CSE 5500 Advanced Sequential and Parallel Algorithms
Exam I; October 17, 2012

Note: You are supposed to give proofs to the time bounds of your algorithms. Read the questions carefully before attempting to solve them.

1. (18 points) Input is an array $A[1:n]$ where there are $\sqrt{n}$ copies of one element and all the other elements are distinct. Present a Las Vegas algorithm to identify the repeated element in $\tilde{O}(\sqrt{n} \log^2 n)$ time.
2. (17 points) $G(V, E)$ is an undirected graph with $|V| = n$ and $|E| = m$. We are given as input the set $E$ of edges in the graph and $n$ pairs of vertices $(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)$. The problem is to check if there is a path from $u_i$ to $v_i$ (of any length) in $G$, for every $i$, $1 \leq i \leq n$. Show how to solve this problem using the Union-Find data structure. What is the run time of your algorithm?
3. (16 points) Solve the following recurrence relations:

(a) 
\[
T(n) = \begin{cases} 
1 & n \leq 3 \\
\frac{1}{27} T\left(\frac{n}{3}\right) + n^3 & n > 3 
\end{cases}
\]

(b) 
\[
T(n) = \begin{cases} 
1 & n \leq 2 \\
T(n - 1) + \log_e n & n > 2 
\end{cases}
\]
4. (16 points) Consider the following algorithm for sorting:

Algorithm NewSort(A, i, j)

if i + 1 ≥ j then return;
k := ⌊(j − i + 1)/3⌋;
NewSort(A, i, j − k); /* sort the first two-thirds */
NewSort(A, i + k, j); /* sort the last two-thirds */
NewSort(A, i, j − k); /* sort the first two-thirds */

Provide an informal proof for the correctness of this algorithm. What is the run time, T(n), of this algorithm (where n = j − i + 1)?
5. (16 points) Show that any algorithm for merging two sorted sequences of length \( n \) each will have to make \( 2n - o(n) \) comparisons in the worst case. (*Hint: You could employ a comparison tree*).
6. (17 points) Input are two arrays $A$ and $B$ (not necessarily in sorted order) with $n$ elements each. The array elements are integers in the range $[0, n^{100}]$. Input is also another integer $x$. Present an $O(n)$ time algorithm to check if there exist two integers $a$ and $b$ such that $a \in A$, $b \in B$, and $a + b = x$. 
