Note: You are supposed to give proofs to the time bounds of your algorithms. Read the questions carefully before attempting to solve them.

1. (16 points) Input is a sequence of $n$ real numbers whose values are uniformly distributed in the range $[0, R]$ ($R$ being an arbitrary real number). Show how to sort these numbers in an expected $O(n)$ time.
2. (17 points) Input is a sequence \( X \) of \( n \) real numbers, not necessarily in sorted order. It is known that \( n = 4k + 3 \) for some integer \( k \) and that \( X \) has \( k \) elements such that each of them appears 4 times and another element that appears 3 times. For example \( X \) could be 7.2, 6.15, 6.15, 8.4, 8.4, 7.2, 7.2, 8.4, 7.2, 6.15, 8.4. Here \( k = 2 \). The elements 7.2 and 8.4 appear 4 times each and the element 6.15 appears thrice. The problem is to identify the element that occurs three times. Present an \( O(n) \) time algorithm for this problem.
3. (16 points) Find a minimum spanning tree for the following graph $G(V, E)$ either using Prim’s algorithm or using Kruskal’s algorithm: $V = \{1, 2, 3, 4, 5, 6, 7\}$. The edge weights are: $W(1, 2) = 8; W(1, 3) = 5; W(1, 4) = 7; W(1, 6) = 4; W(1, 7) = 5; W(2, 4) = 4; W(3, 4) = 2; W(3, 5) = 3; W(3, 6) = 6; W(4, 5) = 2; W(4, 7) = 4; W(5, 6) = 8; W(5, 7) = 11; W(6, 7) = 5.$
4. (17 points) Let \( O_1, O_2, \ldots, O_n \) be a set of \( n \) objects that are to be put into a knapsack of capacity \( m \). The volume of object \( O_i \) is \( w_i \), \( 1 \leq i \leq n \). If \( \sum_{i=1}^{n} w_i \leq m \), then clearly all the objects can be put into the knapsack. So, assume \( \sum_{i=1}^{n} w_i > m \). The problem is to select a maximum subset \( Q \) of the objects to be put into the knapsack. (A maximum subset is one with the maximum number of objects in it). Present a greedy algorithm for this problem and prove that this algorithm will always find an optimal solution. What is the run time of your algorithm? Let \( Q \) be the subset obtained using your algorithm. How small can the knapsack utilization ratio \( (\sum_{O_i \in Q} w_i)/m \) get?
5. Consider the integer knapsack problem obtained by replacing the 0/1 constraint by $x_i \geq 0$ and integer. Generalize $f_i(y)$ to this problem in the obvious way. (Note that $f_i(y)$ is the maximum profit to the problem KNAP(1, i, y).)

(a) (8 points) Obtain the dynamic programming recurrence relation corresponding to $f_i(y)$.

(b) (4 points) Show how to solve this relation in $O(m^2n)$ time assuming that the weights are integers. Here $m$ is the capacity of the knapsack and $n$ is the number of objects.

(c) (5 points) Show that the real knapsack problem with $x_i \geq 0$ can be solved in $O(n)$ time.
6. (17 points) Input is an undirected graph \( G(V, E) \). The problem is to check if \( G \) is a tree or not. Present an \( O(|V|) \) time algorithm for this problem. The input is given in the form of adjacency lists.