

Performance Analysis for Transmit-Beamforming with Finite-Rate Feedback

Shengli Zhou¹, Zhengdao Wang², and Georgios B. Giannakis³

¹ Dept. of ECE, Univ. of Connecticut, 371 Fairfield Road, Storrs, Connecticut 06269

² Dept. of ECE, Iowa State University, 3134 Coover Hall, Ames, IA 50011

³ Dept. of ECE, Univ. of Minnesota, 200 Union St. SE, Minneapolis, MN 55455

Abstract — Transmit beamforming achieves optimal performance in systems with multiple transmit- and single receive- antennas, when perfect channel knowledge is available at the transmitter. In practical systems where the feedback link can only convey a finite number of bits, the beamformer design has been extensively investigated from both the outage probability and the average signal to noise ratio (SNR) perspectives. In this paper, we study the symbol error rate (SER) for transmit beamforming with finite-rate feedback. We derive a SER lower bound, which is tight for good beamformer designs. Comparing this bound with the SER of the ideal case, we quantify the power loss due to the finite rate constraint, across the whole SNR range.

I. INTRODUCTION

Multi-antenna diversity has by now been well established as an effective fading counter measure for wireless communications. In certain application scenarios, e.g., cellular downlink, the number of receive antennas at the mobile is limited due to size and cost constraints, which endeavors transmit-diversity systems. Our attention in this paper is thereby focused on downlink application scenarios dealing with single receive- but multiple transmit- antennas.

To further improve the system performance, the receiver can feed channel state information (CSI) back to the transmitter. With perfect CSI, transmit-beamforming achieves the optimal performance in the multi-input single-output (MISO) setting [8]. However, in practical wireless systems, the feedback channel may only be able to communicate a finite number of feedback bits per block [1, 3, 5–8]. With finite rate feedback, power control is investigated in [1] to reduce the outage probability that the mutual information falls below a certain rate; while transmit beamforming has been investigated based on the average signal to noise ratio (SNR) criterion [6, 8], and on the outage probability criterion [7], respectively. The extension to spatial multiplexing systems with finite-rate unitary precoding is available in [5]. Subject to finite-rate feedback, optimal transmission is also pursued in [3] to maximize the average channel capacity, while adaptive modulation together with transmit beamforming has been pursued in [12] to enhance the transmission rate.

We now elaborate on the results of [7] and [6] on transmit-beamforming with finite-rate feedback. A universal lower bound on the outage probability, that is applicable to all beamformers, is derived in [7]. Good beamformers are then constructed to minimize the outage probability. On the other hand, the beamforming vector codebook is designed in [6] to mini-

mize the reduction on the average SNR due to finite-rate feedback. Albeit starting from different perspectives, the beamformer design criteria in [7] and [6] are equivalent, that the maximum correlation between any pair of beamforming vectors should be minimized. Interestingly, the beamformer design is linked to the line packing problem in Grassmannian manifold, where the minimum chordal distance between any pair of straight lines passing through the origin should be maximized [2].

Distinct from [7] and [6], we in this paper investigate the symbol error rate (SER) performance of transmit beamforming with finite rate feedback. We derive a SER lower bound, that is applicable to any beamformer design. This bound is tight for the good beamformers constructed in [6, 7], thus serves as a good performance indicator. Furthermore, comparing the lower bound with the SER corresponding to the perfect CSI case enables us to quantify the power loss due to the finite rate constraint. Our result is valid across the whole SNR range.

The rest of this paper is organized as follows. We present the system model in Section II, derive the SER lower bound in Section III, and quantify the power loss in Section IV. We then collect numerical results in Section V, and draw concluding remarks in VI.

Notation: Bold upper and lower case letters denote matrices and column vectors, respectively; $\|\cdot\|$ denotes vector norm; $(\cdot)^*$, $(\cdot)^T$, and $(\cdot)^H$ stand for conjugate, transpose, and Hermitian transpose, respectively; $E\{\cdot\}$ denotes expectation; \mathbf{I}_N stands for an identity matrix of size N , and $\mathbf{0}_{M \times N}$ for an $M \times N$ all zero matrix; \mathcal{C}^N stands for the N -dimensional complex vector space; $\mathcal{CN}(\mathbf{b}, \Sigma)$ denotes the complex Gaussian distribution with mean \mathbf{b} and covariance Σ .

II. SYSTEM MODEL

Fig. 1 depicts the considered multi-input single-output (MISO) system with N_t transmit- and single receive- antennas. Let h_μ denote the channel coefficient between the μ -th transmit antenna and the receive antenna, and collect h_μ 's into the channel vector $\mathbf{h} := [h_1, \dots, h_{N_t}]^T$. As in [6, 7], we assume that h_μ 's are independently and identically distributed (i.i.d.) with a complex Gaussian distribution with zero mean and unit variance; i.e.,

$$\mathbf{h} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{N_t}). \quad (1)$$

The extension to correlated channels need further investigation, and is out of this paper's scope.

The adopted transmission scheme is transmit-beamforming (without temporary power control) [6, 7]. Specifically, the information symbol s is multiplied by a beamforming vector \mathbf{w}^* ,

⁰G. B. Giannakis is supported by ARL/CTA under grant no. DAAD19-01-2-011.

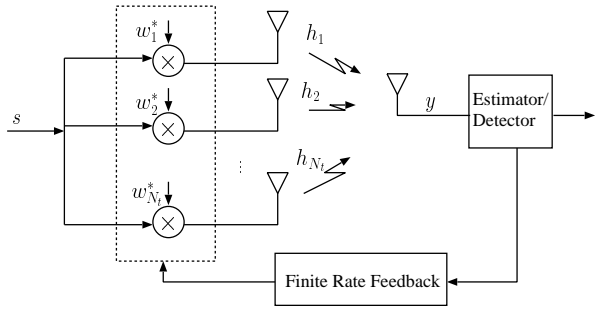


Figure 1: The system model

where $\mathbf{w} := [w_1, w_2, \dots, w_{N_t}]^T$ has unit norm $\|\mathbf{w}\| = 1$. The N_t entries from the vector $s\mathbf{w}^*$ are transmitted simultaneously from N_t antennas, which leads to the received signal y as:

$$y = \mathbf{w}^H \mathbf{h} s + \eta, \quad (2)$$

where η is the additive complex Gaussian noise with zero mean and variance N_0 . With E_s denoting the average symbol energy, the instantaneous SNR in (2) is

$$\gamma = |\mathbf{w}^H \mathbf{h}|^2 E_s / N_0. \quad (3)$$

As in [6, 7], we consider the system where the receiver is able to feedback a finite number (say B) of bits back to the transmitter. We assume that the feedback link is error-free and delay-free. The feedback bits will be used to select the beamforming vector. With B bits, the transmitter has $N = 2^B$ choices on the beamforming vectors. Let us define these vectors as $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_N$, and collect them into a matrix (the codebook of beamforming vectors):

$$\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_N]. \quad (4)$$

The beamforming vector is selected as follows. The receiver is assumed to have perfect knowledge of \mathbf{h} , and chooses the beamforming vector to maximize the instantaneous SNR as:

$$\mathbf{w}_{\text{opt}} = \arg \max_{\mathbf{w} \in \{\mathbf{w}_i\}_{i=1}^N} |\mathbf{w}^H \mathbf{h}|^2. \quad (5)$$

The index of \mathbf{w}_{opt} is coded into B feedback bits. After receiving the B feedback bits, the transmitter switches to the optimal beamforming vector \mathbf{w}_{opt} .

With the selection process in (5), the instantaneous SNR in (3) becomes

$$\gamma = \max_{1 \leq i \leq N} \{|\mathbf{w}_i^H \mathbf{h}|^2\} E_s / N_0, \quad (6)$$

which indicates that system performance depends critically on the design of the beamforming vectors $\{\mathbf{w}_i\}_{i=1}^N$.

A. GOOD BEAMFORMERS

The codebook optimization on $\{\mathbf{w}_i\}_{i=1}^N$ has been thoroughly investigated in [7] and [6], where [7] relies on the outage probability as the figure of merit, and [6] on the average SNR, respectively. Albeit starting from different perspectives, these

two papers arrive at the same beamformer design. Specifically, with i.i.d. channels in (1), a good beamformer should minimize the maximum correlation between any pair of beamforming vectors; i.e.,

$$\mathbf{W}_{\text{opt}} = \min_{\mathbf{W} \in \mathcal{C}^{N_t \times N}} \max_{1 \leq i < j \leq N} |\mathbf{w}_i^H \mathbf{w}_j|. \quad (7)$$

In [6], the beamformer design problem is explicitly linked to the Grassmannian line packing problem [2]. Specifically, \mathbf{w}_i can be viewed as coordinates of a point on the surface of a hypersphere with unit radius centered around origin. This point dictates a straight line in a complex space \mathcal{C}^{N_t} , that passes through the origin. The two lines generated by \mathbf{w}_i and \mathbf{w}_j have a distance defined as:

$$d(\mathbf{w}_i, \mathbf{w}_j) = \sin(\theta_{i,j}) = \sqrt{1 - |\mathbf{w}_i^H \mathbf{w}_j|^2}, \quad (8)$$

where $\theta_{i,j}$ denotes the angle between these two lines, and the distance $d(\mathbf{w}_i, \mathbf{w}_j)$ is termed as ‘‘chordal distance’’ [2]. So the beamformer design in (7) is equivalent to:

$$\mathbf{W}_{\text{opt}} = \max_{\mathbf{W} \in \mathcal{C}^{N_t \times N}} \min_{1 \leq i < j \leq N} d(\mathbf{w}_i, \mathbf{w}_j). \quad (9)$$

The beamformer design is challenging only when $N > N_t$. When $N \leq N_t$, \mathbf{W} can be chosen as N columns of an arbitrary $N_t \times N_t$ unitary matrix, which leads to the minimal chordal distance to be $\sin(90^\circ) = 1$. The resulting beamforming system has identical performance as a selection combining system with N diversity branches [6, 7].

Example codes from [6]: For the illustration purpose, we here give the code design examples in [6].

1) $N_t = 2$ and $N = 4$. The matrix \mathbf{W} is constructed as:

$$\mathbf{W}^T = \begin{bmatrix} -0.1612 - 0.7348j & -0.5135 - 0.4128j \\ -0.0787 - 0.3192j & -0.2506 + 0.9106j \\ -0.2399 + 0.5985j & -0.7641 - 0.0212j \\ -0.9541 & 0.2996 \end{bmatrix}. \quad (10)$$

With this beamformer codebook, the maximum correlation is $\max_{1 \leq i < j \leq N} |\mathbf{w}_i^H \mathbf{w}_j| = 0.57735$, and the minimum chordal distance is $\min_{1 \leq i < j \leq N} d(\mathbf{w}_i, \mathbf{w}_j) = \sqrt{0.6713} = \sin(55^\circ)$.

2) $N_t = 2$ and $N = 8$. The matrix \mathbf{W} is:

$$\mathbf{W}^T = \begin{bmatrix} 0.8393 - 0.2939j & -0.1677 + 0.4256j \\ -0.3427 + 0.9161j & 0.0498 + 0.2019j \\ -0.2065 + 0.3371j & 0.9166 + 0.0600j \\ 0.3478 - 0.3351j & 0.2584 + 0.8366j \\ 0.1049 + 0.6820j & 0.6537 + 0.3106j \\ 0.0347 - 0.2716j & 0.0935 - 0.9572j \\ -0.7457 + 0.1181j & -0.4533 - 0.4719j \\ -0.7983 + 0.3232j & 0.5000 + 0.0906j \end{bmatrix}.$$

With this beamformer codebook, the maximum correlation is $\max_{1 \leq i < j \leq N} |\mathbf{w}_i^H \mathbf{w}_j| = 0.84152$, and the minimum chordal distance is $\min_{1 \leq i < j \leq N} d(\mathbf{w}_i, \mathbf{w}_j) = \sqrt{0.2918} = \sin(32^\circ)$.

III. PERFORMANCE ANALYSIS

Our objective in this paper is to evaluate the average performance for the beamformed transmission with finite-rate feedback. Towards this objective, we assume that s is drawn from a PSK constellation; similar derivations can be carried out for other constellations. Conditioned on the instantaneous SNR γ , the symbol error rate (SER) is [9]:

$$\text{SER}(\gamma) = \frac{1}{\pi} \int_0^{\frac{(M-1)\pi}{M}} \exp\left(-\frac{g_{\text{PSK}}\gamma}{\sin^2\theta}\right) d\theta, \quad (11)$$

where M is the constellation size and $g_{\text{PSK}} := \sin^2(\pi/M)$ is a constellation-dependent constant. With \mathbf{h} a random vector, the average SER is expressed as:

$$\overline{\text{SER}} = E_{\mathbf{h}}\{\text{SER}(\gamma)\}. \quad (12)$$

We want to find exact, or approximate, expressions for $\overline{\text{SER}}$.

A. SER FORMULATION

We define a normalized channel vector as:

$$\tilde{\mathbf{h}} = \frac{\mathbf{h}}{\|\mathbf{h}\|}, \quad (13)$$

such that $\mathbf{h} = \|\mathbf{h}\| \cdot \tilde{\mathbf{h}}$ and $\|\tilde{\mathbf{h}}\| = 1$. We then decompose the SNR in (6) as:

$$\begin{aligned} \gamma &= \|\mathbf{h}\|^2 \frac{E_s}{N_0} \max_i \left| \mathbf{w}_i^H \cdot \frac{\mathbf{h}}{\|\mathbf{h}\|} \right|^2 \\ &= \|\mathbf{h}\|^2 \frac{E_s}{N_0} \left[1 - \min_i d^2(\mathbf{w}_i, \tilde{\mathbf{h}}) \right]. \end{aligned} \quad (14)$$

To simplify notation, we define the average transmit SNR $\bar{\gamma} := E_s/N_0$, and two random variables:

$$\gamma_h := \|\mathbf{h}\|^2, \quad (15)$$

$$Z := \min_i d^2(\mathbf{w}_i, \tilde{\mathbf{h}}), \quad (16)$$

such that $\gamma = \gamma_h(1-Z)\bar{\gamma}$. Based on (1), γ_h is Gamma distributed with parameter N_t and mean $E\{\|\mathbf{h}\|^2\} = N_t$; hence, its probability density function (pdf) is [9]:

$$p(\gamma_h) = \frac{\gamma_h^{N_t-1}}{\Gamma(N_t)} e^{-\gamma_h}. \quad (17)$$

On the other hand, Z is random variable within the interval $Z \in [0, 1]$. Denote $p(z)$ and $F_Z(z)$ as the pdf and cdf (cumulative distribution function) of Z . Due to the assumption in (1), γ_h is independent of $\tilde{\mathbf{h}}$ [7], thus γ_h and Z are independent.

With the definitions of $\{\gamma_h, Z, \bar{\gamma}\}$, we simplify (12) to

$$\overline{\text{SER}} = \int_{\gamma_h=0}^{\infty} \int_{z=0}^1 \text{SER}(\gamma_h(1-z)\bar{\gamma}) p(\gamma_h) p(z) d\gamma_h dz. \quad (18)$$

Substituting (11) and (17) into (18), and utilizing the fact that the moment generating function of γ_h is $E\{e^{s\gamma_h}\} = (1-s)^{-N_t}$ [9], we obtain:

$$\overline{\text{SER}} = \frac{1}{\pi} \int_0^{\frac{(M-1)\pi}{M}} \int_0^1 \left[1 + \frac{g_{\text{PSK}}\bar{\gamma}}{\sin^2\theta}(1-z) \right]^{-N_t} dF_Z(z) d\theta. \quad (19)$$

Notice that $F_Z(z)$ depends on the particular beamformer design \mathbf{W} . To obtain the exact SER in (19), one has to find the exact $F_Z(z)$ for each particular beamformer. Unfortunately, this task is difficult to accomplish. We next develop a SER lower bound, that is applicable to all beamformers. Our approach here is analogous to [7], where a lower bound on the outage probability is derived.

B. SER LOWER BOUND

To find a SER lower bound, we look for a cdf function $\tilde{F}_Z(z)$, such that:

$$F_Z(z) \leq \tilde{F}_Z(z), \quad 0 \leq z \leq 1. \quad (20)$$

Replacing $F_Z(z)$ in (19) by $\tilde{F}_Z(z)$, we define a SER lower bound as:

$$\overline{\text{SER}}_{\text{lb}} := \frac{1}{\pi} \int_0^{\frac{(M-1)\pi}{M}} \int_0^1 \left[1 + \frac{g_{\text{PSK}}\bar{\gamma}}{\sin^2\theta}(1-z) \right]^{-N_t} d\tilde{F}(z) d\theta. \quad (21)$$

The inequality $\overline{\text{SER}} - \overline{\text{SER}}_{\text{lb}} \geq 0$ can be easily verified using integration by part on the variable z .

We next derive such a function $\tilde{F}_Z(z)$ under the geometrical framework presented in [7]. Around each beam-vector \mathbf{w}_i , we define a spherical cap on the surface of the hypersphere,

$$S_i(z) = \{\tilde{\mathbf{h}} : d^2(\mathbf{w}_i, \tilde{\mathbf{h}}) \leq z\},$$

where $0 \leq z \leq 1$. Define $A\{S_i(z)\}$ as the area of the cap $S_i(z)$. Then $A\{S_i(1)\}$ is the whole surface area of the unit hypersphere. According to [7, Lemma 2], the surface area of the spherical cap $S_i(z)$ is:

$$A\{S_i(z)\} = \frac{2\pi^{N_t} z^{N_t-1}}{(N_t-1)!}. \quad (22)$$

Notice that $F_Z(z) = \Pr(\min_i d^2(\mathbf{w}_i, \tilde{\mathbf{h}}) \leq z)$ equals

$$F_Z(z) = \Pr(d^2(\mathbf{w}_1, \tilde{\mathbf{h}}) \leq z, \text{ or } d^2(\mathbf{w}_2, \tilde{\mathbf{h}}) \leq z, \dots, \text{ or } d^2(\mathbf{w}_{N_t}, \tilde{\mathbf{h}}) \leq z), \quad (23)$$

which can be interpreted as the probability that $\tilde{\mathbf{h}}$ falls in the union of the regions $\{S_i(z)\}_{i=1}^{N_t}$ (denoted by $\cup_{i=1}^{N_t} S_i(z)$). Based on the key observation that $\tilde{\mathbf{h}}$ is uniformly distributed on the surface of the unit hypersphere, when \mathbf{h} is i.i.d. [7], we can find $F_Z(z)$ as:

$$F_Z(z) = \frac{A\{\cup_{i=1}^{N_t} S_i(z)\}}{A\{S_i(1)\}}. \quad (24)$$

Due to the union operation, it is true that: $A\{\cup_{i=1}^{N_t} S_i(z)\} \leq \sum_{i=1}^{N_t} A\{S_i(z)\}$. Based on the area computation of (22), we obtain:

$$F_Z(z) \leq \frac{\sum_{i=1}^{N_t} A\{S_i(z)\}}{A\{S_i(1)\}} = N z^{N_t-1}. \quad (25)$$

Taking into account that $F_Z(z) \leq 1$, we define an upper bound $\tilde{F}_Z(z)$ as

$$\tilde{F}_Z(z) = \begin{cases} N z^{N_t-1}, & 0 \leq z < (1/N)^{1/(N_t-1)} \\ 1, & z \geq (1/N)^{1/(N_t-1)} \end{cases}. \quad (26)$$

Substituting (26) into (21), we obtain the SER lower bound as:

$$\begin{aligned}\overline{\text{SER}}_{\text{lb}} &= \frac{1}{\pi} \int_0^{\frac{(M-1)\pi}{M}} \int_0^{\left(\frac{1}{N}\right)^{\frac{1}{N_t-1}}} \left[1 + \frac{g_{\text{PSK}}\bar{\gamma}}{\sin^2\theta} (1-z) \right]^{-N_t} \\ &\quad \times N(N_t-1)z^{N_t-2} dz d\theta \\ &= \frac{1}{\pi} \int_0^{\frac{(M-1)\pi}{M}} \left(1 + \frac{g_{\text{PSK}}\bar{\gamma}}{\sin^2\theta} \right)^{-1} \\ &\quad \times \left[1 + \left[1 - \left(\frac{1}{N} \right)^{\frac{1}{N_t-1}} \right] \frac{g_{\text{PSK}}\bar{\gamma}}{\sin^2\theta} \right]^{1-N_t} d\theta,\end{aligned}\quad (27)$$

where the integration of z can be simply carried out by defining a new variable $t = 1/z$.

Notice that the lower bound is independent of any particular beamformer design. A good beamformer shall try to come as close as possible to the lower bound. This confirms the design guidelines in (7) and (9), as follows.

Denote z_o as the critical value where all the spherical caps $S_1(z_o), S_2(z_o), \dots, S_N(z_o)$ do not overlap. For a given beamformer matrix \mathbf{W} , the value of z_o is available by interpreting the result of [7, Lemma 5] as:

$$z_o = 1 - \frac{1 + \max_{1 \leq i < j \leq N_t} |\mathbf{w}_i^H \mathbf{w}_j|}{2} = \frac{1 - \max_{1 \leq i < j \leq N_t} |\mathbf{w}_i^H \mathbf{w}_j|}{2}.$$

For $z < z_o$, since all the spherical caps do not overlap, we have

$$F_Z(z) = \tilde{F}_Z(z) = Nz^{N_t-1}, \quad z \leq z_o. \quad (29)$$

Hence, $\tilde{F}_Z(z) \neq F_Z(z)$ only when $z > z_o$. To minimize the difference between $\tilde{F}_Z(z)$ and $F_Z(z)$, a good beamformer design tries to maximize z_o as much as possible. If z_o reaches the maximum of $(1/N)^{1/(N_t-1)}$, the true SER in (19) would coincide with the SER lower bound of (27). Therefore, the maximum correlation $\max_{1 \leq i < j \leq N_t} |\mathbf{w}_i^H \mathbf{w}_j|$ should be minimized, agreeing with (7) that was obtained based on either the outage probability, or the average SNR, criterion.

Numerical results show that the universal lower bound in (27) is tight for good beamformer designs. This is consistent with [7], where the lower bound on outage probability is tight.

IV. QUANTIFICATION OF POWER LOSS DUE TO FINITE-RATE FEEDBACK

When N goes to infinity, the performance of beamformed transmission with finite-rate feedback should approach the ideal case with perfect channel knowledge at the transmitter. With perfect CSI, thus $Z = 0$ in (19), one can derive the SER for PSK as:

$$\overline{\text{SER}}_{\text{perfect CSI}} = \frac{1}{\pi} \int_0^{\frac{(M-1)\pi}{M}} \left(1 + \frac{g_{\text{PSK}}\bar{\gamma}}{\sin^2\theta} \right)^{-N_t} d\theta. \quad (30)$$

Therefore, the lower bound in (27) is tight with respect to N , since it coincides with (30) when $N = \infty$.

Comparing (27) of a finite N with (30) of $N = \infty$, we are now able to quantify the power loss due to the finite-rate

constraint. Towards this objective, we assume that the SER lower bound can be approximately achieved by some good beamformer designs. The performance of such beamformers will depend on N_t , N , and $\bar{\gamma}$; hence we explicitly express the SER as $\overline{\text{SER}}_{\text{lb}}(N_t, N, \bar{\gamma})$. On the other hand, the SER with perfect CSI depends only on N_t and $\bar{\gamma}$, denoted as $\overline{\text{SER}}_{\text{perfect CSI}}(N_t, \bar{\gamma})$. Due to the finite-rate constraint,

$$\begin{aligned}\overline{\text{SER}}_{\text{lb}}(N_t, N, \bar{\gamma}) &\geq \overline{\text{SER}}_{\text{lb}}(N_t, \infty, \bar{\gamma}) \\ &= \overline{\text{SER}}_{\text{perfect CSI}}(N_t, \bar{\gamma}).\end{aligned}\quad (31)$$

To compensate for this performance loss, one has to increase the transmission power. Suppose when we increase the average SNR from $\bar{\gamma}$ to $\bar{\gamma}_{\text{new}}$, we arrive:

$$\overline{\text{SER}}_{\text{lb}}(N_t, N, \bar{\gamma}_{\text{new}}) = \overline{\text{SER}}_{\text{lb}}(N_t, \infty, \bar{\gamma}). \quad (32)$$

We define the difference between $\bar{\gamma}_{\text{new}}$ and $\bar{\gamma}$ as the power loss (in decibels) due to the finite-rate constraint as:

$$L(N_t, N, \bar{\gamma}) = 10 \log_{10} \bar{\gamma}_{\text{new}} - 10 \log_{10} \bar{\gamma}. \quad (33)$$

Notice that the power loss in (33) is a function of $\bar{\gamma}$, thus varies within the whole SNR range.

The power loss at high SNR ($\bar{\gamma} \gg 1$) can be easily found. At high SNR, we simplify (27) as:

$$\overline{\text{SER}}_{\text{lb}}(N_t, N, \bar{\gamma}) \approx (G_c \cdot \bar{\gamma})^{-G_d}, \quad (34)$$

where $G_d = N_t$ is termed as the *diversity gain*, and

$$\begin{aligned}G_c &= g_{\text{PSK}} \left[\frac{1}{\pi} \int_0^{\frac{(M-1)\pi}{M}} (\sin\theta)^{2N_t} d\theta \right]^{-\frac{1}{N_t}} \\ &\quad \times \left[1 - \left(\frac{1}{N} \right)^{\frac{1}{N_t-1}} \right]^{1-\frac{1}{N_t}}.\end{aligned}\quad (35)$$

is referred to as the *coding gain* (see e.g., [10, 11]). Eq. (34) implies that the SER versus the average SNR curve in fading channels is well approximated by a straight line at high SNR, when plotted on a log-log scale. The diversity gain G_d determines the slope of the curve, while G_c (in decibels) determines the shift of the curve in SNR relative to a benchmark SER curve of $(\bar{\gamma}^{-G_d})$. Substituting (34) into (32), we obtain:

$$L(N_t, N, \infty) = 10 \log_{10} \left[1 - \left(\frac{1}{N} \right)^{\frac{1}{N_t-1}} \right]^{\frac{1}{N_t} - 1}. \quad (36)$$

For the power loss at arbitrary SNR, we prove in Appendix:

$$L(N_t, N, \bar{\gamma}) \leq L(N_t, N, \infty). \quad (37)$$

Therefore, the power loss across the whole SNR range is bounded to be less than or equal to $L(N_t, N, \infty)$. In other words, the distance (or, the horizontal shift), between $\overline{\text{SER}}_{\text{perfect CSI}}(N_t, \bar{\gamma})$ and $\overline{\text{SER}}_{\text{lb}}(N_t, N, \bar{\gamma})$ is *no larger* than $L(N_t, N, \infty)$ decibels. The maximum power loss

$L(N_t, N, \infty)$ is thus valid across the *whole* SNR range. When $N = 2^B$ is large, we use $\ln(1+x) \approx x$ for small x to have:

$$L(N_t, N, \infty) \approx \frac{10}{\ln 10} \left(1 - \frac{1}{N_t}\right) 2^{-\frac{B}{N_t-1}}. \quad (38)$$

Hence, the power loss in decibels due to the finite-rate constraint decays exponentially with the number of feedback bits B for sufficiently large B .

On the other hand, if a system requires the power loss to be within L_0 decibels relative to the perfect CSI case, we can easily identify how many beamforming vectors are needed as:

$$N \geq \left(1 - 10^{\frac{L_0}{10} \cdot \frac{N_t}{1-N_t}}\right)^{-(N_t-1)}. \quad (39)$$

Notice that only high SNR analysis is carried out in [7] based on the outage probability. Moreover, the distortion measure defined in [7, eq. (52)] does not translate into the power loss. For a given reduction on the average capacity, or, the average SNR, the minimum N is also calculated in [6]. However, the calculation in [6] relies on a *loose* bound on the minimum chordal distance in line packing; thus it is only approximate.

V. NUMERICAL RESULTS

In this section, we collect some numerical results.

Case 1): We compute the power loss in (36) for various cases and list the results in Table 1, where we are interested in non-trivial configurations with $N \geq N_t$. On the other hand,

Table 1: The power loss (dB) due to the finite-rate constraint

	$N = 2$	$N = 4$	$N = 8$	$N = 16$	$N = 64$
$N_t = 2$	1.51	0.62	0.29	0.14	0.03
$N_t = 3$	-	2.01	1.26	0.83	0.39
$N_t = 4$	-	3.24	2.26	1.65	0.94
$N_t = 6$	-	-	3.90	3.09	2.07
$N_t = 8$	-	-	5.16	4.25	3.05

Table 2: The minimum codebook size N for power loss ≤ 1 dB

N_t	2	3	4	5	6	7	8
$N \geq$	3	12	55	256	1220	5851	28170
$\lceil \log_2 N \rceil$	2	4	6	8	11	13	15

if we want to limit the power loss to be within 1dB relative to the perfect CSI case, the minimum number of beamforming vectors N can be computed from (39); the results are listed in Table 2. We clearly see that as N_t increases, the needed number of beamforming vectors increases considerably in order to have performance close to the optimum.

Case 2): We now compare the SER lower bound in (27) with the actual SER of (12), which is obtained via Monte Carlo simulations. We use the QPSK constellation in all cases. Fig. 2 depicts the results with $N_t = 2$. The beamformer codebooks for $N = 4, 8$ are listed in Section II-A. The codebook with $N = 2$ is $\mathbf{W} = \mathbf{I}_2$, which corresponds to the selection diversity. As shown in Fig. 2, the SER lower bound for the $N_t = 2$ case is almost identical to the actual SER, even with small N .

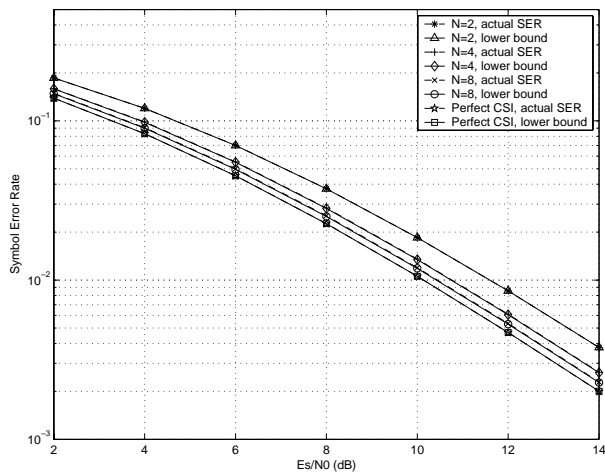


Figure 2: The actual SER versus the lower bound ($N_t = 2$)

Case 3): We now compare the SER lower bound with the actual SER for $N_t = 3$. We use the beamformers listed in [4] with $N = 4, 8, 16$. As shown in Fig. 3, the bound is tight for the given beamformers, although the difference between the bound and the actual SER increases relative to the $N_t = 2$ case.

Case 4): We now compare the SER lower bound with the actual SER for the $N_t = 4$ case. We use the beamformers listed in [4] with $N = 16, 64$. The codebook with $N = 4$ is $\mathbf{W} = \mathbf{I}_4$, corresponding to the selection combining. As shown in Fig. 4, the bound is also tight for the given beamformers, with the difference between the bound and the actual SER larger than those corresponding to $N_t = 2, 3$. The increasing difference could be due to either of the following two reasons, or both: i) the bound becomes less tight for large N_t ; ii) the beamformer currently found for large N_t is not as close to the optimum as in the $N_t = 2$ case.

Figs. 2, 3, and 4 show that the distance between the SER lower bound and the SER with perfect CSI only slightly increases as the average SNR increases, yet always bounded by the maximum power loss listed in Table 1. This confirms (37), and more importantly, shows that high SNR analysis is often accurate even in the low and medium SNR range [11].

VI. CONCLUSIONS

In this paper, we developed a symbol error rate (SER) lower bound for transmit-beamforming systems with finite-rate feedback. This bound applies to all beamformer designs, and is tight for well-constructed beamformers. Comparing this bound with the SER corresponding to the ideal case with perfect channel knowledge, we quantified the power loss due to the finite rate constraint, across the whole SNR range.

Performance analysis in this paper will facilitate future work on adaptive modulation in beamformed transmissions with finite-rate feedback. Especially interesting is the applicability of these results to low or medium SNRs, since the target error rate for an uncoded adaptive system is usually not very low, and the selected transmission mode will not operate in a high SNR scenario.

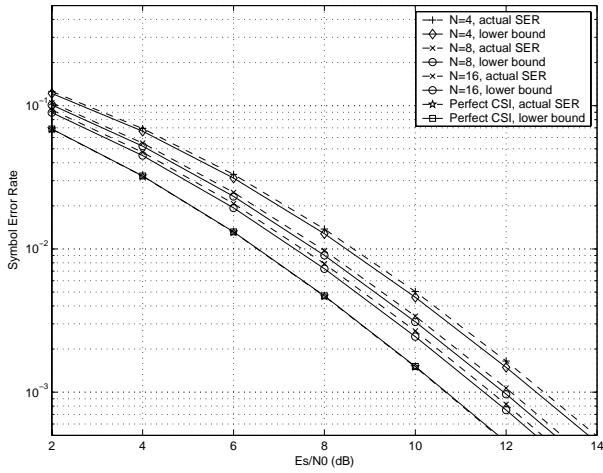


Figure 3: The actual SER versus the lower bound ($N_t = 3$)

APPENDIX: PROOF OF (37)

Define $\xi(N_t, N, \bar{\gamma}) = \bar{\gamma}_{\text{new}}/\bar{\gamma}$, such that $L(N_t, N, \bar{\gamma}) = 10 \log_{10} \xi(N_t, N, \bar{\gamma})$. Eq. (32) implies that:

$$\overline{\text{SER}}_{\text{perfect CSI}}(N_t, \bar{\gamma}) = \overline{\text{SER}}_{\text{lb}}(N_t, N, \bar{\gamma} \cdot \xi(N_t, N, \bar{\gamma}))$$

Notice that $\overline{\text{SER}}_{\text{lb}}$ is a decreasing function of the average SNR $\bar{\gamma}$. In order to prove $\xi(N_t, N, \bar{\gamma}) \leq \xi(N_t, N, \infty)$, we need:

$$\overline{\text{SER}}_{\text{perfect CSI}}(N_t, \bar{\gamma}) \geq \overline{\text{SER}}_{\text{lb}}(N_t, N, \bar{\gamma} \cdot \xi(N_t, N, \infty)). \quad (40)$$

Denote $\beta = g_{\text{PSK}}/\sin^2 \theta$ and $\alpha = [1 - (1/N)^{1/(N_t-1)}]$ for notational convenience. Substituting (30), (27), and (36) into (40), we need to prove that:

$$(1 + \beta\bar{\gamma})^{-N_t} \geq \left(1 + \beta\bar{\gamma}\alpha^{\frac{1}{N_t}-1}\right)^{-1} \left(1 + \beta\bar{\gamma}\alpha^{\frac{1}{N_t}}\right)^{1-N_t} \quad (41)$$

for each θ . Taking logarithm on (41), we shall prove:

$$\ln(1 + \beta\bar{\gamma}) \leq \frac{1}{N_t} \ln \left(1 + \beta\bar{\gamma}\alpha^{\frac{1-N_t}{N_t}}\right) + \frac{N_t-1}{N_t} \ln \left(1 + \beta\bar{\gamma}\alpha^{\frac{1}{N_t}}\right). \quad (42)$$

We verify that the function $f(x) = \ln(1 + \beta\bar{\gamma}e^{x \ln \alpha})$ is convex, since $f''(x) \geq 0$ regardless of the values of β , α , and $\bar{\gamma}$. Eq. (42) holds true due to $f(0) \leq \frac{1}{N_t} f(-\frac{N_t-1}{N_t}) + \frac{N_t-1}{N_t} f(\frac{1}{N_t})$.

References

- [1] S. Bhashyam, A. Sabharwal, and B. Aazhang, "Feedback gain in multiple antenna systems," *IEEE Transactions on Communications*, vol. 50, no. 5, pp. 785–798, May 2002.
- [2] J. H. Conway, R. H. Hardin, and N. J. A. Sloane, "Packing Lines, Planes, etc.: Packings in Grassmannian Space," *Experimental Math.*, no. 5, pp. 139–159, 1996.
- [3] V. Lau, "Optimal transmission design for MIMO block fading channels with feedback capacity constraint," in *Proc. of Info. Theory Workshop*, Apr. 2003, pp. 94–98.

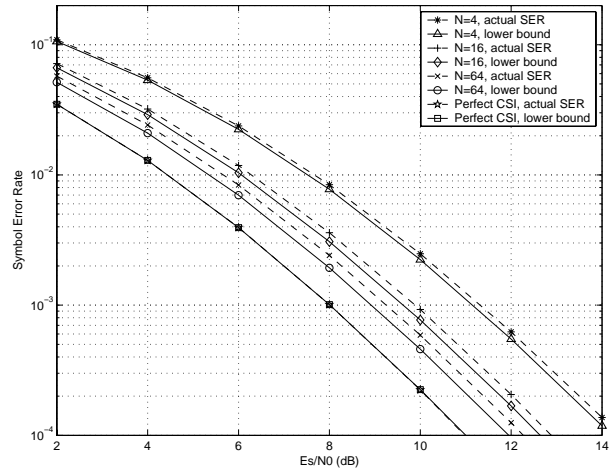


Figure 4: The actual SER versus the lower bound ($N_t = 4$)

- [4] D. Love, *Personal Webpage on Grassmannian Subspace Packing*, <http://www.ece.utexas.edu/~djlove/grass.html>.
- [5] D. J. Love and R. W. Heath Jr., "Limited feedback precoding for spatial multiplexing systems," in *Proc. of Globecom*, San Francisco, CA, Dec. 2003.
- [6] D. J. Love, R. W. Heath Jr., and T. Strohmer, "Grassmannian beamforming for multiple-input multiple-output wireless systems," *IEEE Transactions on Information Theory*, vol. 49, no. 10, pp. 2735–2747, Oct. 2003.
- [7] K. K. Mukkavilli, A. Sabharwal, E. Erkip, and B. Aazhang, "On Beamforming with Finite Rate Feedback in Multiple Antenna Systems," *IEEE Trans. on Information Theory*, vol. 49, pp. 2562–2579, Oct. 2003.
- [8] A. Narula, M. J. Lopez, M. D. Trott, and G. W. Wornell, "Efficient use of side information in multiple-antenna data transmission over fading channels," *IEEE Journal on Selected Areas in Communications*, vol. 16, no. 8, pp. 1423–1436, Oct. 1998.
- [9] M. K. Simon and M.-S. Alouini, *Digital Communication over Generalized Fading Channels: A Unified Approach to the Performance Analysis*, John Wiley & Sons, Inc., 2000.
- [10] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communication: performance criterion and code construction," *IEEE Trans. on Info. Theory*, vol. 44, pp. 744–765, Mar. 1998.
- [11] Z. Wang and G. B. Giannakis, "A simple and general parameterization quantifying performance in fading channels," *IEEE Transactions on Communications*, vol. 51, no. 8, pp. 1389–1398, Aug. 2003.
- [12] P. Xia, S. Zhou, and G. B. Giannakis, "Multi-Antenna Adaptive Modulation with Transmit-Beamforming based on Bandwidth-Constrained Feedback," in *Proc. of 37th Asilomar Conf. on Signals, Systems, and Computers*, Pacific Grove, CA, November 9–12 2003.