Thus, we may consider the preimage of the basis of \( T_{S(p)} \) diagonalizing \( h_p \). This preimage is also a basis for \( T_p \). Considering \( T_p \) as an affine translation of \( \mathbb{R}^2 \) to have origin \( p \), we may consider \( h_p \) as a \( 2 \times 2 \) symmetric real matrix. To be precise here, suppose \( \{v_1, v_2\} \subset \mathbb{R}^2 \) is the preimage of the basis for \( T_{S(p)} \) under the differential where we have consistently ordered them so that \( \kappa_1(p) \geq \kappa_2(p) \). We interpret \( h_p \) to be the real \( 2 \times 2 \) matrix which is diagonalized by \( \{v_1, v_2\} \) and with eigenvalue \( \kappa_1(p) \) along \( v_1 \) and \( \kappa_2(p) \) along \( v_2 \) provided both \( \kappa_i(p) \geq \alpha \). If either \( \kappa_i(p) < \alpha \) then we clip these shape operators so that eigenvalue is replaced by \( \alpha \).

The reason for the clipping is for technical simplicity. For a manifold with boundary, we have a natural upper bound on the size of any piecewise-linear approximant. To avoid detailed consideration of what happens at the boundary, we insist that no piecewise-linear element be too large: equivalently, we insist that our sample set have density no less than \( 1/\sqrt{\ln \eta^{-1}} \). This will be achieved by a our choice \( \alpha = 1/\sqrt{\ln \eta^{-1}} \) and will ensure that a sample point (and, so, a linear element) appear within \( \sqrt{\ln \eta^{-1}} \) of every point of the boundary; beyond this consideration, guaranteeing a reasonable approximation to the boundary, we ignore its effects.

Let \( B_r \) be the Euclidean unit ball of radius \( r \) about the origin and denote by \( B_r(p) \subset [0, 1]^2 \) the translate of \( B_r \) to the point \( p \). At points \( p \in [0, 1]^2 \) such that \( h_p \) is not degenerate, our
approximation is based on sets of the form
\[ h_p^{-1}B_\gamma(p) := \{ p + v : \|h_p(v)\|_2 < \gamma \}. \]

Since \( h_p \) is orthogonally diagonalizable, this set is an ellipse oriented along the eigenspaces of \( h_p \).

For any ellipse \( E \) with major and minor axes given as vectors \( v_1, v_2 \), we define \( \sqrt{E} \) to be the ellipse with major and minor axes
\[ \frac{v_1}{\sqrt{\|v_1\|_2}} \quad \text{and} \quad \frac{v_2}{\sqrt{\|v_2\|_2}}. \]

When \( p \) is a point with non-zero principal curvatures, \( h_p^{-1}B_\gamma \) is an ellipse along the eigenspaces of \( h_p \) with lengths along the major and minor axis \( \gamma/\kappa_1(p), \gamma/\kappa_2(p) \) provided \( \kappa_1(p) \) and \( \kappa_2(p) \) are both larger or equal to \( \alpha \). In the case that either of the principal curvatures are too small, by the way we have interpreted \( h_p \) as a \( 2 \times 2 \) matrix, these directions have been clipped and we always have \( h_p^{-1}B_\gamma \subset B_{\sqrt{\gamma/\alpha}}(p) \).

**Definition 4.2.** Let \( \gamma, \alpha > 0 \). For a point \( p \in [0,1]^2 \) with non-zero Gauss curvature set
\[ E_p := \sqrt{h_p^{-1}B_\gamma(p)}. \]

Note that when \( p \in [0,1] \) is a point of zero curvature, set \( E_p = B_{\sqrt{\gamma/\alpha}}(p) \). We define a finite subset \( X \subset [0,1]^2 \) to be a \((\gamma, \alpha)\)-hitting set provided that \( E_p \cap X \neq \emptyset \) for every point \( p \in [0,1]^2 \) with non-zero Gauss curvature. For a fixed \((\gamma, \alpha)\)-hitting set \( X \), define \( Y_X \subset S = \{ S(p) : p \in X \} \) the associated sample set of points on the surface \( S \).

We first relate the notion of our hitting sets to Clarkson’s \( \epsilon \)-covers.

**Lemma 4.3.** Let \( X \) be a \((\gamma, \alpha)\)-hitting set and assume \( \beta < 1/\alpha \). Let \( p \in X \) and set \( E_p \) be defined as Definition 4.2. For any \( q, q' \in E_p \), we have \( |d_{II}(q, q')| \leq (\lambda/\alpha)\sqrt{\gamma} \).

**Proof.** Let \( v_{qq'} \) be the vector in \( I^2 \) starting at \( q \) and ending at \( q' \). Consider the parameterization \( \tilde{\sigma} : [0,\|v_{qq'}\|_2] \rightarrow I^2 \) be given by \( \tilde{\sigma}(t) = q + t(v_{qq'}/\|v_{qq'}\|_2) \). Let \( \sigma = S \circ \tilde{\sigma} \). We have
\[ |d_{II}(q, q')| \leq \left| \int_0^{\|v_{qq'}\|_2} \sqrt{q_{II}(\sigma'(t), \sigma(t))} dt \right| \]
which is at most \( \|v_{qq'}\|_2 \sqrt{q_{II}(\sigma'(t), \sigma(t))} \) so it suffices to estimate \( |q_{II}(\sigma'(t), \sigma(t))| \). Applying the Cauchy-Schwartz inequality one obtains
\[ |q_{II}(\sigma'(t), \sigma(t))| \leq \|\sigma'(t)\|^2_2 \|h_{\sigma(t)}\|_{op} \leq \Lambda^2/\alpha \]
where the last inequality comes from the assumptions on the derivative of \( S \) and that \( \beta < 1/\alpha \). Therefore
\[ |d_{II}(q, q')| \leq \Lambda(1/\sqrt{\alpha})\sqrt{\gamma/\alpha} = \Lambda\sqrt{\gamma}/\alpha. \]

**Corollary 4.4.** For a \((\gamma, \alpha)\)-hitting set \( X \), the associated sample set \( Y_X = \{ S(p) : p \in X \} \) is a \((\Lambda\sqrt{\gamma}/\alpha)\)-cover.

**Remark 4.5.** It is important to note that when \( \epsilon \) is small enough for Liebon-Letcher triangulations to exist, [9, proof of Thm. 4.2] gives an estimation on the Hausdorff distance between his triangulation and the surface for an \( \epsilon \)-cover. In particular, the discussion at the end of the proof ensures a \( \epsilon \)-cover has Hausdorff distance \( 9\epsilon^2 \). Also, Liebon-Letscher [18, Lem. 3.3]
give precisely the density for their triangulations to exist. In particular, it is required that 
\( \varepsilon < \min\{\text{inj}(S)/10, \pi/(10\sqrt{\beta})\} \). However, since our surface is embedded in \( \mathbb{R}^3 \), the Fery-Milnor theorem shows us that any closed geodesic \( \sigma \) satisfies 
\( 4\pi \leq \int_{\sigma} \kappa(\sigma) \) where \( \kappa(\sigma) \) is the curvature along \( \sigma \). Using the estimate 
\( \int_{\sigma} \kappa(\sigma) \leq (\text{len } \sigma) \beta \) where \( \text{len } \sigma \) is the length of \( \sigma \), we have therefore 
\( 4\pi/\beta \leq 2 \cdot \text{inj}(S) \) and so 
\( 2\pi/\beta < \text{inj}(S) \).

If we assume \( \beta > 1 \) then \( \sqrt{\beta} < \beta \) and we have 
\( \pi/(10\beta) < \pi/(10\sqrt{\beta}) \) and so 
\( \pi/10\beta < 2\pi/10\beta < \text{inj}(S) \).

Therefore Liebon-Letscher triangulations exist for \( \varepsilon \)-covers with \( \varepsilon < \pi/(10\beta) \). Since \( \beta \) is assumed to be an upper bound for curvature for \( S \) it is harmless to assume that \( \beta > 1 \). The resulting effect will only change lower order terms and not the overall form of our estimation on the number of needed samples (see Theorem 5.1.)

Thus, when \( \beta > 1 \) we have \( (\Lambda/\alpha)\gamma < \pi/(10\beta) \), any \( (\gamma, \alpha) \)-hitting set is an \( \varepsilon = (\Lambda/\alpha)\sqrt{\gamma} \)-cover for which Liebon-Letscher triangulations exist and the Hausdorff bound between \( S \) and the Clarkson triangulation is at most \( 9(\Lambda/\alpha)^2\gamma \).

We provide a summary of the algorithm where some of the subtleties are explained in more detail in Section 5, in particular Step 3.

**Input:** A parametric surface \( S : [0, 1]^2 \rightarrow \mathbb{R}^3 \), an upper bound \( \beta \) on the curvature of the surface \( (\beta \geq \max\{\|h_p\|_{\text{op}}, 1\}) \) and an upper bound \( \Lambda \) on the directional derivative of curvature and on the first derivative of the manifold \( (\Lambda \geq \max\{\|\partial h/\partial u\|_{\text{op}}, \|\partial S/\partial u\|_2\}) \) and a tolerance \( \eta \) so that 
\[
0 < \eta < \min\left\{ \frac{9\pi^2}{50\beta^2}, e^{-\left(\frac{10\beta}{\pi\Lambda}\right)^2}\right\}.
\]

**Output:** A triangulation \( \Sigma \) with Hausdorff distance between \( \Sigma \) and \( S \) at most \( \eta \).

1. For any \( \rho > 0 \), we determine a family \( \{A_k\} \) of linear operators which partition the parameter space into level sets for which all of the shape operators \( h_p \) at points in a fixed the level set satisfy \( \|h_p A_k^{-1} - I\|_{\text{op}} < \rho \), there are no more than 18 such operators (see Lemma 5.2.)

2. From \( A_k \), construct a subset \( N_k \subset [0, 1]^2 \) with discrepancy \( D(N_k) \) inversely proportional to \( 1/\sqrt{\det|A_k|} \). From each \( N_k \) select the subset of sample points lying in the level set \( k \) and the adjacent ones.

3. The union of the selected points is a \( (\eta/(9\Lambda^2 \ln \eta^{-1}), (\ln(\eta^{-1}))^{-1/2}) \)-hitting set \( X \subset [0, 1]^2 \) (Lemma 5.3.)

4. Take the sample set \( Y_X = \{S(p)|p \in X\} \) and build the triangulation \( \Sigma \) as [9].

**Algorithm 1:** Outline of algorithm to produce approximation of surface

**Remark 4.6.** We do not provide an explicit computation of the operators \( A_k \) nor are their computations truly necessary. Only \( \det A_k \) is needed to construct the sets \( N_k \).
5 Constructing the sample sets and ensuring ambient isotopy

Our next goal is to describe precisely how to construct our hitting sets (see Subsection 5.1) and with some further restriction on \( \eta \) describe precise bounds on the size of the sample set required with respect to the imposed bounds on Hausdorff distance (see Section 5.2.) In particular, we will show our main theorem.

**Theorem 5.1.** Let \( \eta < \frac{9\alpha^2}{50\beta^2} \) be small enough so that \( (\ln \eta)^4 \leq 9/(32\beta^5) \) and \( \eta < e^{-\left(\frac{10\beta}{\pi\Lambda}\right)^2} \).

Algorithm 1 produces a triangulation with Hausdorff distance no more than \( \eta \) using a vertex set of cardinality at most

\[
C_1(\Lambda) \frac{(\ln(\eta^{-1}))^2}{\eta} \int_S \kappa + C_2(\beta, \Lambda) \frac{(\ln(\eta^{-1}))^{3/2}}{\eta} = C_1(\Lambda) \frac{(\ln(\eta^{-1}))^2}{\eta} (1 + o(1)) \int_S \kappa,
\]

where \( C_1 \) is a constant that depends only on \( \Lambda \) and \( C_2 \) is a constant depending on \( \beta \) and \( \Lambda \).

As the choice of \( \alpha \) is consistent throughout, we refer to an \((\gamma, \alpha)\)-hitting set simply as a \( \gamma \)-hitting set. We follow this with a discussion of how to efficiently construct sets sufficient for the approximation, along with upper bounds on their cardinalities. Estimating the sample density of our sample set \( Y_X \) is easily established by the following lemma.

### 5.1 Construction of \( \gamma \)-hitting sets

Our explicit construction of hitting sets is built upon partitioning our parameter space into finitely many regions on which all of the shape operators at points in the same region are close in operator norm. We call these regions level sets and one may easily visualize them as regions where the shape operator is roughly constant. For each level set, one may create a subset of \([0, 1]^2\) with low discrepancy easily with established methods such as the Halton-Hammersley construction [19, p. 70]. At this stage one has a finite collection of finite discrete subsets of the parametric domain. We then give a selection procedure to select our hitting set from their union and estimate the number of points that are selected.

We now describe in more detail Step 2 of Algorithm 1. In particular, we partition the parameter space into regions upon which the surface has approximately constant curvature, i.e., approximately constant shape operator. To control overlap, we have need to control the sign of the eigenvalues involved. By Remark 4.1, the eigenvalues of each shape operator \( h_p \) are ordered. We define the **type** of a \( 2 \times 2 \) matrix with ordered eigenvalues to be a symbol in \((\pm, \pm)\) with the first slot corresponding to the sign of the first eigenvalue, and similarly for the second. Lemma 5.2 assures us that there is a sufficiently dense finite collection of operators to provide a partition of the shape operators into level sets according to type and operator norm.

**Lemma 5.2.** Let \( \rho, \alpha > 0 \) and assume \( \rho < \ln 2 \). There is a set \( \{A_k\} \) of self-adjoint operators of size \( 4(2\ln(\beta/\alpha)\beta^2/\rho)^3 \) such that for each point \( p \) with the eigenvalues of \( h_p \) in \([-\beta, -\alpha] \cup [\alpha, \beta] \), there is a \( A_k \) such that \( ||h_p A_k^{-1} - I||_{op} < \rho \). Furthermore for each \( p \),

\[
|\{k : ||h_p A_k^{-1} - I||_{op} < \rho \text{ and } \text{type}(A_k) = \text{type}(h_p)\}| \leq 8.
\]

**Proof.** The argument breaks into four cases depending on the signs of the eigenvalues of \( h_p \). We give the argument in the case that both eigenvalues are positive. The argument in the other cases is similar.
Let $S_{\alpha}^\beta$ be the space of symmetric real $2 \times 2$ matrices with eigenvalues in $[\alpha, \beta]$. We first demonstrate a family of $2 \times 2$ real matrices which additively approximate any element of $S_{\alpha}^{\ln \beta}$ in operator norm by at most a fixed $\delta > 0$. We then translate this family to a multiplicative approximating family by exponentiation using the matrix exponential $\exp: S_{\alpha}^{\ln \beta} \to S_{\alpha}^\beta$ which is an injection.

Let $\delta > 0$ and suppose $\lambda_0, \ldots, \lambda_t$ form a uniform subdivision of $[\ln \alpha, \ln \beta]$ with $t = (\ln \beta - \ln \alpha)/\delta$. Define $\{A_k\}$ to be the collection of matrices with entries in set $\{\lambda_0, \ldots, \lambda_t\}$. The set $\{A_k\}$ has size $((\ln \beta - \ln \alpha)/\delta)^3$.

Now suppose $\delta < \ln 2$. If $A$ and $B$ lie in $S_{\alpha}^{\ln \beta}$, and $\|A - B\|_{\text{op}} < \delta < \ln 2$, then we claim $\|e^B e^{-A} - I\|_{\text{op}} < 2\delta \beta^2$. Indeed, write $A - B = S$ so $B = A - S$. Set $R = e^{A - S} - e^A$. We have that

$$e^B e^{-A} = e^{A - S} e^{-A} = (e^A + R) e^{-A} = I + Re^{-A}$$

We estimate $\|Re^{-A}\|_{\text{op}} \leq \|R\|_{\text{op}} e^{\|A\|_{\text{op}}} \leq \|R\|_{\text{op}} \beta$. So it suffices to estimate the operator norm of $R = e^{A - S} - e^A$. We do so by

$$\|e^{A - S} - e^A\|_{\text{op}} \leq \| - S\|_{\text{op}} e^{\|A\|_{\text{op}}} e^{\|S\|_{\text{op}}} \leq \delta \beta e^{\delta}.$$

Therefore, $\|e^B e^{-A} - I\|_{\text{op}} = \|Re^{-A}\|_{\text{op}} \leq \beta \|R\|_{\text{op}} \leq \beta \cdot \delta \beta e^{\delta} \leq 2\delta \beta^2$.

Now, realize a shape operator $h_p$ as an element $2 \times 2$ symmetric matrix as in Remark 4.1 and note that its operating norm lies in $[\alpha, \beta]$. Since $0 < \alpha$, there is a matrix logarithm $B \in S_{\alpha}^{\ln \beta}$ such that $e^B = h_p$. By above, we may choose $\tilde{A}_k$ so that $\|\tilde{A}_k - B\|_{\text{op}} < \delta$. Set $A_k = e^{\tilde{A}_k}$ and $\delta = \frac{\sqrt{2\beta^2}}{2\beta^2} < \ln 2$ then we have $\|h_p A_k^{-1} - I\|_{\text{op}} \leq 2\delta \beta^2 = \rho$. So the set $\{A_k\}$ where $A_k = e^{\tilde{A}_k}$ suffices to multiplicatively approximate each $h_p$ with eigenvalues in $[\alpha, \beta]$.

We may view the approximating set $\{\tilde{A}_k\}$ as a lattice in $\mathbb{R}^3$. Each point in $S_{\alpha}^{\ln \beta}$ lines in some cube in this lattice, and so each element in $S_{\alpha}^{\ln \beta}$ is additively within no more than $\delta$ of 8 elements of $\{\tilde{A}_k\}$. Since exp: $S_{\alpha}^{\ln \beta} \to S_{\alpha}^\beta$ is injective, this means every element of $S_{\alpha}^\beta$ is multiplicatively covered by no more than 8 elements of $\{A_k\}$. For the choice $\delta = \frac{\sqrt{2\beta^2}}{2\beta^2} < \ln 2$ the size of $\{A_k\}$ is $(2\ln(\beta/\alpha) \beta^2/\rho)^3$. Taking the union of the sets $\{A_k\}$ defined for the other signed cases gives the desired approximation set with the desired size $4(2\ln(\beta/\alpha) \beta^2/\rho)^3$.

We proceed to explain Step 2 of Algorithm 1. Constructing for each $k$, a set $N_k$ such that the discrepancy of $N_k$ is $D(N_k) = \gamma/\sqrt{|\det A_k|}$ aligned to the principal frame of $A_k$ (so the set $D(N_k)$ has small discrepancy with respect to rectangles aligned to the diagonal basis of $A_k$). Also construct a set $N_\alpha$ with discrepancy $\gamma/\alpha$. This last set is constructed to take care of the clipping that occurs in areas of extremely low curvature, a subtlety that is not readily apparent in the previous summary of Algorithm 1. For each $\rho > 0$ set

$$N_\rho := \{p \in N_k : \|h_p A_k^{-1} - I\|_{\text{op}} < 2\rho \text{ and } \text{type}(h_p) = \text{type}(A_k)\}.$$ 

Our hitting set is the set of the form $N_\rho \cup N_\alpha$ for sufficiently chosen $\rho$.

**Lemma 5.3.** Let $\gamma, \rho, \alpha > 0$. Assume $\rho < 1$ and $\gamma < \left((\alpha^3 \rho)/(2\sqrt{2} \beta^3/2\Lambda)\right)^2$. The set $X = N_\rho \cup N_\alpha$, defined above, is a $2\gamma$-hitting set.
Proof. Our hypothesis for a bound on $\gamma$ becomes useful in the analysis where we need to ensure that an operator $D$ for which we can estimate the operator norm $||D||_{op} < \sqrt{2\gamma}A\beta^{3/2}/\kappa^3$ actually has $||D||_{op} \leq \frac{\gamma}{2}$.

By our definition of $N_\rho$, sample points near a point $p$ depend only on the shape operator at $p$ and so we may handle each type separately. Let $q \in [0,1]^2$. We argue in the case that type$(q) = (+, +)$, the other cases are similar. If the eigenvalues of $h_q$ are larger than $\alpha$ in size, then by Lemma 5.2 there is a value $k$ so that $q \in N_k$. Let $B_\gamma$ be the Euclidean unit ball of radius $\gamma$ centered at the origin. The ellipse $\sqrt{A_k^{-1}\gamma}$ contains a rectangle of area $\gamma/\sqrt{|\det A_k|}$ and thus the affine translate of $\sqrt{A_k^{-1}\gamma}$ to $q$ meets $N_k$ non-trivially. We claim that $\sqrt{A_k^{-1}\gamma} \subset \sqrt{h_q^{-1}B_2}$.

It will be sufficient to show

$$A_k^{-1}B_\gamma \subset h_q^{-1}B_{\gamma + \rho} \subset h_q^{-1}B_{2\gamma}$$

where the last inclusion follows trivially as $\rho < 1$.

Let $w \in B_\gamma$ consider $A_k^{-1}w \in A_k^{-1}B_\gamma$. Set $v = h_qA_k^{-1}w$. We have $h_q^{-1}v = A_k^{-1}w$ and

$$||v||_2 = ||h_qA_k^{-1}w||_2 \leq ||h_qA_k^{-1}||_{op} \gamma.$$  

Since $||h_qA_k^{-1} - I||_{op} < \rho$, $||h_qA_k^{-1}||_{op} < 1 + \rho$ and so $||v||_2 \leq \gamma + \rho \gamma$.

Consider $q' \in h_q^{-1}B_{2\gamma} \cap N_k$. It now suffices to show $q' \in N_\rho$, i.e., $||h_qA_k^{-1} - I||_{op} < 2\rho$. Set $D := h_qA_k^{-1} - I$. We have $h_q' - h_q = Dh_q$ and so $||h_q' - h_q||_{op} = ||D||_{op}||h_q||_{op}$. Since $q' \in h_q^{-1}B_{2\gamma}$, the Euclidean distance between $q$ and $q'$ is at most $\gamma/\kappa_2(q)$. For any point in $I^2$, $\sqrt{\kappa_1\kappa_2} \leq \beta$ for $i = 1, 2$, and so $\kappa^2/\beta \leq \kappa_i$. So we have $\kappa^2/\beta < ||h_q||_{op}$. Using the upper estimate on $||h_q - h_q'||_{op}$ we have

$$||D||_{op}||h_q||_{op} = ||h_q - h_q'||_{op} \leq \Lambda \sqrt{\frac{\gamma}{\kappa_2(q)}} \leq \Lambda \sqrt{\frac{2\gamma}{\kappa}}.$$  

However, we also have that $||h_q - h_q'||_{op} = ||D||_{op}||h_q||_{op} > ||D||_{op}\kappa^2/\beta$.

Therefore, $||D||_{op} < \sqrt{\gamma}A\beta^{3/2}/\kappa^3$. Write $h_qA_k^{-1} - I = T$, then $||T||_{op} < \rho$. Now consider $(h_qh_q^{-1})(h_qA_k^{-1}) = (I + D)(I + T) = I + D + T + DT$. Now $||D + T + DT||_{op} < ||D||_{op} + \rho + \rho ||D||_{op}$ and

$$||h_qA_k^{-1} - I||_{op} \leq ||D||_{op} + \rho + \rho ||D||_{op} \leq 2\rho.$$  

This occurs provided $||D||_{op} < \min\{\rho/2, 1/2\}$ however as $\rho < 1$, it suffices to ensure $||D||_{op} < \rho/2$ which is satisfied by the hypothesis.

Lemma 5.3 shows that $X = N_\rho \cup N_\alpha$ is indeed a hitting set in the sense of Definition 4.2. So, in order to prove Theorem 5.1, we need a bound on the size of $N_\rho$. This, in turn, controls the number of approximating triangles appearing in the approximant as the number of triangles in a two-dimensional Delaunay triangulation is linear in the number of vertices [17].

We determine the size of $N_\rho$ by using the Koksma-Hlawka Theorem which we have included in the desired form for convenience.
Theorem 5.4. (see [16, Thm 5.5, page 151]) Let \( f : [0, 1]^2 \to \mathbb{R} \) be a function of bounded Hardy-Krause variation \( V(f) \) and \( X = x_1, \ldots, x_N \) be a finite point set in \([0, 1]^2\). We have

\[
\left| \int_{[0,1]^2} f d\lambda - \frac{1}{N} \sum_{i=1}^{N} f(x_i) \right| \leq V(f) D(X),
\]

where \( D(X) \) is the discrepancy of \( X \) and \( \lambda \) denotes Lebesgue measure.

For each pair \((t,k)\) we consider level sets of the form

\[ L_{t,k} := \{ p \in [0, 1]^2 : ||h_p A_k^{-1} - I||_{op} < t \rho \} \]

Define \( \varphi_{t,k}(p) = \max\{0, 1 - \frac{1}{\rho} d(p, L_{t,k})\} \) where \( d(p, L_{t,k}) \) denotes the distance between \( p \) and the set \( L_{t,k} \) in the plane. Clearly, \( \varphi_{t,k} \) is differentiable almost everywhere, and we note this function has zero derivative outside \( L_{t+1,k} \setminus L_{t,k} \) and inside \( L_{t,k} \) and has a derivative at most \( \frac{1}{\rho} \) on \( L_{t+1,k} \setminus L_{t,k} \); the same is true for any projection. Therefore, the Hardy-Krause variation is estimated

\[
V(\varphi_{t,k}) \leq \int_{[0,1]^2} \left| \frac{\partial^2}{\partial x \partial y} \varphi_t \right| \lambda + \int_I \left| \frac{d}{dx} \varphi_t \right|_{[0,1]^2} \lambda + \int_I \left| \frac{d}{dy} \varphi_t \right|_{1 \times [0,1]^2} \lambda \leq \frac{3}{\rho} \lambda(L_{t+1,k} \setminus L_{t,k}).
\]

Applying the Koksma-Hlawka Theorem we find:

\[
\left| \int_{[0,1]^2} \varphi_{t,k} d\lambda - \frac{1}{|N_k|} \sum_{p \in N_k} \varphi_{t,k}(p) \right| \leq 4V(\varphi_{t,k}) D(N_k) \text{ and }
\]

\[
|L_{2,k} \cap N_k| \leq |N_k| \lambda(L_{3,k}) + 4|N_k| V(\varphi_{2,k}) D(N_k).
\]

Finally, since we can write \(|N_{\rho}| \leq \sum_k |(L_{2,k} \cap N_k)|\), the Koksma-Hlawka Theorem gives an estimate of the size of \( N_{\rho} \).

A key step in the proof of Theorem 5.6 is to compare how far the selected sample points deviate in curvature from the roughly constant representative curvature of the level set in which they sit. Lemma 5.5 provides the technical comparison.

Lemma 5.5. Let \( A, B : \mathbb{R}^2 \to \mathbb{R}^2 \) be self-adjoint operators with eigenvalues in \([-\beta, \beta]\). If we have \( ||AB^{-1} - I||_{op} < \rho < 1 \), then \( \sqrt{\det A} \leq \sqrt{2(\sqrt{\det B} + \sqrt{\rho^2})} \).

Proof. Write \( AB^{-1} = D + I \) where \( ||D||_{op} < \rho \). Note

\[
\det A = \det(DB + B) \leq \det(D) \det(B) + \det(B) + \rho \text{Tr}(B) \leq \rho \det(B) + \det(B) + \rho \text{Tr}(B).
\]

When \( \rho < 1 \) and \( \text{Tr}(B) \leq \beta \) we see \( \det A \leq 2(\det(B) + \rho \beta) \). Taking square roots gives the result. \( \square \)

We now precisely estimate the size of \( X \).
Lemma 5.3. We have

\[ |X| \leq 32\sqrt{2} \frac{\ln(\gamma^{-1})}{\alpha} \int \kappa d\lambda + 32\sqrt{2} \frac{\ln(\gamma^{-1})}{\gamma^2} \alpha + \frac{24\alpha^3}{\sqrt{2} \beta^3/2 \Lambda} \ln(\gamma^{-1}) + \frac{\alpha}{\gamma} \ln\frac{\alpha}{\gamma}. \]

Proof. First we explain our hypothesis on the values \( \gamma \) and \( \rho \). Rewriting the identity defining \( \rho \) in terms of \( \gamma \) gives us

\[ \gamma = \left( \frac{\alpha^3 \rho}{(4\sqrt{2} \beta^{3/2} \Lambda)} \right)^2 \leq \left( \frac{(\alpha^3 \rho)}{(2\sqrt{2} \beta^{3/2} \Lambda)} \right)^2 \]

and this ensures that Lemma 5.3 may be applied and so our sample set will indeed be a \( 2\gamma \)-hitting set. The choice of bound \( \gamma < \alpha^{10}/(32\Lambda^2 \beta^{5}) \) is done to ensure \( \rho \leq \alpha^2/\beta \).

Note \( D(N_k) = \gamma / \sqrt{\det A_k} \), \( |N_k| = (\sqrt{\det A_k} / \gamma) \ln(\sqrt{\det A_k} / \gamma) \). To estimate \( N_\rho \), consider the bound on the hitting set, \( X \), given by \( |X| \leq |N_\rho| + |N_\alpha| = |N_\rho| + (\alpha/\gamma) \ln(\alpha/\gamma) \).

\begin{align*}
|N_\rho| &= \sum_k |(L_{2,k} \cap N_k)| \\
&\leq \sum_k |N_k| \lambda(L_{3,k}) + 4|N_k| V(\varphi_{2,k}) D(N_k) \\
&\leq \sum_k |N_k| \lambda(L_{3,k}) + |N_k| D(N_k) \frac{12}{\rho} \lambda(L_{3,k}/L_{2,k}) \\
&\leq \sum_k |N_k| \lambda(L_{3,k}) + |N_k| D(N_k) \frac{12}{\rho} \lambda(L_{3,k}) \\
&\leq \sum_k \left[ \frac{\sqrt{\det A_k}}{\gamma} \ln\left( \frac{\sqrt{\det A_k}}{\gamma} \right) + \frac{12}{\rho} \ln\left( \frac{\sqrt{\det A_k}}{\gamma} \right) \right] \lambda(L_{3,k}).
\end{align*}

If \( p \in L_{3,k} \) then \( \sqrt{\det A_k} \leq \sqrt{2} (\kappa(p) + \sqrt{\rho} \beta) \) by Lemma 5.5. If there is no such \( p \) then \( \lambda(L_{3,k}) = 0 \). If we set \( \delta = \sqrt{2} (\kappa(p) + \sqrt{\rho} \beta) / \gamma \), then

\[ |N_\rho| = \sum_k \left[ \delta \ln(\delta) + \frac{12}{\rho} \ln(\delta) \right] \lambda(L_{3,k}). \]

By Lemma 5.2 each \( p \) is in \( L_{3,k} \) for at most 8 different \( k \), therefore

\[ |N_\rho| \leq 8 \left[ \int \int \delta \ln(\delta) + \frac{12}{\rho} \ln(\delta) d\lambda \right]. \]

Since \( \rho \geq \frac{\alpha^2}{\beta} \geq \frac{\kappa^2}{\beta} \) we have

\[ |N_\rho| \leq 8 \left[ \int \int \frac{2\sqrt{2} \kappa(p)}{\gamma} \ln\left( \frac{2\sqrt{2} \kappa(p)}{\gamma} \right) + \frac{12}{\rho} \ln\left( \frac{2\sqrt{2} \kappa(p)}{\gamma} \right) d\lambda \right]. \]
Notice that $1/\gamma \gg 2\sqrt{2}\kappa$ therefore $\ln(2\sqrt{2}\kappa/\gamma) \leq 2\ln(1/\gamma)$. Also as we have only been counting points in $N_\gamma$ we have those points whose shape operator has spectra less than $\alpha$. To correct this, notice $[0,1]^2$ has measure 1 and thus we pick up an additive factor of $\alpha$. Finally, note that $\rho = (4\sqrt{2}\beta^3/2\Lambda \sqrt{\gamma}/\alpha^2)$ so that $12/\rho = 3\alpha^2/(\sqrt{2}\beta^3/2\Lambda \sqrt{\gamma})$, to complete the proof.

We conclude this section with the proof of Theorem 5.1.

Proof. (of Theorem 5.1)

Given $\eta$ meeting the hypothesis, to produce the approximation we set $\gamma = \eta \alpha^2/18\Lambda^2$ and $\alpha = 1/\sqrt{\ln \eta^{-1}}$. The requirement that $\eta < \frac{9\pi^2}{50\beta}$ ensure that

$$\eta = 18\gamma \frac{\Lambda^2}{\alpha^2} < \frac{18\pi^2}{100\beta^2}$$

and so $\sqrt{\gamma} \frac{\Lambda}{\alpha} < \frac{\pi}{10\beta}$.

Thus the construction of the sample set $X$ which is a $2\gamma$-hitting set by Lemma 5.3 has a Lieb-Letscher triangulation by Remark 4.5 and has the desired Hausdorff distance $18\gamma(\Lambda/\alpha)^2 = \eta$ by construction.

The hypothesis $\eta < e^{-\left(\frac{100\gamma}{\pi \alpha}\right)^2}$ ensures $\gamma$ is small enough so that we apply Theorem 5.6 for our $2\gamma$-hitting set and that it will have the desired Theorem 5.1. Since $\eta < e^{-\left(\frac{100\gamma}{\pi \alpha}\right)^2}$ we have $(10\beta/9\pi \Lambda)^2 < \ln \eta^{-1}$ and so $\alpha < (9\pi \Lambda)/(10\beta)$. Note that $\eta < 1$ and $(\ln \eta^{-1})^4 < 9/(32\beta^5)$ therefore and we have $\eta(\ln \eta^{-1})^4 < 9/(32\beta^5)$. Rewriting with $\alpha = 1/\sqrt{\ln \eta^{-1}}$ we have $\eta < 9\alpha^8/(32\beta^5)$ which means $18\gamma(\Lambda/\alpha)^2 < 9\alpha^8/(32\beta^5)$ and so $2\gamma < \alpha^{10}/(32\beta^5)$. It is now routine substitution to transform the estimate in Theorem 5.6 into the desired form of Theorem 5.1.

5.2 Verifying Ambient Isotopic Approximation

We are now ready to give conditions on $\eta$ to ensure an ambient isotopic approximant. Notice that almost the same proof of Lemma 4.3 shows for each point $q$ in $E_p$, then $d_M(S(q,p)) \leq \Lambda \sqrt{\gamma/\alpha}$. This shows that the image on the surface of our $\gamma$-hitting sets are in fact $\Lambda \sqrt{\gamma/\alpha}$-sample sets on the surface as described [11]. Therefore, the set $Y_X$ defined in Definition 4.2 as a subset of the manifold is an $\varepsilon$-sample set for $\varepsilon = (\Lambda/\alpha)\sqrt{\gamma}$ in the sense previously defined [11].

Additional data from the surface is needed to ensure topological guarantees. In particular, denote by $d(\cdot, \cdot)$ the Euclidean distance and $MA(S)$ the medial axis [4]. The parameter of interest is the local feature size $\chi = d(S, MA(S))$. We emphasize that, like [11], our dependence is only on the distance and not on the medial axis itself, hence our ambient isotopic approximation avoids known numerical instabilities [10].

Theorem 5.7. Let $S$ be a surface and denote by $\chi$ the distance to the medial axis. Denote by $\tau$ the approximant given by Algorithm 1 with tolerance $\eta$ meeting the bounds of Theorem 5.1.

- If $\eta \leq 9(\chi/4)^2$ then $\tau$ is homeomorphic to $S$.
- If $\eta \leq \chi^2/(9\beta^2)$ then $\tau$ is ambient isotopic to $S$.

Proof. To prove the first claim, we recall that $\eta = 9\varepsilon^2$ and that the condition $\varepsilon < \chi/4$ is sufficient [11, Thm. 3]. It is also established in [11, Thm. 3] for an $\varepsilon$-sample the normal distance (maximal distance from $S$ to the approximant along normals) is at most $9\varepsilon$ where $\beta$ is maximal curvature.
So when $\eta \leq \chi^2/(9\beta^2)$, it is evident that $9\beta \varepsilon \leq \chi$ or $\varepsilon \leq \chi/(9\beta)$ and so the normal distance from $S$ to the approximant is less than the distance from $S$ to the medial axis of $S$ which is sufficient to guarantee ambient isotopy [4].

We remark that the net change in the estimate of Theorem 5.1 after meeting the bounds of Theorem 5.7 is a change in the leading coefficient of the $\eta^{-1}(\ln \eta^{-1})^2 \int_S \kappa$ which becomes dependent on $\chi$ (or more coarsely $\beta$) in addition to $\Lambda$.

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References


