A Reduced Solution for Fully Anisotropic Circular Cylindrical Shells subject to Axial Compression Buckling

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The buckling of laminated circular cylindrical shells has received much attention due to its broad use in engineering applications. However, most of the studies have been done in the dimensional form, which complicates their usage due to the huge range of material properties available. The current study develops the non-dimensional scheme for a fully anisotropic structure. In addition, a novel technique is employed to reduce the solution of the buckling equation to just one variable. The main assumption used is that all variables are continuous, thus giving the lower bound of the buckling load. This allows significant savings in the computational time, increasingly so in complex optimization studies. Thus it is suitable for use in the early design stage. For restricted classes of laminates, very simple expressions can be obtained for the buckling load.

Nomenclature

\[ [a] \] = Compliance matrix of \([A]\)
\[ [b] \] = Eccentricity matrix \([A]^{-1}[B]\)
\[ [d^*] \] = Modified bending stiffness \([D] - [B][A]^{-1}[B]\)
\( F \) = Non-dimensional Airy’s stress function
\( F_0 \) = Non-dimensional amplitude of Airy’s stress function
\( K_1 \) = Buckling coefficient
\( L_x, L_y \) = Characteristic dimensions of shell in \(x\) and \(y\) direction respectively
\( l \) = Length of the cylinder
\( M_x, M_y, M_{xy} \) = Moment resultants per unit length
\( M \) = Number of longitudinal half waves
\( m^* \) = Slope of spiral in a spiral mode shape
\( N_1 \) = Non-dimensional buckling load per unit length
\( N_{1,0} \) = Buckling load per unit length
\( N_x, N_y, N_{xy} \) = In-plane force resultants per unit length
\( n \) = Number of circumferential waves
\( Q_x, Q_y \) = Out-of-plane force resultants
\( r \) = Radius of the cylinder
\( W \) = Non-dimensional radial displacement
\( W_{0} \) = Non-dimensional amplitude of radial displacement
\( w \) = Radial displacement

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\[ X = \text{Non-dimensional axial coordinate}, \quad X = x/L_x \]
\[ x = \text{Axial coordinate} \]
\[ Y = \text{Non-dimensional circumferential coordinate}, \quad Y = y/L_y \]
\[ y = \text{Circumferential coordinate} \]
\[ Z = \text{Modified Batdorf-Stein Z parameter} \]
\[ \Phi = \text{Airy's stress function} \]
\[ \Lambda_m = \text{Non-dimensional axial wavelength} \]
\[ \Lambda_n = \text{Non-dimensional circumferential wavelength} \]
\[ \alpha_i = \text{Non-dimensional parameters for stiffness element 11, 22(except B’s)} \]
\[ \beta_i = \text{Non-dimensional parameters for stiffness element 12, 66} \]
\[ \gamma_i = \text{Non-dimensional parameters for stiffness element 16} \]
\[ \delta_i = \text{Non-dimensional parameters for stiffness element 26} \]
\[ \varepsilon_i = \text{Non-dimensional parameters for stiffness element 22(only B’s)} \]
\[ \varepsilon_x, \varepsilon_y = \text{In-plane direct strain resultants} \]
\[ \kappa_x, \kappa_y, \kappa_{xy} = \text{Out-of-plane curvatures} \]
\[ \eta = \text{Ratio of axial wavelength to circumferential wavelength} \]
\[ \lambda_m = \text{Axial wavelength} \]
\[ \lambda_n = \text{Circumferential wavelength} \]

**Subscript**

\[ A = \text{Non-dimensional parameters for in-plane stiffnesses} \]
\[ B = \text{Non-dimensional parameters for ext/bend stiffnesses} \]
\[ D = \text{Non-dimensional parameters for bending stiffnesses} \]

**I. Introduction**

The high strength to weight ratio of laminated composite materials has attracted many designers of engineering structures, especially in the aerospace and also the racing automotive industries. This has fuelled interest in investigating the behaviour of laminated structures under different loads, and to the identification of their failure modes. One important failure mode is local buckling as it is usually occurs when a thin-walled structure is under compression. The present analysis looks specifically on the buckling load of a circular cylindrical shell under axial compression. However, subsequent analysis can be easily extended to incorporate different loads. This type of structure can be found in numerous structures such as airframes, rockets and missiles.

In the literature, many of the expressions for buckling are presented in dimensional form, which makes it difficult to characterise the entire range of different material properties and also the fibre orientation effects on the buckling load. One report where non-dimensional parameters are introduced is by Khot and Venkayya\(^1\), but the expressions are rather complicated and difficult to use. This restricts the full usage of the elastic tailoring capability of the laminated material in accommodating different engineering applications. It is the purpose of the current study to develop a non-dimensional scheme for a simple model incorporating full anisotropy\(^2\). This then allows the characterisation of the instability problem into a few non-dimensional parameters, thus simplifying the effort in designing a laminated composite structure and helps gaining increased physical insight.

A novel solution methodology is employed here to reduce the number of variables needed in solving the buckling load equation from the conventional number of two wavelengths (the axial and circumferential) in all other solutions, into just one variable. The computational time in calculating the buckling load, for example in optimization of design, could be reduced significantly since the number of variables is halved.

The current work derives the simplified buckling expressions starting from the non-dimensional expressions of the equation of motion and the equation of compatibility based on the Donnell’s shell assumptions. The reduction procedure is then applied to the derived buckling expression and a few special classes of laminates are examined in detail.
II. Non-dimensional scheme

The work presented here extends those that are found in Wong and Weaver. Without re-deriving the equations of motion and compatibility expressions, the following equations are quoted directly from Eq. (8) in Ref. 2.

\[
\begin{align*}
\frac{a_{22}}{L} \frac{\partial^4 \Phi}{\partial x^4} & - 2a_{22} \frac{\partial^4 \Phi}{\partial x \partial y^3} + (2a_{12} + a_{16}) \frac{\partial^4 \Phi}{\partial x^3 \partial y^2} - 2a_{16} \frac{\partial^4 \Phi}{\partial x^2 \partial y^3} + a_{11} \frac{\partial^4 \Phi}{\partial y^4} - \frac{1}{r} \frac{\partial^4 W}{\partial y^4} \\
\left[ b_{21} \frac{\partial^4 W}{\partial x^2} + (2b_{26} - b_{64}) \frac{\partial^4 W}{\partial x \partial y^3} + (b_{11} + b_{22} - 2b_{64}) \frac{\partial^4 W}{\partial x^2 \partial y^2} + (2b_{16} - b_{64}) \frac{\partial^4 W}{\partial x^3 \partial y} + b_{21} \frac{\partial^4 W}{\partial y^4} \right] &= 0
\end{align*}
\]  

(1a)

\[
\begin{align*}
d_{11} \frac{\partial^4 W}{\partial x^4} & + 4d_{11} \frac{\partial^4 W}{\partial x \partial y^3} + 2(d_{12} + 2d_{66}) \frac{\partial^4 W}{\partial x^3 \partial y^2} + 4d_{26} \frac{\partial^4 W}{\partial x^2 \partial y^3} + d_{22} \frac{\partial^4 W}{\partial y^4} \\
\left[ b_{21} \frac{\partial^4 W}{\partial x^2} - (b_{64} - 2b_{26}) \frac{\partial^4 W}{\partial x \partial y^3} + (b_{11} + b_{22} - 2b_{64}) \frac{\partial^4 W}{\partial x^2 \partial y^2} - (b_{64} - 2b_{26}) \frac{\partial^4 W}{\partial x^3 \partial y} + b_{21} \frac{\partial^4 W}{\partial y^4} \right] \\
+ \frac{1}{r} \frac{\partial^4 \Phi}{\partial x^2} + N_1^n \frac{\partial^4 W}{\partial x^2} &= 0
\end{align*}
\]  

(1b)

Equation (1a) is the compatibility equation, and Eq. (1b) is the moment equilibrium equation of a circular cylindrical shell under axial compression. The procedures that are applied to transform these equations into the non-dimensional form are similar to those introduced by Nemeth, with slight modification to the notation of the parameters and the parameters themselves.

\[
\alpha_a = \frac{L_y}{L_x} \sqrt{\frac{a_{22}}{a_{11}}} , \quad \beta_a = \frac{2a_{12} + a_{16}}{2\sqrt{a_{11}a_{22}}} , \quad \gamma_a = -\frac{a_{26}}{\sqrt{a_{11}a_{22}}} , \quad \delta_a = -\frac{a_{26}}{\sqrt{a_{11}a_{22}}} , \quad K_1 = \frac{N_1^n L_x^2}{\pi^3 \sqrt{a_{11}a_{22}}}
\]

\[
\alpha_d = \frac{L_y}{L_x} \sqrt{\frac{d_{11}^*}{d_{11}^*}} , \quad \beta_d = \frac{d_{11}^* + 2d_{66}^*}{\sqrt{d_{11}^* d_{22}^*}} , \quad \gamma_d = \frac{d_{11}^*}{\sqrt{d_{11}^* d_{22}^*}} , \quad \delta_d = \frac{d_{11}^*}{\sqrt{d_{11}^* d_{22}^*}}
\]  

(2)

Instead of the usual bending stiffness, \([D]\), the reduced bending stiffness, \([d^*]\), is used. Also, the subscript notation makes it easy to identify which stiffness is being referred to, whilst retaining the Greek symbols introduced by Nemeth.

By applying Nemeth techniques and the parameters as shown in Eq. (2), the governing equations can be transformed into the following form,

\[
\begin{align*}
\alpha_a^3 \frac{\partial^4 F}{\partial X^4} + 2\alpha_a \delta_a \frac{\partial^4 F}{\partial X^3 \partial Y} + 2\beta_a \frac{\partial^4 F}{\partial X^2 \partial Y^2} + 2\gamma_a \frac{\partial^4 F}{\partial X \partial Y^3} + \frac{1}{\alpha_a^2} \frac{\partial^4 F}{\partial Y^4} &- \frac{L_x^2}{L_y} \frac{1}{\sqrt{a_{11}a_{22}}} \frac{\partial^4 W}{\partial X^2} \\
+ \left[ L_x \left( b_{21} \frac{\partial^4 W}{\partial X^2} + L_x \left( \frac{2b_{26} - b_{64}}{\sqrt{a_{11}a_{22} d_{11}^* d_{22}^*}} \right) \frac{\partial^4 W}{\partial X \partial Y^3} + \frac{b_{11} + b_{22} - 2b_{64}}{\sqrt{a_{11}a_{22} d_{11}^* d_{22}^*}} \frac{\partial^4 W}{\partial X^2 \partial Y^2} \right) \frac{\partial^4 W}{\partial X^2 \partial Y^2} \right] &= 0
\end{align*}
\]  

(3a)
As can be seen from Eq. (3), the only terms that remain to be transformed into the non-dimensional form are those involving the eccentricity matrix (Geier et al). These terms assume the same form in both equations, Eqs. (3a) and (3b), thus making it convenient to set the extra parameters as:

$$\alpha = \frac{b_3}{\sqrt{a_1a_2d_1d_2}} L_i^2, \quad \gamma = \frac{2b_{2a}-b_{2b}}{\sqrt{a_1a_2d_1d_2}} L_i, \quad \beta = \frac{b_3-b_{2b}}{\sqrt{a_1a_2d_1d_2}},$$

$$\epsilon = \frac{b_3}{\sqrt{a_1a_2d_1d_2}} L_i, \quad \delta = \frac{2b_{2a}-b_{2b}}{\sqrt{a_1a_2d_1d_2}} L_i, \quad Z = \frac{L_i^2}{r} \frac{1}{\sqrt{a_1a_2d_1d_2}}$$

In solving the problem, the following non-dimensional displacement and Airy stress functions are

$$W = W_0 \sin(\Lambda_0 Y - \Lambda_0 X), \quad F = F_0 \sin(\Lambda_0 Y - \Lambda_0 X)$$

where

$$\Lambda_0 = L_i \lambda_0, \quad \Lambda_0 = L_0 \lambda_0 \quad \lambda_0 = \frac{m'n}{r}, \quad \lambda_0 = \frac{n}{r}$$

These functions are used because the laminate may develop a spiral deformation due to the presence of membrane anisotropy (extension/shear couplings), extension/twist and flexural anisotropies (flexural/twist couplings). By substituting those parameters in Eq. (4) and the assumed functions in Eq. (5) into Eq. (3), and applying the solution technique by Cheng and Ho (which is by grouping the equations into matrix form and solve for the determinant of the coefficient matrix), the final buckling equation in non-dimensional form is

$$N_i = K_0 \pi^2 = \alpha \Lambda^2 - 4 \alpha_d \gamma \Lambda \Lambda_0 + 2 \beta \delta \Lambda + 4 \delta_D \Lambda_0^2 + \frac{1}{\alpha_d} \Lambda_0^4$$

This is analogous to the dimensional form as derived in Wong and Weaver, and it can be grouped into the bending terms, the couplings terms, and the membrane terms. Like many of the models found in the literature, there are two variables that need to be solved in order to calculate the actual buckling load. However, the technique shown in the following section will reduce this to only one variable.

Note that if there is no angle-ply anisotropy present, i.e., $\gamma = \delta = 0$, Eq. (6) remains valid except the basis functions become
\[ W = W_0 \sin \Lambda_n Y \sin \Lambda_m X, \quad F = F_0 \sin \Lambda_n Y \sin \Lambda_m X \]

where

\[ \Lambda_m = L_m \Lambda_m, \quad \Lambda_n = L_n \Lambda_n \quad \text{and} \quad \lambda_m = \frac{m \pi}{l}, \quad \lambda_n = \frac{n}{r} \]

### III. Reduced solution methodology

The method is based on the minimization method used in calculus. Despite the non-continuous nature of the wave numbers the method still deserve practical merit. For the calculated load to be a minimum in Eq. (6), it has to be calculated at points of zero gradient, or turning points, which can be satisfied by differentiating the load against the two variables and equating them to zero.

\[ \frac{\partial N_i}{\partial \Lambda_m} = 0, \quad \frac{\partial N_i}{\partial \Lambda_n} = 0 \quad \text{where} \quad N_i = K_i \pi^2 \]  

(7)

Since at the turning points, these two differentials equate to zero, it is legitimate to combine them as

\[ \frac{\partial N_i}{\partial \Lambda_m} \cdot \frac{\partial \Lambda_n}{\partial k} + \frac{\partial N_i}{\partial \Lambda_n} \cdot \frac{\partial \Lambda_m}{\partial k} = 0 \]  

(8)

After some tedious manipulation, the resulting expression reduces to

\[ \begin{align*}
\alpha^2 \Lambda_n^2 \Lambda_m - 4 \alpha \Lambda_m \Lambda_n \Lambda_n \Lambda_n + 2 \beta \Lambda_m \Lambda_n \Lambda_n \Lambda_n - 4 \frac{\delta_d}{\alpha} \Lambda_m \Lambda_n \Lambda_n \Lambda_n + \frac{1}{\alpha^2} \Lambda_n \Lambda_n \Lambda_n \Lambda_n - \\
\left[ \alpha^2 \left( \delta_g \Lambda_m \Lambda_m \Lambda_m \Lambda_m - \delta_g \Lambda_m \Lambda_m \Lambda_m \Lambda_m \right) \right] + \\
\left[ \left( \delta_g \Lambda_n \Lambda_n \Lambda_n \Lambda_n - 2 \alpha \delta_g \Lambda_n \Lambda_n \Lambda_n \Lambda_n - 2 \alpha \beta \Lambda_n \Lambda_n \Lambda_n \Lambda_n - 2 \alpha \beta \Lambda_n \Lambda_n \Lambda_n \Lambda_n - 2 \alpha \beta \Lambda_n \Lambda_n \Lambda_n \Lambda_n - 2 \alpha \beta \Lambda_n \Lambda_n \Lambda_n \Lambda_n \right) \right] = 0
\end{align*} \]

(9)

Before examining the case of general anisotropy, it is interesting to investigate the behaviour when the laminate is symmetric. In this case, all the terms with a subscript “B” will vanish, giving

\[ \begin{align*}
\alpha^2 \Lambda_n^2 \Lambda_m - 4 \alpha \Lambda_m \Lambda_n \Lambda_n \Lambda_n + 2 \beta \Lambda_m \Lambda_n \Lambda_n \Lambda_n - 4 \frac{\delta_d}{\alpha} \Lambda_m \Lambda_n \Lambda_n \Lambda_n + \frac{1}{\alpha^2} \Lambda_n \Lambda_n \Lambda_n \Lambda_n - \\
\left[ \alpha^2 \left( \delta_g \Lambda_m \Lambda_m \Lambda_m \Lambda_m - \delta_g \Lambda_m \Lambda_m \Lambda_m \Lambda_m \right) \right] = 0
\end{align*} \]

(10)

It is realized that Eq. (10), when compared to Eq. (6), without coupling terms (terms with subscript “B”), contains exactly the same terms, with only the difference in sign of the right hand side of Eq. (10). Moreover, with the coupling matrix equals to zero, the parameters for bending stiffness are now based on the original \([D]\) matrix and not on the modified bending stiffness of \([d^*]\). This leads to the conclusion that the contribution of bending stiffness towards the buckling load is exactly the same as the contribution from the membrane stiffness. Or in other words, one could say that the energies stored in resisting the bending and membrane deformation are exactly the same in the ideal case. It only happens in the ideal case because in this case, the cylinder is allowed to take any arbitrary shape without the limitation of the physical world (the circumferential wavelength is assumed to be definite continuous). In the real world, the cylinder is only allowed to develop a full number of waves around its circumference due to the closed boundary in this direction. This is where the so-called “integer effect” comes into play. This limitation on one of the variables would mean that the structure buckles with a less efficient deformation shape than that of a
comparable, yet hypothetical, structure in which there are continuous variables. Thus, the buckling load found by
assuming both variables as continuous will provide simplified yet useful design data since the actual buckling load
is not significantly greater (as shown later). Similar arguments apply to general laminates.

In the case of general laminates, the relation between Eq. (6) and Eq. (9) is somewhat obscured by the large
number of terms involved. A more elegant presentation is achieved when some of the terms are grouped together.
The following groupings are suggested:

\[ f_b = \alpha_b^2 \Lambda_m^6 - 4\alpha_\theta \gamma_{\theta \phi} \Lambda_m \Lambda_\phi + 2\beta_\phi \Lambda_\phi^2 - 4\frac{\delta_\phi}{\alpha_\phi} \Lambda_m^4 + \frac{1}{\alpha_\phi^2} \Lambda_m^4 \]

\[ f_c = \alpha_b \Lambda_m^6 - \gamma_b \Lambda_m^4 \Lambda_\phi + \beta_\phi \Lambda_\phi^2 \Lambda_m^2 - \delta_\phi \Lambda_m \Lambda_\phi^3 + \epsilon_\phi \Lambda_\phi^4 \]

\[ f_{MN} = Z\Lambda_m^2 \]

\[ f_{MD} = \alpha_b \Lambda_m^6 - 2\alpha_\phi \delta_\phi \Lambda_m \Lambda_\phi + 2\alpha_\phi \beta_\phi \Lambda_\phi^2 \Lambda_m^2 - 2\alpha_\phi \beta_\phi \Lambda_\phi \Lambda_m \Lambda_\phi + \Lambda_m^4 \]

where:

- \( f_b \): the bending deformation function
- \( f_c \): the function related to the coupling between bending and membrane
- \( f_{MN} \): the membrane numerator deformation function
- \( f_{MD} \): the membrane denominator function

The chief advantage of these groupings will become more apparent in the derivations that follow. As a
consequence, Eqs. (6) and (9) are written as

\[ N_i = f_b + \frac{\alpha_b^2 (f_c + f_{MN})^2}{\Lambda_m^2 f_{MD}} \]  \hspace{1cm} (12)

\[ f_b = -\frac{\alpha_b^2 (f_c + f_{MN})(f_c - f_{MN})}{\Lambda_m^2 f_{MD}} = -\frac{\alpha_b^2 (f_c^2 - f_{MN}^2)}{\Lambda_m^2 f_{MD}} \]  \hspace{1cm} (13)

Substituting Eq. (13) into Eq. (12) yields

\[ N_i = 2\frac{\alpha_b^2}{\Lambda_m^2 f_{MD}} f_{MN} (f_c + f_{MN}) \]  \hspace{1cm} (14)

which is still in terms of two variables. To reduce this into just one variable, a new variable, \( \eta \), is defined which
combines the two variables.

\[ \eta = \frac{\Lambda_m}{\Lambda_\phi} \]  \hspace{1cm} (15)

Using this definition in the groupings in Eq. (11), results in the following expressions, where the new notation
can be easily distinguished from that in Eq. (11) by noting that those with a bar on top are defined with the new
variable, \( \eta \).

\[ f_c = \Lambda_\phi^4 (\alpha_b \eta - \gamma_b \eta^3 + \beta_\phi \eta^2 - \delta_\phi \eta + \epsilon_\phi) = \Lambda_\phi^4 \cdot \bar{f}_c \]

\[ f_{MN} = \Lambda_\phi^2 (Z \eta^2) = \Lambda_\phi^2 \cdot \bar{f}_{MN} \]

\[ f_{MD} = \Lambda_\phi^4 (\alpha_b \eta^2 - 2\alpha_\phi \delta_\phi \eta^2 + 2\alpha_\phi \beta_\phi \eta^2 - 2\alpha_\phi \gamma_\phi \eta + 1) = \Lambda_\phi^4 \cdot \bar{f}_{MD} \]  \hspace{1cm} (16)
Equation (14) is then transformed into:

\[
N_i = 2 \frac{\alpha_i^2}{\eta^2} f_{MD} \left( \tilde{f}_C + \frac{1}{\Lambda_n^*} \tilde{f}_{MN} \right)
\]  

(17)

As observed in Eq. (17), the buckling load equation is now in terms of \( \eta \) except for an additional \( \Lambda_n^* \). So the task required is to find an expression for \( \Lambda_n^* \) in terms of \( \eta \) only. This is done through further algebraic manipulation.

The buckling load expression is once more differentiated with respect to \( \Lambda_n \), but this time using Eq. (12),

\[
\frac{\partial N_i}{\partial \Lambda_n} = \frac{1}{2\Lambda_n} \left( \frac{\partial f_y}{\partial \Lambda_n} \right) + \frac{\alpha_i^2}{\gamma_M} \left[ \frac{\partial (f_C + f_{MN})}{\partial \Lambda_n} \right] f_{MD} (f_C + f_{MN}) + \frac{\partial f_y}{\partial \Lambda_n} \left[ f_C + f_{MN} \right]^2 = 0
\]  

(18)

By introducing the following definitions, involving only \( \eta \),

\[
\frac{-\partial f_y}{\partial \Lambda_n} \left[ \frac{1}{\Lambda_n^2} f_C + f_{MN} \right] = R_1
\]

\[
\frac{\partial (f_C + f_{MN})}{\partial \Lambda_n} \left[ \frac{1}{\Lambda_n^2} f_C + f_{MN} \right] = R_2
\]

\[
\frac{\partial f_y}{\partial \Lambda_n} \left[ \frac{1}{\Lambda_n^2} f_C + f_{MN} \right] = R_3
\]

(19)

Eq. (18) becomes

\[
R_1 + \frac{\alpha_i^2}{\gamma_M} \left[ \left( f_C + f_{MN} \right) f_{MD} \cdot R_2 + \left( f_C + f_{MN} \right)^2 \cdot R_1 \right] = 0
\]

\[
\Rightarrow R_1 + \frac{\alpha_i^2}{\gamma_M} \left[ \left( \tilde{f}_C + \frac{1}{\Lambda_n^*} \tilde{f}_{MN} \right) f_{MD} \cdot R_2 + \left( \tilde{f}_C + \frac{1}{\Lambda_n^*} \tilde{f}_{MN} \right)^2 \cdot R_1 \right] = 0
\]

(20)

The result is a quadratic equation in \( \tilde{f}_C + \tilde{f}_{MN} \) as is required, and explicitly shown in Eq. (17), with all the coefficients in terms of \( \eta \). Thus, the next step in our solution procedure is to solve this quadratic equation, using the standard solution,

\[
\left( \tilde{f}_C + \frac{1}{\Lambda_n^*} \tilde{f}_{MN} \right) = \frac{\tilde{f}_{MD} \left( -\alpha_i R_2 \pm \sqrt{\alpha_i^2 R_2^2 - 4R_i R_3} \right)}{2\alpha_i R_2}
\]

(21)

Substituting this into Eq. (17) and expanding the definitions in Eq. (19) gives

\[
N_i = \alpha_i Z \frac{-\alpha_i R_2 \pm \sqrt{\alpha_i^2 R_2^2 - 4R_i R_3}}{R_i}
\]

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The final equation, Eq. (22), contains only one continuous variable, \( \eta \). As a result, buckling loads are found relatively economically by iterating through different values of \( \eta \). Furthermore, it is possible to derive a polynomial equation in \( \eta \) for the solution to represent a turning point. Before this can be done, another differentiation, similar to Eq. (18), is required, but now with respect to \( \Lambda_m \).

\[
\frac{\partial N_1}{\partial \Lambda_m} \frac{1}{2\Lambda_m^2} = \left( \frac{\partial f}{\partial \Lambda_m} \right) \left( \frac{\partial (f_c + f_{MN})}{\partial \Lambda_m} \right)_{f_c + f_{MN}} + \left( -\frac{f_{MN}}{\Lambda_m^2} \right) \left( \frac{\partial f_{MN}}{\partial \Lambda_m} \right)_{f_c + f_{MN}} = 0 \tag{23}
\]

Here, it is not as straightforward as in Eq. (19) where all the expressions are automatically in terms of \( \eta \). One of the differentiated terms results in a \( \Lambda_m \) in the expression as can be seen below.

\[
\frac{\partial (f_c + f_{MN})}{\partial \Lambda_m} \frac{1}{\Lambda_m^2} = 4\alpha_C - 3\gamma_A \frac{1}{\eta} + 2\beta_S \frac{1}{\eta^2} - \delta_S \frac{1}{\eta^3} + 2\Lambda_m \frac{1}{\Lambda_m} \tag{24a,b}
\]

In order to have all expressions in terms of \( \eta \), Eq. (24a) can be rearranged to have the term with \( \Lambda_m \) inside \((f_c + f_{MN})\). Furthermore, to obtain a similar expression to Eq. (20), Eq. (24b) needs to be reformed as well.

\[
\frac{\partial (f_c + f_{MN})}{\partial \Lambda_m} \frac{1}{\Lambda_m^2} = \frac{2(f_c + f_{MN})}{\Lambda_m} - 2\alpha_C \frac{1}{\eta} + \gamma_A \frac{1}{\eta^2} - \delta_S \frac{1}{\eta^3} + 2\epsilon_S \frac{1}{\eta^4} \tag{25}
\]

By introducing the following definitions,

\[
\frac{\partial f}{\partial \Lambda_m} \frac{1}{2\Lambda_m} = S_1 = \alpha_C - 2\alpha_S \frac{1}{\eta} + \beta_S \frac{1}{\eta^2} - \delta_S \frac{1}{\eta^3} + \frac{1}{\eta^4} \tag{26}
\]

and substituting them into Eq. (25), then Eq. (23), an expression similar to Eq. (20) is now possible.
\[
\frac{\partial N_i}{\partial \lambda_m} + \frac{1}{2\Lambda_m} \left[ \frac{2(f_c + f_{MN})}{\Lambda_m} + S_2 \left(f_{MD} + f_{MN}\right) + \left(-\frac{2f_{MD} + S_1}{\Lambda_m}\right)\left(f_c + f_{MN}\right) \right] = 0
\]

\[
\Rightarrow \frac{\partial N_i}{\partial \lambda_m} + \frac{1}{2\Lambda_m} \left[ f_{MD} \left(f_c + f_{MN}\right) + S_2 \left(f_{MD} + f_{MN}\right)\cdot S_1 \right] = 0
\]

\[
\Rightarrow S_1 \left( f_c + \frac{1}{\Lambda_m} f_{MN}\right)^2 + S_2 \cdot f_{MD} \left(f_c + \frac{1}{\Lambda_m} f_{MN}\right) + \frac{f_{MD}^2}{\alpha_d^2} = 0
\]  

(27)

There are now two equations with \( \left( f_c + f_{MN} / \Lambda_m^2 \right) \), Eqs. (20) and (27), with all their coefficients in term of \( \eta \). The objective is to derive an expression with only \( \eta \), therefore the term \( \left( f_c + f_{MN} / \Lambda_m^2 \right) \) would have to be eliminated from both of the equations. For clarity, let \( \xi = \frac{f_c + f_{MN} / \Lambda_m^2} \), thus Eqs. (20) and (27) become

\[
\begin{align*}
(a) & \quad R_1 \xi^2 + R_2 \cdot f_{MD} \cdot \xi + R_3 \frac{f_{MD}^2}{\alpha_d} = 0 \quad \Rightarrow -\xi^2 = \frac{R_2}{R_1} \cdot f_{MD} \cdot \xi + \frac{R_3}{R_1} \frac{f_{MD}^2}{\alpha_d} \\
(b) & \quad S_1 \xi^2 + S_2 \cdot f_{MD} \cdot \xi + S_3 \frac{f_{MD}^2}{\alpha_d} = 0 \quad \Rightarrow -\xi^2 = \frac{S_2}{S_1} \cdot f_{MD} \cdot \xi + \frac{S_3}{S_1} \frac{f_{MD}^2}{\alpha_d} \\
\therefore (a) = (b) & \Rightarrow \frac{R_2}{R_1} \cdot f_{MD} \cdot \xi + \frac{R_3}{R_1} \frac{f_{MD}^2}{\alpha_d} = \frac{S_2}{S_1} \cdot f_{MD} \cdot \xi + \frac{S_3}{S_1} \frac{f_{MD}^2}{\alpha_d}
\end{align*}
\]  

(28)

By comparing coefficients, two equations are obtained as

\[
\begin{align*}
\frac{R_3}{S_3} & = \frac{S_1}{S_1} \quad \Rightarrow R_1 S_1 = R_3 S_3 \\
\frac{R_2}{R_1} & = \frac{S_2}{S_1} \quad \Rightarrow R_1 S_1 = R_2 S_1
\end{align*}
\]  

(29a,b)

Furthermore, these two equations can be combined into just one equation,

\[
\begin{align*}
(a) & \quad R_1 S_1 = R_3 S_3 \quad \Rightarrow R_1 = \frac{R_3 S_1}{S_3} \\
(b) & \quad R_1 S_1 = R_2 S_1 \\
\Rightarrow & \quad \frac{R_1 S_1}{S_3} = R_3 S_3 \\
\Rightarrow & \quad R_1 S_1 - R_3 S_1 = 0
\end{align*}
\]  

(30)

Expanding this expression creates a sixth order polynomial in \( \eta \).
In general there are no exact solutions for this polynomial and numerical methods must be used. However, solutions become possible for certain classes of laminate. For example, careful observation of this expression will reveal that for a symmetric laminate with $B$ coupling matrix equal to zero, this polynomial is satisfied automatically. In this special case, another equation is needed, which is Eq. (29a). Once expanded, it again takes the form of another sixth order polynomial,

$$\eta^6\left(-\alpha_D^2\delta_A^2 + 2\alpha_D\delta_A\alpha_A^4\right) + \eta^6\left(2\alpha_D^2\alpha_A^2\delta_A - 2\beta_D\alpha_A^4\right) + \eta^4\left(3\alpha_D^2\delta_B^2 + 4\alpha_D\beta_B + 2\beta_D\delta_B - 12\frac{\delta_B}{\alpha_D}\alpha_B\right) + \eta^4\left(12\alpha_D\delta_B\alpha_B + 2\beta_D\delta_B - 3\frac{1}{\alpha_D}\delta_B\right) + \eta\left(4\delta_D\alpha_B + 1\delta_B^2 - \frac{1}{\alpha_D}\delta_B\right) = 0 \quad (31)$$

Depending on whether the laminate is symmetric or not, one only needs to find the values of $\eta$ that satisfy these equations, possibly using numerical methods. The buckling load is then just the lowest load given by one of these $\eta$ found.

In conclusion, by using this reduced solution, involving one buckling equation and one polynomial (chosen according to whether $B$ matrix is zero or not), only six $\eta$ values need to be calculated to find the buckling load. In the realm of optimization in which many different laminates are chosen as possible candidates for a particular design, when all their buckling loads need to be calculated, this solution procedure could present a significant amount of saving in terms of computational time. Thus reducing the lead-time needed in the design phase considerably. For example, using the usual two variables method, the number of operations needed is the range given by $n$ multiplied by the range of $m$ or $m'$. Using this solution, only six values are needed. The saving can be calculated as

$$\text{No. of steps saved} = \left(\frac{\text{range of } m'}{\text{interval of } m' \text{used}} \times n\right) - 6$$

In the next sections special classes of laminates are considered.

**IV. Special classes of laminates**

Numerical methods are required to solve the polynomials in Eqs. (31) and (32). However, for special classes of laminates, the polynomials can be simplified further and solved analytically, either in part or in full. The following classes of laminates are considered in the current analysis.

1. Symmetric, and orthotropic laminate;
2. Symmetric, orthotropic, homogeneous, and quasi-isotropic laminate;
3. Symmetric, quasi-isotropic, and homogeneous laminates with membrane and flexural anisotropies;

These classes of lay-ups are used because they exhibit certain properties that would allow the simplification of the solution. Furthermore, they are either commonly used or they are known to improve the performance of buckling. For example, Onoda showed that the maximum local buckling load for symmetric laminates can be achieved using a quasi-isotropic lay-up (with no couplings) consisting of an infinite number of infinitesimally thin
layers creating an effectively homogeneous laminate. However, this lay-up is impractical and Weaver et al.\textsuperscript{7} showed that this can be achieved by using 0°, 90°, ±45° plies. The minimum number of layers that this is possible is 48 if symmetrically laminated.

Laminates with membrane (extension/shear) and flexural (flexural/twist) anisotropies are also considered because the present of the membrane anisotropies may increase the buckling load, as shown in Wong and Weaver\textsuperscript{2}. The flexural anisotropies will always be present with membrane anisotropies for the class of laminates considered. Although anti-symmetric flat plates have a problem of warping during cool down from curing, it is less pronounced in cylindrical shells because they are closed sections. Furthermore, it was found in Weaver et al.\textsuperscript{8} that the present of membrane/flexural anisotropies may enhance the buckling load.

Table 1. The different properties of different classes of laminates

<table>
<thead>
<tr>
<th>Class</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symmetric</td>
<td>(\alpha_A = \beta_B = \gamma_B = \delta_A = 0)</td>
</tr>
<tr>
<td>Orthotropic</td>
<td>(\alpha_A = \beta_B = \gamma_B = \delta_A = \gamma_D = \delta_D = 0)</td>
</tr>
<tr>
<td>Homogeneous</td>
<td>(\alpha_A = \alpha_D)</td>
</tr>
<tr>
<td>Quasi-isotropic</td>
<td>(\alpha_A = 0)</td>
</tr>
<tr>
<td>Anti-symmetric</td>
<td>(\gamma_A = \delta_A = \gamma_D = \delta_D = 0)</td>
</tr>
</tbody>
</table>

Table 1 gives the different properties that will be used in the following analysis.

A. Symmetric laminate

For a symmetric laminate where the \([B]\) matrix equals zero, Eq. (22) can be simplified quite significantly,

\[
N_1 = 2\alpha_A Z \left[ \begin{array}{c}
-2\alpha_d\beta_d\eta^3 + 2\beta_d\beta_d\eta^2 - 6\delta_d\eta + \frac{1}{\alpha_d} \\
-\alpha_A\delta_d\eta^3 + 2\alpha_A\beta_d\eta^2 - 3\alpha_A\gamma_d\eta + 2
\end{array} \right]
\]

(33a)

whilst satisfying the appropriate polynomial, Eq. (32). It is noted that if \(\gamma_A = \delta_D = 0\) then one solution is \(\eta = 0\) and one possible solution for the buckling load is

\[
N_1 = 2\frac{\alpha_L Z}{\alpha_D}
\]

(33b)

1. Orthotropic

If the laminate is symmetric and orthotropic, then the buckling load is given by the lower value of Eq. (33b) and

\[
N_1 = 2Z \left[ \begin{array}{c}
\beta_d\eta^2 + \frac{1}{\alpha_D} \\
\beta_d\eta^2 + \frac{1}{\alpha_A}
\end{array} \right]
\]

(34)

The characteristic polynomial is

\[
\eta^4\left(\alpha_D^2\alpha_A^2\beta_A - \beta_D\alpha_A^4\right) + \eta^3\left(\alpha_D^2 - \frac{\alpha_A^4}{\alpha_D^2}\right) + \eta^2\left(\beta_D - \frac{\alpha_A^2}{\alpha_D^2}\beta_A\right) = 0
\]

(35)

The solution of \(\eta^2\) can be determined using the standard quadratic solution, which give
This can be substituted directly into Eq. (34) to find the buckling load.

2. Homogeneous

If the laminate is homogeneous as well, the relation \(\alpha_t = \alpha_d\) can be used to further simplified the solution in Eq. (36). The solution of \(\eta\) is then

\[
\eta = 0, \frac{1}{\alpha_d}, \quad (37a)
\]

and the buckling load is the lower of Eq. (33b) and

\[
N_1 = 2Z \frac{\beta_d + 1}{\beta_d + 1} \quad (37b)
\]

3. Quasi-isotropic

For this case, the solution of the buckling load simplifies greatly. As \(\alpha_t = \alpha_d = 1\), Eq. (37a) gives

\[
\eta = 0, \pm 1, \pm i \quad (38)
\]

Furthermore \(\beta_t = \beta_d = 1\), thus the buckling equation reduces to

\[
N_1 = 2Z \quad (39)
\]

4. Symmetric, quasi-isotropic and homogeneous with membrane and flexural anisotropies

If a laminate is symmetric, quasi-isotropic and homogeneous, it exhibits the following properties. The term quasi-isotropic refers to \(\alpha_t = \alpha_d = \beta_d = 1\)

The parameter \(\beta_t\) does not equal unity as the extension/shear coupling is non-zero. The membrane and flexural anisotropies have the following properties

\[
\gamma_t = \delta_t, \quad \gamma_d = \delta_d
\]

respectively. With these properties, the solution can be further simplified. Two unique solutions of the polynomial in Eq. (32) can be found and the polynomial can be reduced to the following form.

\[
(\eta - 1)(\eta + 1) \left[ \eta^5 (2\gamma_d - \gamma_t) + \eta^4 (2\beta_t - 2) + \eta^3 (8\gamma_d - 4\gamma_t \beta_t - 2\gamma_t) + \eta (2\beta_t - 2) + (2\gamma_d - \gamma_t) \right] = 0 \quad (40)
\]

It is noted that the axisymmetric buckling mode given by \(\eta = 0\) is no longer a solution. Equation (33a) can be further simplified to

\[
N_1 = 2Z \sqrt{-\frac{2\gamma_t \eta^3 + 2\eta^2 - 6\gamma_t \eta + 2}{-\gamma_t \eta^3 + 2\beta_t \eta^2 - 3\gamma_t \eta + 2}} \quad (41)
\]
and when the solution of $\eta$ is assumed to be $\pm1$, this further reduces to:

$$N_i = 4Z \frac{1 \mp 2\gamma_D}{2 + 2\beta_D \mp 4\gamma_A}$$

(42)

The validity of assuming $\eta = \pm1$ will become clear later.

**B. Anti-symmetric laminate**

Similar to the symmetric case, the solution can be simplified for the anti-symmetric case. It is known that in this case, the membrane and flexural anisotropies, $\gamma_A$, $\delta_A$, $\gamma_D$, and $\delta_D$, are zero. Equation (22) can then be simplified as

$$N_i = \alpha_A Z \left[ \alpha_A \left( -\gamma_B \eta + 2\beta_B - 3\delta_B \frac{1}{\eta_0} + 4\epsilon_B \frac{1}{\eta^2} \right) \right]$$

whilst satisfying Eq. (31), with those terms containing the membrane and flexural anisotropies ignored. Note that Eq. (33b) is also a solution as the axisymmetric buckling mode, $\eta = 0$, is a solution. Its value should be compared with the following formulae and the lowest value chosen for the buckling load.

**5. Homogeneous, quasi-isotropic with membrane/flexural anisotropies**

If a laminate is anti-symmetric, quasi-isotropic and homogeneous, it exhibits the following properties:

$$\alpha_A = \beta_A = \alpha_D = 1, \quad \gamma_A = \delta_A = \gamma_D = \delta_D = 0, \quad \alpha_B = \beta_B = \epsilon_B = 0$$

The parameter $\beta_B$ does not equal unity in this case, for reasons given for the symmetric case in Sec. IV.A.4. The membrane/flexural anisotropies have the following properties:

$$\gamma_B = \delta_B$$

If this is substituted into Eq. (31), it is found again that $\eta = \pm1$ are solutions of the polynomial and the resulting polynomial is simpler than that of the symmetric case,

$$(\eta - 1)(\eta + 1) \left[ \eta^4 + 2\eta^3(2 - \beta_D) + 1 \right] = 0$$

(44)

In this polynomial, the only material property that is needed is $\beta_D$, which is surprisingly simple as the membrane/flexural anisotropies do not affect $\eta$. In this case, the fourth order polynomial can be further solved and thus the remaining solutions are

$$\eta^2 = \beta_D - 2 \pm \sqrt{(\beta_D - 3)(\beta_D - 1)}$$

(45)

Equation (43) can then be further simplified to
\[ N_1 = Z \left\{ \left( -\gamma_a \eta - 3 \gamma_b \frac{1}{\eta} \right) \pm \left( -\gamma_a \eta - 3 \gamma_b \frac{1}{\eta} \right)^2 + 16 \left( \beta_D + \frac{1}{\eta^2} \right)^2 \left( 1 + \frac{1}{\eta^2} \right) \right\}^{1/2} \]

(46)

and for the solution of \( \eta = \pm 1 \), this reduces to

\[ N_1 = Z \left[ \gamma_b \pm \sqrt{\gamma_b^2 + 2(\beta_D + 1)} \right] \]

(47)

Again, the validity of assuming \( \eta = \pm 1 \) will be clarified when tested on laminates.

V. Results

In order to validate the resultant expression found in Sec. IV, some lay-ups have been devised and solved using different methods. The lay-ups are obtained using the thirty lay-ups found in the Appendix B of Ref. 8 as a basis. These lay-ups are modified by making the number of \(+45^\circ\) and \(-45^\circ\) layers unequal, but keeping the total number of \(\pm 45^\circ\) layers unchanged. This would give lay-ups that are homogeneous, quasi-isotropic with flexural and membrane anisotropies. The final number of different distinct lay-ups that can be formed is 377. The radius and the thickness of the cylinder used were 200 mm and 6 mm (with ply thickness of 0.125 mm) respectively.

These lay-ups are solved using both MAPLE§ and Microsoft Excel Solver**. The different methods of solution are

Maple: Equation (6) is used with integer \( n \)⁵ and \( m' \) of range from -3 to 3 with increment of 0.025.

Excel: (a) Equation (42) is used where \( \eta = \pm 1 \) is used with the lower value of the two taken to be the buckling load.
   (b) Equation (6) is used with the Solver tool using two different constraint settings:
      (i) Integer \( n \) with \( 1 \leq n \leq 10^{17} \);
      (ii) Continuous \( n \) with \( 1 \leq n \leq 10 \).

As hoped, it was found that the results obtained using Excel (a) and Excel (b)(ii) are equal, thus providing confidence that \( \eta = \pm 1 \). The buckling load found using these two methods is lower in magnitude than both Maple and Excel (b)(i) results. This provides strong evidence that the ideal case, where \( n \) is assumed to be continuous, will always give the lower bound of the buckling load. Besides, this also proves the validity of assuming \( \eta = \pm 1 \) in Eq. (42) without considering the other four possible solution of Eq. (40).

For these 377 lay-ups tested, the maximum difference between the buckling loads found using integer \( n \) and those found using continuous \( n \) is 1.46%. This in practical terms is small and can be ignored.

For the special case of anti-symmetric, quasi-isotropic and homogeneous laminate in Sec. B, the laminates used for testing is the same 377 laminates used above but with anti-symmetric lamination. The value of \( \beta_D \) was found to range from 0.79 to 1. For this range of \( \beta_D \), the solution of \( \eta^2 \) in Eq. (43) ranges from -0.53 to -1, in which case all the remaining solutions of Eq. (42) are imaginary in nature. Thus, for this special case, it is certain that the critical buckling load will be obtained with \( \eta = \pm 1 \).

§ Maple 8.00, Copyright© 1981-2002 by Waterloo Maple Inc.
** The solver add-in provided in Microsoft Excel uses the simplex methods with bounds on the variables for linear problem and the branch-and-bound method for integer problem.
⁵ The range of \( n \) used is from 0 to \((t/r)^{0.5}\).
⁶ The maximum \( n \) is set to be 10 as it is known for the geometry of shells examined that \( n \) will not be larger.

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VI. Conclusions

A non-dimensional scheme is extended to the full anisotropic case with the introduction of non-dimensional parameters for the eccentricity matrix. The result is a non-dimensional buckling equation. Using this non-dimensional buckling equation, the two wavelengths needed to solve the equation are reduced by mathematical manipulation to just one variable. This is made possible by assuming that the circumferential wave number is continuous and the use of calculus in finding the minimum. In solving the reduced buckling equation, one only needs to find the solution of a sixth order polynomial, and the critical buckling load will be the solution that will give the lowest load in the reduced buckling equation. If the laminate is symmetric, another sixth order polynomial should be used. The resulting solution procedure allows significant savings in time during the early design stage where many possible configurations are considered. For the special case of quasi-isotropic, homogeneous, symmetric or anti-symmetric lay-ups, the buckling load can be found easily with only two possible answers. As this type of orientation is very commonly used, it provides a very quick and easy way to calculate the buckling load.

References


