Quantifying the Performance Gain of Direction Feedback in a MISO System

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***Abstract*— We consider finite-rate channel-direction feedback in a system with multiple transmit-antennas and single receive-antenna. We address how the symbol error rate performance improves as a function of the amount of feedback. We conclude that when the number of feedback directions is equal to or larger than the number of transmit antennas, transmit beamforming is optimal even with limited feedback. On the other hand, when the number of feedback directions is smaller than the number of transmit antennas, the antennas are divided into two groups, where antenna selection is used in the first group to choose the strongest channel and equal power allocation is used in the second group. At high SNR, the optimal power allocation between these two antenna groups is proportional to the number of antennas in each group. Based on high SNR analysis, we fully characterize the power gain of each feedback bit.**

I. Introduction

Multi-antenna communication has attracted a lot of attention lately because of its promise of high transmission rate and much improved performance in fading channels. We in this paper consider a multi-input single-output (MISO) system with N_t transmit-antennas and one receive-antenna.

When the transmitter does not have any channel state information (CSI), a low-rate repetition transmission achieves the optimal error performance, where the information symbol is transmitted N_t times with one antenna at a time [15]. Here, the optimality refers to uncoded error performance, i.e., the system performance without channel coding. On the other hand, if the transmitter has complete channel knowledge, the optimal transmission (without power control) is beamforming, where the information symbol is weighted by a beam-steering vector matched to the channel and transmitted from N_t antennas. Compared with the repetition transmission with no CSI, transmit beamforming with perfect CSI achieves an N_t-fold increase on the received signal-to-noise-ratio (SNR). Intuitively speaking, transmit beamforming concentrates all transmission power to the channel direction, while repetition transmission spreads the transmission power to N_t orthogonal directions due to lack of channel knowledge. This N_t-fold power advantage is often termed as array gain in the literature.

In practical wireless systems, however, the feedback link is often rate-limited. For example, the feedback link can only convey a finite number of bits (say B bits). The channel direction, then, has to be quantized at the receiver side using a codebook with 2^B entries. The receiver picks the quantized channel direction from the codebook, and communicates the codeword index to the transmitter via the B feedback bits. For such a practical setup, interesting questions include:

1. *With finite-rate direction feedback, what is the optimal transmission in terms of uncoded error performance?*
2. *How much performance gain is achieved with a given number of feedback bits?*

We in this paper aim to address these two questions. Assuming that channel coefficients are independent and identically distributed (i.i.d.) according to a complex Gaussian distribution, we derive a tight lower-bound on the symbol error rate (SER). We then optimize the SER lower-bound based on the amount of feedback. We conclude that

- when $B \geq \log_2(N_t)$, transmit beamforming is optimal in terms of minimizing the tight SER lower-bound.
- when $B < \log_2(N_t)$, the antennas are divided into two groups, where antenna selection is used in the first group of $N = 2^B$ antennas to choose the strongest channel, and equal power allocation is used in the second group of $N_t - N$ antennas. At high SNR, the optimal power allocation between these two antenna groups is proportional to the number of antennas in each group.
- Based on high SNR analysis, we quantify the power gain of direction feedback relative to the benchmark with no feedback. When $B < \log_2(N_t)$ each additional bit brings more incremental feedback gain, while when $B > \log_2(N_t)$, the "diminishing returns" effect shows up. The $\log_2(N_t)$-th bit provides the most feedback gain.

There are several existing works relevant to this paper:

- Orthogonal space time block coding (OSTBC) achieves the same performance as single-symbol repetition transmission [14] [15]. Our development in this paper thus applies to the precoded OSTBC setting. Precoded OSTBC has been considered based on statistical CSI in the forms of channel mean feedback [2], [14], and channel covariance feedback [15]. It has also been investigated based on limited direction feedback in [4], [5], [7].
- The works in [6], [8], [16] have focused on transmit-beamforming, and investigated how the average SNR, the outage probability and the SER performance change as a function of the number of feedback bits.
- The conditions on when beamforming is optimal are

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specified in [3] from the ergodic capacity point of view. In particular, it is proved that beamforming is optimal with channel direction feedback if $B = \log_2(N_t)$.

Notation: Bold upper and lower letters denote matrices and column vectors, respectively; $(\cdot)^T$, $(\cdot)^*$, and $(\cdot)^{\mathcal{H}}$ denote transpose, conjugate, and Hermitian transpose, respectively; $\|\cdot\|$ is the vector norm; $\mathbf{I}_N$ is the $N \times N$ identity matrix; $\mathcal{CN}(\cdot)$ stands for complex Gaussian distribution.

II. System Model

We consider a MISO system with N_t transmit-antennas and single receive-antenna. The channel coefficients are collected in an $N_t \times 1$ channel vector $\mathbf{h} := [h_1, \ldots, h_{N_t}]^T$. We assume that h_μ's are independent and identically distributed (i.i.d.) according to a complex Gaussian distribution with zero mean and unit variance, i.e.,

$$\mathbf{h} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{N_t}). \tag{1}$$

A. Space-time spreading and precoded OSTBC

We define a space time spreading (STS) matrix $\mathbf{T}$ of size $N_t \times N_t$. With $\mathbf{X} = \mathbf{T}s$ transmitted from N_t antennas, the receiver obtains

$$\mathbf{y}^T = \mathbf{h}^T \mathbf{T} s + \mathbf{v}^T, \tag{2}$$

where $\mathbf{v}$ is the additive white Gaussian noise with each entry having zero-mean and variance N_0. The instantaneous SNR at the maximum-ratio-combiner (MRC) output is

$$\gamma = \|\mathbf{h}^T \mathbf{T}\|^2 \frac{E_s}{N_0}, \tag{3}$$

where E_s is the average symbol energy.

Precoded OSTBC achieves the same performance as STS thus improving the transmission rate with no cost [14], [15]. Let $\boldsymbol{\mathcal{O}}_{N_t}$ denote the OSTBC for N_t transmit-antennas. For example, when $N_t = 2$ and $N_t = 4$, the OSTBCs are, respectively [1], [10]:

$$\boldsymbol{\mathcal{O}}_2 = \begin{bmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{bmatrix}, \quad \boldsymbol{\mathcal{O}}_4 = \begin{bmatrix} s_1 & 0 & -s_2^* & s_3^* \\ 0 & s_1 & -s_3 & -s_2 \\ s_2 & s_3^* & s_1^* & 0 \\ -s_3 & s_2^* & 0 & s_1^* \end{bmatrix}. \tag{4}$$

The transmitted space-time code-matrix corresponding to precoded OSTBC is

$$\mathbf{X} = \mathbf{T}\boldsymbol{\mathcal{O}}_{N_t}, \tag{5}$$

where $\mathbf{T}$ now serves the role of precoder. The received vector corresponding to the transmitted $\mathbf{X}$ is

$$\mathbf{y}^T = \mathbf{h}^T \mathbf{X} + \mathbf{v}^T = \mathbf{h}^T \mathbf{T} \boldsymbol{\mathcal{O}}_{N_t} + \mathbf{v}^T. \tag{6}$$

Linear processing on $\mathbf{y}$ leads to separate decoding on each information symbol, and the SNR per symbol is given in (3).

Note that STS transmits only one symbol in N_t time slots, thus the transmission rate is extremely low. However, STS provides an error-performance bound for any linear space time block coding, where multiple symbols are multiplexed for transmission [14], [15]. Studying STS, or equivalently precoded OSTBC, serves our purpose of transmitter optimization with error performance as the criterion.

B. Finite-rate direction feedback

Due to the feedback bandwidth constraint, perfect knowledge of $\mathbf{h}$ at the transmitter side is not available. As in [6], [8], [16], we consider finite-rate direction feedback. Suppose that the feedback link can support B feedback bits. The receiver can quantize the channel direction, which is the normalized channel $\tilde{\mathbf{h}} = \mathbf{h}/\|\mathbf{h}\|$, into one out of $N = 2^B$ possible choices from a pre-defined codebook:

$$\mathbf{W} := [\mathbf{w}_1, \ldots, \mathbf{w}_N]. \tag{7}$$

The quantization is based on

$$\mathbf{w}^{\text{opt}} = \arg \max_{\mathbf{w} \in \mathbf{W}} |\mathbf{w}^{\mathcal{H}} \tilde{\mathbf{h}}|. \tag{8}$$

Codebook design of $\mathbf{W}$ has been addressed in [6], [8], [13], which aims to minimize the maximum cross-correlation between any two vectors

$$\min_{\mathbf{W}} \rho_{\max} \quad \text{where } \rho_{\max} := \max_{i \neq j} |\mathbf{w}_i^{\mathcal{H}} \mathbf{w}_j|. \tag{9}$$

Let us specify the optimal codebook for some special cases.

- When $N = N_t$, the optimal codebook is $\mathbf{W} = \mathbf{I}_{N_t}$, which leads to $\rho_{\max} = 0$. Direction feedback amounts to antenna selection that specifies the strongest channel from N_t antennas [6], [13].
- When $N < N_t$, the optimal codebook can be taken as first N columns of $\mathbf{I}_{N_t}$ with $\rho_{\max} = 0$ [6], [13]. Direction feedback amounts to specifying the strongest channel from the first N antennas.
- When $N > N_t$, optimal codebook based on (9) is usually pursued by numerical search [6], [8], [13].

C. Precoding based on direction feedback

Based on the feedback direction $\mathbf{w}^{\text{opt}}$, the transmitter responds with a precoder $\mathbf{T}$. We now specify the design of $\mathbf{T}$. Represent $\mathbf{T}$ by its singular value decomposition (SVD) as

$$\mathbf{T} = \mathbf{U} \boldsymbol{\Delta}^{\frac{1}{2}} \mathbf{V}^{\mathcal{H}}, \tag{10}$$

where $\mathbf{U}$ and $\mathbf{V}$ contain left and right singular vectors, and $\boldsymbol{\Delta}^{\frac{1}{2}} = \text{diag}(\delta_1^{\frac{1}{2}}, \ldots, \delta_{N_t}^{\frac{1}{2}})$ contains singular values of $\mathbf{T}$. We impose a power constraint as

$$\text{tr}(\boldsymbol{\Delta}) = 1. \tag{11}$$

The instantaneous SNR in (3) reduces to

$$\gamma = \|\mathbf{h}^T \mathbf{U} \boldsymbol{\Delta}^{\frac{1}{2}} \mathbf{V}^{\mathcal{H}}\|^2 \overline{\gamma}, \tag{12}$$

where we define $\overline{\gamma} = \frac{E_s}{N_0}$ for notational brevity. The matrices $\mathbf{U}, \boldsymbol{\Delta}$, and $\mathbf{V}$ shall be chosen adaptively based on the direction feedback. We observe that:

- $\mathbf{V}$ does not affect the SNR, thus the system performance. Hence, we set $\mathbf{V} = \mathbf{I}_{N_t}$.
- With only one direction $\mathbf{w}^{\text{opt}}$ available at the transmitter, we will set the first column of $\mathbf{U}$ to be $\mathbf{w}^{\text{opt}}$, and the remaining columns to be orthogonal to $\mathbf{w}^{\text{opt}}$. This is

consistent with [14] based on channel mean feedback. In other words, we have

$$\mathbf{U} = [\mathbf{w}^{\mathrm{opt}}, \mathbf{u}_2, \ldots, \mathbf{u}_{N_t}], \tag{13}$$

where $\mathbf{u}_2, \ldots, \mathbf{u}_{N_t}$ are orthogonal to each other, and orthogonal to $\mathbf{w}^{\mathrm{opt}}$.

- When $N \geq N_t$, the optimal codebook $\mathbf{W}$ has full rank, thus the channel coefficients from all N_t antennas affect the direction quantization. As only the first direction in $\mathbf{U}$ is distinct from the remaining columns, the power allocation among directions should be

$$\mathbf{\Delta} = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_2), \tag{14}$$

because the last $N_t - 1$ directions have to be treated equally (note that the last $N_t - 1$ directions can be arbitrarily chosen). This agrees with the power loading solution derived in [14] with channel mean feedback.

- When $N < N_t$, the antennas are divided into two groups, with the first N antennas in the first group and and the last $N_t - N$ antennas in the second group. The direction feedback is only relevant to first N antennas, while independent of the last $N_t - N$ antennas. The first group should follow the power allocation in (14). On the second antenna group, power should be equally distributed due to lack of channel information. Therefore, the power allocation among directions should be

$$\mathbf{\Delta} = \mathrm{diag}(\underbrace{\delta_1, \delta_2, \ldots, \delta_2}_{N}, \underbrace{\delta_3, \ldots, \delta_3}_{N_t - N}). \tag{15}$$

In summary, corresponding to $N = 2^B$ different directions $\mathbf{w}_i$, we have N precoding matrices as

$$\mathbf{T}_i = \mathbf{U}_i \mathbf{\Delta}^{\frac{1}{2}}, \quad i = 1, \ldots, N, \tag{16}$$

where the quantized direction feedback affects $\mathbf{U}_i$. We will optimize power allocation $\mathbf{\Delta}$ in later sections. Notice that:

- When $\delta_1 = 1$, the system reduces to transmit beamforming studied in [6], [8], [16].
- When $\delta_1 = \delta_2 = \delta_3 = 1/N_t$, the system reduces to the plain OSTBC system [1], [10]; the SNR in (12) becomes $\gamma = \frac{1}{N_t}\|\mathbf{h}\|^2 E_s/N_0$, which is independent of $\mathbf{U}$.

With this general formulation, we would like to address the following issues:

- What is the system performance for any finite feedback, where $B = 0, 1, \ldots, \infty$?
- How do we optimize the system performance by judicious power allocation?
- Under what conditions, transmit beamforming ($\delta_1 = 1$) is optimal?

III. Performance Analysis with $N \geq N_t$

We first consider the case $N \geq N_t$, where $\mathbf{\Delta}$ takes the form of (14). For exposition simplicity, we consider M-PSK constellation, where M is the constellation size. We underscore that similar analysis applies to any two-dimensional constellation [16]. The instantaneous SER is

$$\mathrm{SER}(\gamma) = \frac{1}{\pi}\int_0^{\frac{(M-1)\pi}{M}} \exp\left(-\frac{g_{\mathrm{PSK}}\gamma}{\sin^2\theta}\right) d\theta, \tag{17}$$

where $g_{\mathrm{PSK}} := \sin^2(\pi/M)$ is a constellation-dependent constant [9]. With random $\mathbf{h}$, the average SER is expressed as:

$$\overline{\mathrm{SER}} = E_{\mathbf{h}}\{\mathrm{SER}(\gamma)\}. \tag{18}$$

We want to find either exact or (tight) approximate expressions for $\overline{\mathrm{SER}}$.

Define the normalized channel vector as: $\tilde{\mathbf{h}} = \frac{\mathbf{h}}{\|\mathbf{h}\|}$. Substituting (13) and (14) into (12), we obtain:

$$\begin{aligned}\gamma &= \overline{\gamma}\, \|\mathbf{h}\|^2 \left[(\delta_1 - \delta_2) \max_i \left|\mathbf{w}_i^{\mathcal{H}} \cdot \frac{\mathbf{h}}{\|\mathbf{h}\|}\right|^2 + \delta_2\right] \\ &= \overline{\gamma}\, \|\mathbf{h}\|^2 \left\{(\delta_1 - \delta_2)\left[1 - \min_i d^2(\mathbf{w}_i, \tilde{\mathbf{h}})\right] + \delta_2\right\}\end{aligned} \tag{19}$$

where $d(\cdot)$ is the chordal distance define as $d(\mathbf{w}, \tilde{\mathbf{h}}) = \sqrt{1 - |\mathbf{w}^{\mathcal{H}}\tilde{\mathbf{h}}|^2}$. To simplify notation, we define:

$$\gamma_h := \|\mathbf{h}\|^2, \quad Z := \min_i d^2(\mathbf{w}_i, \tilde{\mathbf{h}}), \tag{20}$$

such that

$$\gamma = \gamma_h \overline{\gamma}\Big[(\delta_1 - \delta_2)(1 - Z) + \delta_2\Big]. \tag{21}$$

We infer from (1) that γ_h is Chi-square distributed with $2N_t$ degrees of freedom; its probability density function (pdf) is $p(\gamma_h) = \Gamma^{-1}(N_t)\gamma_h^{N_t - 1} e^{-\gamma_h}$, where $\Gamma(\cdot)$ is the Gamma function [9]. Let $p(z)$ and $F_Z(z)$ denote the pdf and cdf (cumulative distribution function) of Z. Due to the i.i.d. assumption in (1), γ_h is independent of $\tilde{\mathbf{h}}$; thus, γ_h and Z are independent. We first average (18) over γ_h to obtain (see similar derivation in [16])

$$\begin{aligned}\overline{\mathrm{SER}} &= \frac{1}{\pi}\int_0^{\frac{(M-1)\pi}{M}} \int_0^1 \\ &\left\{1 + \frac{g_{\mathrm{PSK}}\overline{\gamma}}{\sin^2\theta}\Big[(\delta_1 - \delta_2)(1 - Z) + \delta_2\Big]\right\}^{-N_t} dF_Z(z) d\theta.\end{aligned} \tag{22}$$

An upper bound on $F_Z(z)$ is shown in [16] as:

$$F_Z(z) \leq \tilde{F}_Z(z) = \begin{cases} N z^{N_t - 1}, & 0 \leq z < (1/N)^{1/(N_t - 1)} \\ 1, & z \geq (1/N)^{1/N_t - 1} \end{cases}.$$

As shown in Appendix A of [16], replacing $F_Z(z)$ by $\tilde{F}_Z(z)$ in (22) will lead to a tight lower bound on $\overline{\mathrm{SER}}$ as:

$$\begin{aligned}\overline{\mathrm{SER}}_{\mathrm{lb}} &= \frac{1}{\pi}\int_0^{\frac{(M-1)\pi}{M}} \left(1 + \delta_1 \frac{g_{\mathrm{PSK}}\overline{\gamma}}{\sin^2\theta}\right)^{-1} \cdot \\ &\left[1 + \left[\delta_1 - (\delta_1 - \delta_2)\left(\frac{1}{N}\right)^{\frac{1}{N_t - 1}}\right] \frac{g_{\mathrm{PSK}}\overline{\gamma}}{\sin^2\theta}\right]^{1 - N_t} d\theta.\end{aligned} \tag{23}$$

In carrying out the integration, we have used a variable change of $t = 1/z$. Eq. (23) reduces to eq. (32) of [16] when $\delta_1 = 1$, thus including transmit-beamforming as a special case. On the other hand, it reduces to plain OSTBC when $\delta_1 = \delta_2 = \frac{1}{N_t}$:

$$\overline{\text{SER}}_{\text{lb}} = \frac{1}{\pi} \int_0^{\frac{(M-1)\pi}{M}} \left(1 + \frac{g_{\text{PSK}}\overline{\gamma}}{N_t \sin^2\theta}\right)^{-N_t} d\theta, \tag{24}$$

where each diversity branch has SNR $\overline{\gamma}/N_t$. Our numerical results show that $\overline{\text{SER}}_{\text{lb}}$ in (23) is extremely tight for the entire SNR range.

Based on the $\overline{\text{SER}}_{\text{lb}}$ of (23), we now look for optimal power allocation δ_1 and δ_2. We have the following result:

Proposition 1 *In an MISO system with finite-rate direction feedback, transmit beamforming minimizes a tight SER lower bound as long as the number of feedback bits B satisfies $N = 2^B \geq N_t$.*

Proof: Due to the power constraint (11), we have $\delta_2 = (1 - \delta_1)/(N_t - 1)$. For notational brevity, we define $\beta = g_{\text{PSK}}\overline{\gamma}/\sin^2\theta$. The integrand in (23) is

$$f(\delta_1) = (1 + \delta_1\beta)^{-1} \left[1 + \frac{1}{N_t - 1}\left(\frac{1}{N}\right)^{\frac{1}{N_t-1}} \beta + \left[1 - \frac{N_t}{N_t - 1}\left(\frac{1}{N}\right)^{\frac{1}{N_t-1}}\right]\delta_1\beta\right]^{1-N_t}. \tag{25}$$

We verify that $N_t \geq [N_t/(N_t-1)]^{N_t-1}$ for any $N_t \geq 2$. With our assumption $N \geq N_t$, we have $N \geq [N_t/(N_t-1)]^{N_t-1}$, which leads to $1 - \frac{N_t}{N_t-1}\left(\frac{1}{N}\right)^{\frac{1}{N_t-1}} \geq 0$. Therefore, the integrand $f(\delta_1)$ is always a decreasing function of δ_1 for any N_t and β (thus θ). This reveals that $\overline{\text{SER}}_{\text{lb}}$ is a decreasing function of δ_1. Hence the SER optimal δ_1 is $\delta_1 = 1$. ■

Proposition 1 reveals that transmit beamforming achieves the optimal performance, as soon as the feedback enables full diversity. In a MISO system, there is no need for precoded OSTBC when $N \geq N_t$.

IV. Performance Analysis with $N < N_t$

When $N < N_t$, we divide the antennas into two groups, with the first group containing the first N antennas and the second group containing the remaining $N_t - N$ antennas. The power loading matrix follows the form in (15). The direction feedback specifies the strongest channel within the first group of N antennas. Based on Proposition 1, we conclude that $\delta_2 = 0$, i.e., beamforming (or antenna selection in this case) should be implemented in the first group of antennas. Whatever power is assigned to the first group of antennas, the power should be concentrated to the strongest channel within the group.

We now look for optimal power allocation between the two sets of antennas, with power constraint $\delta_1 + (N_t - N)\delta_3 = 1$. The instantaneous SNR based on feedback is

$$\gamma = \overline{\gamma}\left(\delta_1 \max_{1\leq i\leq N} |h_i|^2 + \delta_3 \sum_{j=N+1}^{N_t} |h_j|^2\right). \tag{26}$$

Applying the bounding technique on the first group of antennas, and carrying out the expectation on the second group of antennas, we obtain the SER lower bound for $N < N_t$ as:

$$\overline{\text{SER}}_{\text{lb}} = \frac{1}{\pi} \int_0^{\frac{(M-1)\pi}{M}} \left(1 + \delta_3 \frac{g_{\text{PSK}}\overline{\gamma}}{\sin^2\theta}\right)^{-(N_t-N)} \cdot \left(1 + \delta_1 \frac{g_{\text{PSK}}\overline{\gamma}}{\sin^2\theta}\right)^{-1} \left[1 + \left[1 - \left(\frac{1}{N}\right)^{\frac{1}{N-1}}\right]\delta_1 \frac{g_{\text{PSK}}\overline{\gamma}}{\sin^2\theta}\right]^{1-N} d\theta. \tag{27}$$

We now optimize δ_1 to minimize the $\overline{\text{SER}}_{\text{lb}}$ in (27).

- At extreme low SNR with $\overline{\gamma} \ll 1$, one can easily verify that the integrand in (27) is a decreasing function of δ_1, by using $(1 + x)^{-N} \approx 1 - Nx$ for $x \ll 1$. Hence, beamforming ($\delta_1 = 1$) is optimal at low SNR.
- At high SNR, we obtain:

$$\overline{\text{SER}}_{\text{lb}} \approx \left[\frac{1}{\pi}\int_0^{\frac{(M-1)\pi}{M}} \left(\frac{g_{\text{PSK}}}{\sin^2\theta}\right)^{-N_t} d\theta\right] \cdot \delta_3^{-(N_t-N)} \delta_1^{-N} \left[1 - \left(\frac{1}{N}\right)^{\frac{1}{N-1}}\right]^{1-N} \overline{\gamma}^{-N_t} \tag{28}$$

  Therefore, our objective becomes

$$\text{minimize } \delta_3^{-(N_t-N)}\delta_1^{-N} \text{ where } \delta_1 + (N_t - N)\delta_3 = 1. \tag{29}$$

  The optimal solution is easily found as

$$\delta_1 = \frac{N}{N_t}, \quad \delta_3 = \frac{1}{N_t} \tag{30}$$

  The optimal power splitting at high SNR is based on the ratio of the number of antennas in each group.
- For a general SNR, we resort to numerical search for δ_1. The optimal δ_1 depends on N_t, N, $\overline{\gamma}$ and g_{PSK}.

We summarize the results in this section as:

Proposition 2 *Consider an MISO system with finite-rate direction feedback. When $N = 2^B < N_t$, transmit beamforming is optimal only at extreme low SNR. At high SNR, the optimal power loading matrix is*

$$\mathbf{\Delta} = \textit{diag}\Big(\frac{N}{N_t}, 0, \ldots, 0, \underbrace{\frac{1}{N_t}, \ldots, \frac{1}{N_t}}_{N_t-N}\Big). \tag{31}$$

With no feedback, each antenna gets power allocation $1/N_t$. With feedback, the first N antennas focuses all energy into the strongest channel at high SNR, and the power allocation in the remaining $N_t - N$ antennas remains unchanged.

V. High SNR Power Gain Quantification

One can approximate system performance at high SNR as

$$\overline{\text{SER}} \approx (G_c \cdot \overline{\gamma})^{-G_d}, \tag{32}$$

where G_d is termed as the diversity gain, and G_c is referred to as the coding gain. Eq. (32) implies that the SER versus average SNR curve in fading channels is well approximated by a straight line at high SNR, when plotted on a log-log scale. The diversity gain G_d determines the slope of the curve, while G_c (in decibels) determines the shift of the curve in SNR relative to a benchmark SER curve of $\overline{\gamma}^{-G_d}$. Based on the SER lower-bound in Section III and in Section IV, we have

$$G_d = N_t, \tag{33}$$

which means the full diversity is always achieved. Define a constellation-specific constant as

$$C_0 = g_{\mathrm{PSK}} \left[\frac{1}{\pi} \int_0^{\frac{(M-1)\pi}{M}} (\sin\theta)^{2N_t} d\theta \right]^{-\frac{1}{N_t}}. \tag{34}$$

From (23), (24), and (28), we infer the coding gains for different N as:

$$G_c = \begin{cases} \frac{1}{N_t} C_0 & N = 1 \text{ (no CSI)} \\ \frac{1}{N_t} N^{\frac{N}{N_t}} \left[1 - \left(\frac{1}{N}\right)^{\frac{1}{N-1}}\right]^{\frac{N-1}{N_t}} C_0 & 1 < N < N_t \\ \left[1 - \left(\frac{1}{N}\right)^{\frac{1}{N_t-1}}\right]^{1-\frac{1}{N_t}} C_0 & N \geq N_t \\ C_0 & N = \infty \end{cases} \tag{35}$$

We define power gain (PG) as the ratio of the coding gain with a particular N relative to that with no feedback ($N = 1$). Feedback improves the effective SNR from $\overline{\gamma}$ to $\mathrm{PG} \cdot \overline{\gamma}$. We obtain PG from (35) as

$$\mathrm{PG} = \begin{cases} N^{\frac{N}{N_t}} \left[1 - \left(\frac{1}{N}\right)^{\frac{1}{N-1}}\right]^{\frac{N-1}{N_t}}, & 1 \leq N < N_t \\ N_t \left[1 - \left(\frac{1}{N}\right)^{\frac{1}{N_t-1}}\right]^{\frac{N_t-1}{N_t}}, & N \geq N_t \end{cases} \tag{36}$$

We proceed to determine the power gain per feedback bit. The power gain due to the Bth bit is the ratio of the power gains with B-bit and $(B-1)$-bit feedback. From (36), we obtain the power gain for the B-th bit (denoted as $\mathcal{G}(B)$) as

$$\mathcal{G}(B) = \begin{cases} \dfrac{2^{\frac{B2^B}{N_t}} \left(1 - 2^{-\frac{B}{2^B-1}}\right)^{\frac{2^B-1}{N_t}}}{2^{\frac{(B-1)2^{B-1}}{N_t}} \left(1 - 2^{-\frac{B-1}{2^{B-1}-1}}\right)^{\frac{2^{B-1}-1}{N_t}}} & B \leq \log_2(N_t) \\ \dfrac{\left(1 - 2^{-\frac{B}{N_t-1}}\right)^{\frac{N_t-1}{N_t}}}{\left(1 - 2^{-\frac{B-1}{N_t-1}}\right)^{\frac{N_t-1}{N_t}}} & B > \log_2(N_t). \end{cases} \tag{37}$$

We verify that the power gain per bit is an increasing function when $B \leq \log_2(N_t)$, while it becomes a decreasing function when $B > \log_2(N_t)$. Therefore, when $B < \log_2(N_t)$ each additional bit brings more incremental feedback gain, while when $B > \log_2(N_t)$, we have "diminishing returns". The $\log_2(N_t)$-th bit yields the most feedback gain.

VI. Numerical Results

In all test cases, we use QPSK constellation ($M = 4$).

Test Case 1) optimal versus suboptimal power allocation. We compare the SER lower bound (27) with optimal power

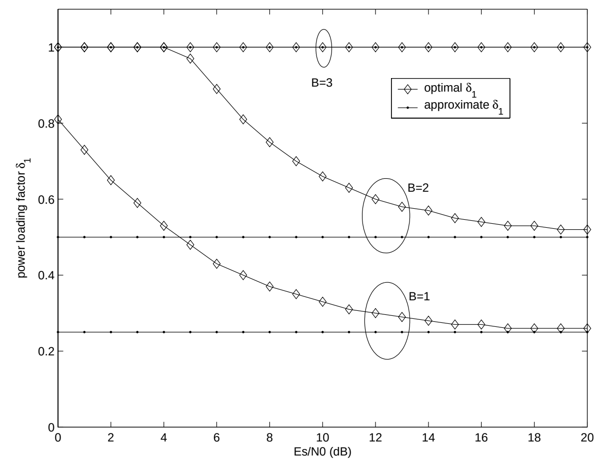

Fig. 1. Optimal versus suboptimal power loading ($N_t = 8$, $B = 1, 2, 3$)

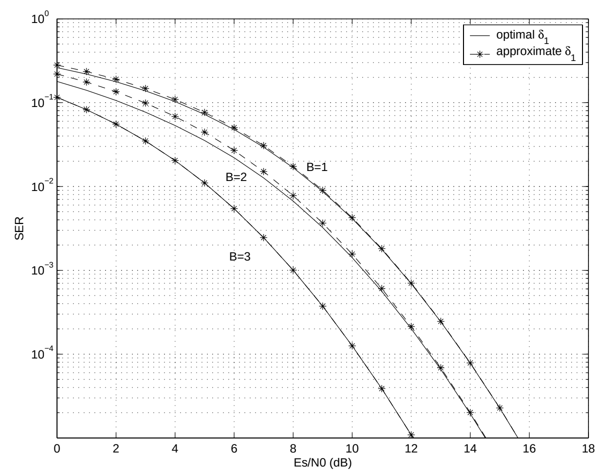

Fig. 2. The SER lower bound with optimal and suboptimal power loadings

allocation and that with the high-SNR power allocation of (30). The optimal power allocation is obtained by numerical search, which is SNR dependent. With $N_t = 8$ and $B = 1, 2, 3$, Fig. 1 shows that the optimal δ_1 changes with respect to SNR when $B < \log_2(N_t)$, and the optimal δ_1 becomes $\delta_1 = 1$ when $B = \log_2(N_t)$. Fig. 2 shows that the SER lower bound with high-SNR δ_1 in (30) is extremely close to that with the optimal power loading. Therefore, in practice, we can use high-SNR power loading with negligible performance loss.

Test Case 2) the accuracy of SER lower bound. In Fig. 3, we compare the SER lower bound in (27) with the exact SER obtained by Monte-Carlo simulation, where $N_t = 8$ and $B = 0, 1, 2, 3, \infty$. We conclude that the SER lower bound is very tight, thus one can rely on the lower bound to approximate the exact SER.

Test Case 3) power gain with respect to the number of feedback bits. With $N_t = 2, 4, 8, 16$, we plot in Fig. 4 the power gain (in decibels) due to direction feedback relative to the no feedback case. Figs. 4 specifies how the SER curves shift when the number of feedback bits changes. In Fig. 5, we plot the power gain corresponding to each bit. When

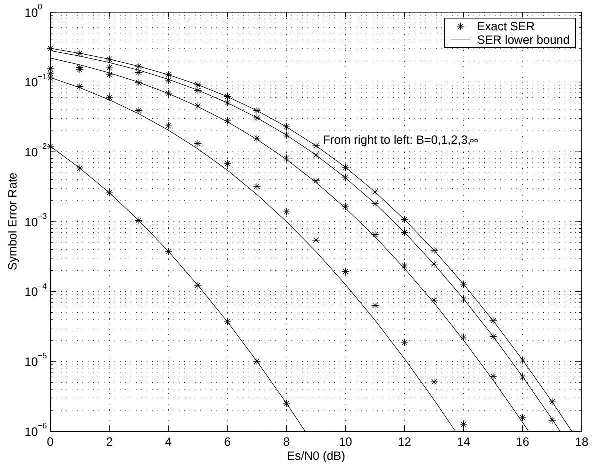

Fig. 3. BER lower bound versus exact SER ($N_t = 8$, $B = 0, 1, 2, 3, \infty$)

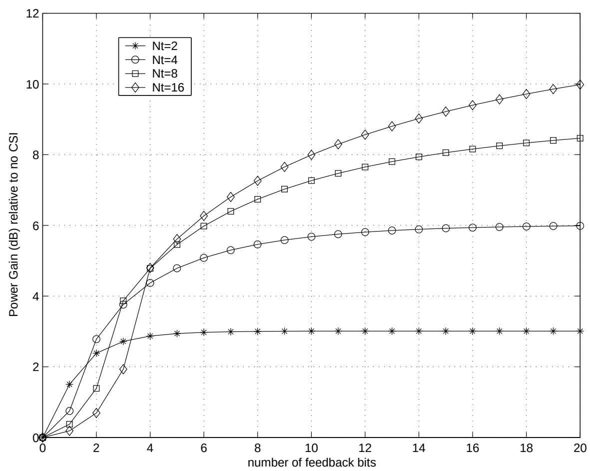

Fig. 4. The power gain due to finite-rate direction feedback

$B < \log_2(N_t)$, each additional bit brings more feedback gain, while when $B > \log_2(N_t)$, we see "diminishing returns". The $\log_2(N_t)$-th bit gives the most feedback gain. These observations agree with theoretical analysis.

VII. Conclusions

In this paper, we quantized the performance gain with finite-rate channel-direction feedback in a MISO system. We observed that when the number of feedback directions is larger than the number of antennas, transmit beamforming is optimal in minimizing a tight SER lower bound. On the other hand, when the number of feedback directions is less than the number of transmit-antennas, a judicious power allocation is needed between two groups of antennas, with antenna selection in one group and uniform power allocation in another group. We quantized the power gain corresponding to each feedback bit.

The results in this paper are only applicable to a MISO system in the absence of feedback error. The extension to systems with multiple receive antennas (e.g., [7]), and systems with feedback errors (e.g., [5], [12]) warrants further investigation.

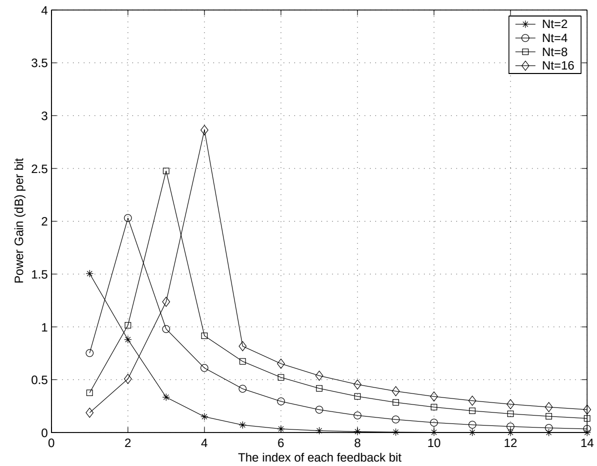

Fig. 5. The power gain corresponding to each feedback bit

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