Random Walks on Graphs

- Input: \( G(V,E) \)

- Repeat
  
  Pick a random neighbor and go

Questions

- What’s the expected time before each node is seen at least once?
- If we start from \( u \), what’s the expected time to see \( v \) for the first time?

Example 1

- Consider a completely connected graph \( C_n \), If we start from \( u \), the probability that the next node is \( v = \frac{1}{n-1} \).
- Expected time to see \( v \) starting from \( u = (n-1) \).
- Probability of not visiting node \( i \) in the fist \( k \) walks \( \leq \left( 1 - \frac{1}{n-1} \right)^k \approx \left( 1 - \frac{1}{n} \right)^k \).
- Prob. \( \exists \) an \( i \) that is not seen in the first \( k \) steps] is \( \leq n \left( 1 - \frac{1}{n} \right)^k \leq n \exp \left( -\frac{k}{n} \right) \).
- \( n \exp \left( -\frac{k}{n} \right) = n^{-\alpha} \Rightarrow -\frac{k}{n} = -\alpha + 1 \log n \Rightarrow k = (\alpha + 1)n \log n \).

Example 2 (2-SAT)

- Let \( x_1, x_2, \ldots, x_n \) be one satisfying assignment to the given formula.
- Let these values be called correct values.
- Start from a random assignment
Repeat

If the given formula F is satisfied
report & quit
else

Pick a clause that is not satisfied
Pick a random literal in the clause and change its value

Forever

The number of the correct values changes by ± 1
The expected time to visit node n is \( O(n^2) \)

**A Markov chain (MC)**

- A discrete time stochastic process with a set S of states & a probability transition matrix \( P \).
- \( P_{ij} = \) Probability of visiting \( j \) from \( i \).
- Let \( x_t \) be the state of the MC at step \( t \) where \( t = 0,1, \ldots \).

\[ \text{Prob.} \left[ x_t = i \mid x_{t-1} = j, x_{t-2} = q_{t-2}, \ldots, x_1 = q_1, x_0 = q_0 \right] = \text{Prob.} \left[ x_t = i \mid x_{t-1} = j \right] = P_{ij}. \]

\( P_{ij}^{(t)} = \) \( t \)-step transition probability.

- Let \( r_{ij}^{(t)} \) be the probability of visiting \( j \) at step \( t \) for the first time starting from \( i \) at time = 0.

\[ r_{ij}^{(t)} = \text{Prob.} \left[ x_t = j, x_s \neq j, 1 \leq s \leq t-1 \mid x_0 = 0 \right]. \]

Probability of ever visiting \( j \) starting from node \( i \) in step 0, \( f_{ij} = \sum_{t>0} r_{ij}^{(t)} \).

Let \( h_{ij} \) be the expected time to see \( j \) starting from node \( i \) at step 0, \( h_{ij} = \sum_{t=0}^{\infty} t \cdot r_{ij}^{(t)} \).

**Note**

- if \( f_{ij} < 1 \) then \( h_{ij} = \infty \).
- if \( f_{ij} < 1 \) then we call the node \( i \) as a TRANSIENT (means that most probably we won’t visit this node again).
- if \( f_{ij} = 1 \) then we say the node is PERSISTENT.
- if \( f_{ij} = 1 \) and \( h_{ij} = \infty \) then the node \( i \) is NULL-PERSISTENT.
- If \( f_{ij} = 1 \) and \( h_{ij} \neq \infty \) then the node \( i \) is NON-NULL-PERSISTENT.
- We can use a graph \( G \) to model a Markov Chain, each node corresponds to a state, an edge corresponds to a transition.
- A strong component of \( G \) is a maximal subgraph \( C \) of \( G \) such that for every pair \( a, b \in C \).
- There are directed paths from \( a \) to \( b \) and from \( b \) to \( a \).
- A strong component \( C \) of \( G \) is a final strong component, if \( C \) has no outgoing edges.
- A Markov chain is irreducible if it consists of a single strong component.
- For any two nodes \( a, b \) in a strong component there is a non-zero probability of visiting node \( b \) in a finite number of steps starting from node \( a \), this probability is \( 1 \) if the strong component is final.

**Fact 1**
- In a finite irreducible Markov chain, all the states are NON-NULL PERSISTENT.
- The state probability vector at time step \( t = q^{(t)} = (q_1^{(t)}, q_2^{(t)}, \ldots, q_n^{(t)}) \) where \( q_i^{(t)} = \text{prob. } [x_t = j] \ 1 \leq i \leq n \).

**Fact 2**
- \( q^{(t)} = q^{(t-1)} \cdot p = q^{(0)} \cdot p^t \).
- \( \pi \) is a stationary probability distribution if \( \pi = \pi \cdot P \).
- \( \pi \) corresponds to the steady state.
- The periodicity of state \( k \) is the largest \( T \) such that \( q_k^{(t)} > 0 \Rightarrow t \in \{a + iT : i \geq 0\} \).
- A state is periodic if its periodicity is \( > 1 \), otherwise it is called aperiodic and NON-NULL PERSISTENT.
- A Markov Chain is Aperiodic if all the states are aperiodic.
- A Markov Chain is ERGODIC if all the states are ERGODIC.
**Theorem**

In a finite aperiodic and irreducible Markov Chain the following hold:

- All the states are ERGODIC
- There exist a stationary probability distribution $\pi$ such that $\pi_i > 0$ for every $1 \leq i \leq n$.
- $f_{ij} = 1$ and $h_{ij} = 1/\pi_i$ for $1 \leq i \leq n$.
- if $N(i,t)$ is the number of steps in which the states of the Markov Chain is $i$ in the first $t$ steps, then $\lim (N(i,t)/t) = \pi_i$ (as $t$ goes to infinity).

**Note**

- If we have a bipartite graph corresponding to a Markov Chain, then the states will be periodic.
- Let $G$ be any connected non Bipartite Graph we can construct a Markov Chain out of as follows:
  - each node corresponds to a state $P_{ij} = 1/d_i$ (where $d_i$ is the degree of node $i$)