CSE 3500 Algorithms and Complexity  
Spring 2013; Exam I; Help Sheet

1. Preliminaries. We say \( f(n) = O(g(n)) \) if \( f(n) \leq cg(n) \) for all \( n \geq n_0 \) for some constants \( c \) and \( n_0 \). We say \( f(n) = \Omega(g(n)) \) if and only if \( g(n) = O(f(n)) \). Also, \( f(n) = \Theta(g(n)) \) if \( f(n) = O(g(n)) \) and \( f(n) = \Omega(g(n)) \).

A partial list of functions in increasing order is: \( O(1), (\log n)^\epsilon, \log n, (\log n)^{1+\mu}, n^\epsilon, n, n^{1+\mu}, 2^n, 2^n, 2n^{1+\mu} \) where \( 0 < \epsilon < 1 \) and \( \mu > 0 \) are constants.

Stirling’s approximation: \( n! \approx (n/e)^n \sqrt{2\pi n} \).
\[
\sum_{i=1}^{n} i = n(n+1)/2. \quad \sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6. \quad \sum_{i=1}^{n} i^3 = n^2(n+1)^2/4.
\]

2. Master theorem. Consider the recurrence relation: \( T(n) = aT(n/b) + f(n) \), where \( a \geq 1 \) and \( b > 1 \) are constants. \textbf{Case1:} If \( f(n) = O(n^{\log_b a-\epsilon}) \) for some constant \( \epsilon > 0 \), then \( T(n) = \Theta(n^{\log_b a}) \). \textbf{Case2:} If \( n^{\log_b a} = \Theta(f(n)) \), then \( T(n) = \Theta(f(n) \log n) \). \textbf{Case3:} If \( f(n) = \Omega(n^{\log_b a + \epsilon}) \) for some constant \( \epsilon > 0 \) and \( af(n/b) \leq cf(n) \) for some constant \( c < 1 \), then \( T(n) = \Theta(f(n)) \).

3. Randomized algorithms. A Monte Carlo algorithm runs for a prespecified amount of time and its output is correct with high probability. By high probability we mean a probability of \( \geq 1 - n^{-\alpha} \), for any constant \( \alpha \). A Las Vegas algorithm always outputs the correct answer and its run time is a random variable. We say the run time of a Las Vegas algorithm is \( \tilde{O}(f(n)) \) if the run time is \( \leq cf(n) \) for all \( n \geq n_0 \) with probability \( \geq (1 - n^{-\alpha}) \) for some constants \( c \) and \( n_0 \).

4. Dictionaries and Priority Queues: A dictionary supports the operations: SEARCH (for an arbitrary element), INSERT (an arbitrary element), and DELETE (an arbitrary element). A (max) priority queue supports: INSERT (an arbitrary element), SEARCH (for the maximum element), and DELETE (the maximum element).

5. Heaps and Heapsort: A (max) heap is a complete binary tree where a key is stored at each node. The key at any node will be greater than the keys of its children.

A (max) heap supports the following operations: SEARCH (for the maximum), INSERT (an arbitrary element), and DELETE (the maximum). Each operation can be completed in \( O(\log n) \) time, \( n \) being the number of elements in the heap. A heap can be used to sort elements. Heapsort on \( n \) elements takes \( O(n \log n) \) time.

6. A 2-3 Tree can be used to support a dictionary as well as a priority queue. Each operation of interest will take \( O(\log n) \) time in the worst case.

7. Binary search on a sorted array of size \( n \) takes \( O(\log n) \) time. Mergesort sorts \( n \) arbitrary keys in \( O(n \log n) \) time. Quicksort takes \( \Omega(n^2) \) time in the worst case to sort \( n \) keys. Its average run time is \( O(n \log n) \).