Note: You are supposed to give proofs to the time bounds of your algorithms. Read the questions carefully before attempting to solve them.

1. (16 points) Input is a sequence of $n$ keys where each key is an integer in the range $[1, n^{\sqrt{\log n}}]$. Show how to sort this sequence in time $O(n^{\sqrt{\log n}})$. 
2. (17 points) Input are a sequence $X$ of $n$ arbitrary and distinct real numbers and an integer $k < n$. The problem is to partition $X$ into $X_1, X_2, \ldots, X_k$ such that $|X_i| = \frac{n}{k}$ for $1 \leq i \leq k$ and all the elements of $X_j$ are less than any element in $X_{j+1}$, for $1 \leq j \leq k - 1$. Present an $O(n \log k)$ time algorithm to solve this problem.
3. (16 points) Input are $n \times n$ matrices $A_1, A_2, \ldots, A_k$ and integers $n_1, n_2, \ldots, n_k$, where $n_i$ is \[ \geq 2 \text{ for } 1 \leq i \leq k. \] Let $N = \Pi_{i=1}^{k} n_i$. The problem is to compute $A_1^{n_1} \times A_2^{n_2} \times \cdots \times A_k^{n_k}$. Show how to compute this product in $O(n \log^2 \log N)$ time.
4. (17 points) Input is a sequence of $n$ jobs $J_1, J_2, \ldots, J_n$. There is only one machine to process the jobs. To complete job $J_i$, it has to be run on the machine for $T_i$ units of time (where $T_i$ is a real number), for $1 \leq i \leq n$. $D$ is a given global deadline. The problem is to find the maximum number of jobs that can be completed within the deadline. Present an $O(n \log n)$ time greedy algorithm for this problem. Show that the output of your algorithm is optimal.
5. (17 points) Consider the 0/1 knapsack problem with $n$ objects whose profits and weights are $p_i, 1 \leq i \leq n$ and $w_i, 1 \leq i \leq n$, respectively. The capacity constraint is $m$. It is also given that the weight of each object is $\geq \epsilon m$ for some constant fraction $\epsilon$. Show how you’ll solve this problem in $O\left(n^{1/\epsilon}\right)$ time.
6. (17 points) Input is a directed weighted graph $G(V,E)$, in which the edge weights are integers in the range $[1, C]$, $C$ being a constant. Present an $O(|V| + |E|)$ time algorithm to solve the single source shortest paths problem on this graph.