Note: You are supposed to give proofs to the time bounds of your algorithms. Read the questions carefully before attempting to solve them.

1. (16 points) Let $X$ be a sequence of $n$ distinct arbitrary real numbers. $X$ need not be in sorted order. If $x$ is an element of $X$, the rank of $x$ in $X$ is defined to be $|\{q \in X : q \leq x\}|$. Present an algorithm that takes as input $X, a, b$ (where $a$ and $b$ are integers in the range $[1, n]$) and outputs all the elements of $X$ whose ranks are in the interval $[a, b]$. Your algorithm should run in $O(n)$ time.
2. (18 points) Input are \( n \) keys where each key is an integer in the range \([1, N]\) where \( N \) is much larger than \( n \). Present an algorithm that sorts the keys in time \( O\left(n \frac{\log N}{\log n}\right)\).
3. (16 points) Input is an $n \times n$ matrix $A$. Present an $O(n^{\log_2 7} \log m)$ time algorithm to compute $A^m$ where $m$ is an integer. If it helps you can assume that $m$ is an integral power of 2.
4. (18 points) Present a greedy approach for the solution of the traveling salesperson problem. Is your algorithm optimal? If so, prove it. If not, specify an upper bound on $\frac{YC}{OC}$ where $YC$ is the total cost of the tour found by your algorithm and $OC$ is the cost of the optimal tour. What is the run time of your algorithm?
5. (16 points) Find a minimum spanning tree for the following graph $G(V, E)$ either using Prim’s algorithm or using Kruskal’s algorithm: $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$. The edge weights are: $W(1, 2) = 2$; $W(1, 3) = 1$; $W(1, 4) = 1$; $W(2, 3) = 8$; $W(3, 4) = 1$; $W(2, 6) = 5$; $W(6, 3) = 4$; $W(6, 7) = 3$; $W(3, 7) = 2$; $W(7, 8) = 6$; $W(3, 8) = 3$; $W(3, 5) = 5$; $W(5, 8) = 8$; $W(5, 4) = 6$. 
6. (16 points) Use Dijkstra’s algorithm to solve the single source shortest path problem on the directed graph $G(V,E)$: $V = \{s, 2, 3, 4, 5, 6, 7, 8\}$. Edge weights are: $W(s, 2) = 5$; $W(s, 4) = 10$; $W(2, 5) = 6$; $W(2, 6) = 2$; $W(3, 5) = 14$; $W(3, 8) = 5$; $W(4, 3) = 3$; $W(4, 8) = 2$; $W(5, 7) = 2$; $W(6, 3) = 6$; $W(6, 4) = 2$; $W(6, 5) = 4$; $W(6, 7) = 3$; $W(8, 5) = 7$. 