

CE 240  
Soil Mechanics & Foundations  
Lecture 2.2

**Weight-Volume Relationships**  
**(Das, Ch. 3)**

# **Outline of this Lecture**

- 1. Density, Unit weight, and Specific gravity ( $G_s$ )**
- 2. Phases in soil (a porous medium)**
- 3. Three phase diagram**
- 4. Weight-volume relationships**

# For a general discussion we have

**Weight  $W$**

$$W = Mg$$

**density  $\rho$**

$$\rho = \frac{M}{V}$$

**Unit weight  $\gamma$**

$$\gamma = \frac{W}{V}$$

**So that**

$$\gamma = \frac{W}{V} = \frac{Mg}{V} = \rho g$$

**Unit weight:  $\gamma = \rho g$**

**Unit weight is the product of density and gravity acceleration.** It is the gravitational force caused by the mass of material within a unit volume (density) in the unit of Newtons per cubic meter in SI system.

## Specific Gravity ( $G_s$ )

Specific gravity is defined as the ratio of unit weight (or density) of a given material to the unit weight (or density) of water since

$$G_s = \frac{\gamma}{\gamma_w} = \frac{\rho g}{\rho_w g} = \frac{\rho}{\rho_w}$$

# Specific Gravity

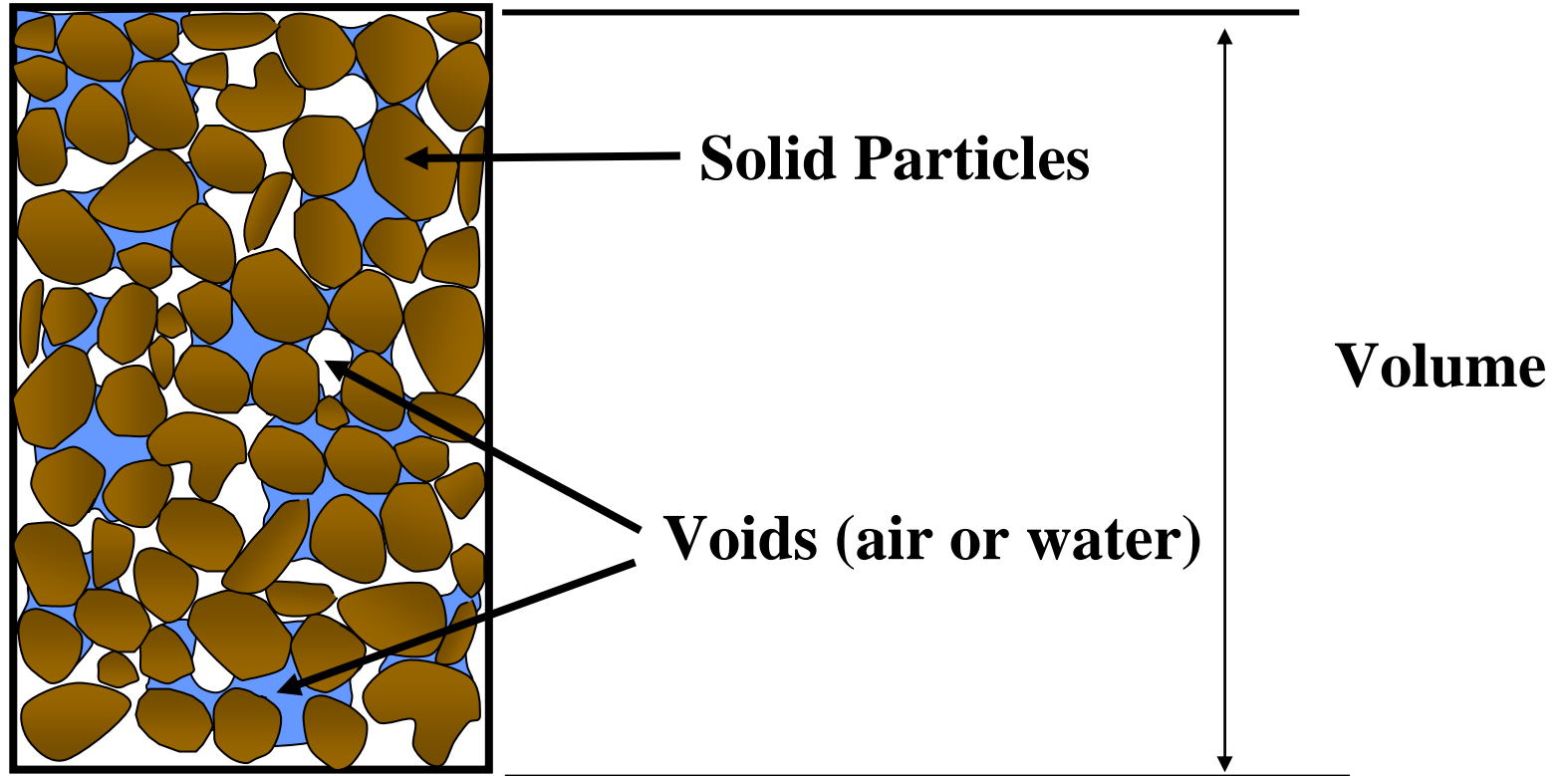
- Expected Value for  $G_s$

Type of Soil	$G_s$
Sand	2.65 - 2.67
Silty sand	2.67 – 2.70
Inorganic clay	2.70 – 2.80
Soils with mica or iron	2.75 – 3.00
Organic soils	< 2.00

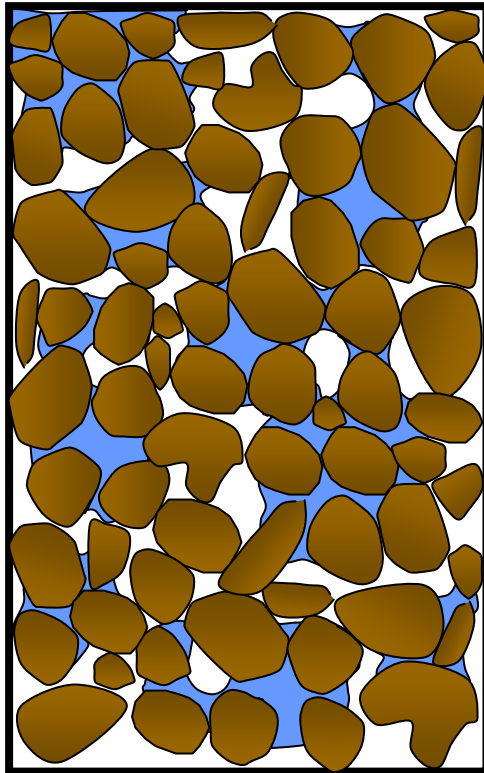
# **Three Phases of Soils**

**Naturally occurred soils always consist of solid particles, water, and air, so that soil has three phases: solid, liquid and gas.**

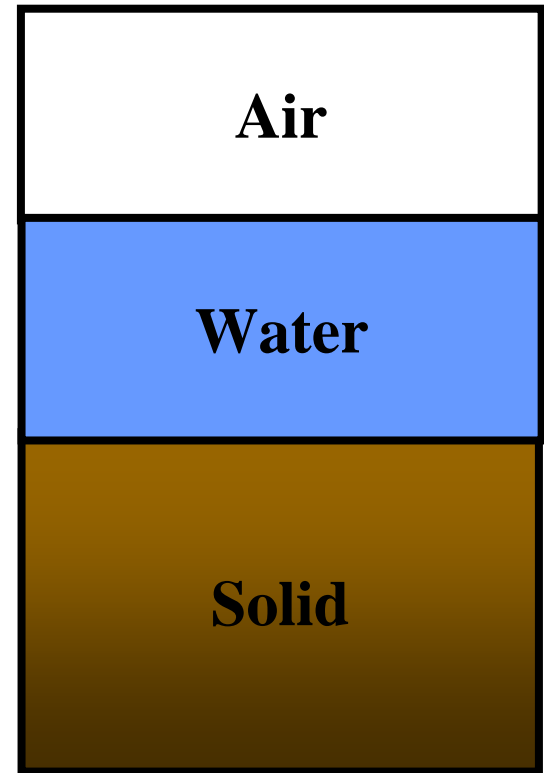
# Soil model



# Three Phase Diagram

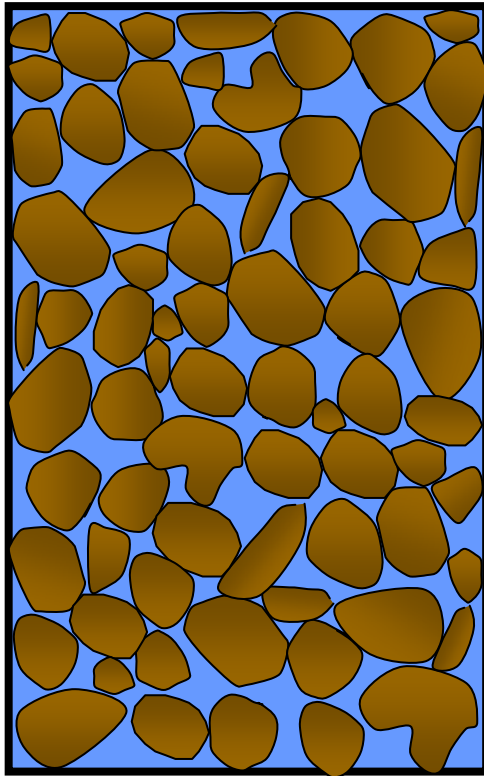


Mineral Skeleton

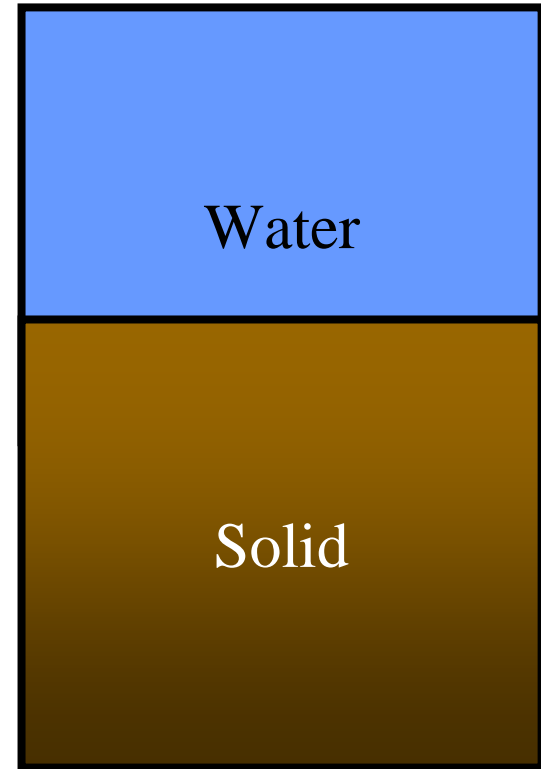


Idealization:  
*Three Phase Diagram*

# Fully Saturated Soils (**Two** phase)

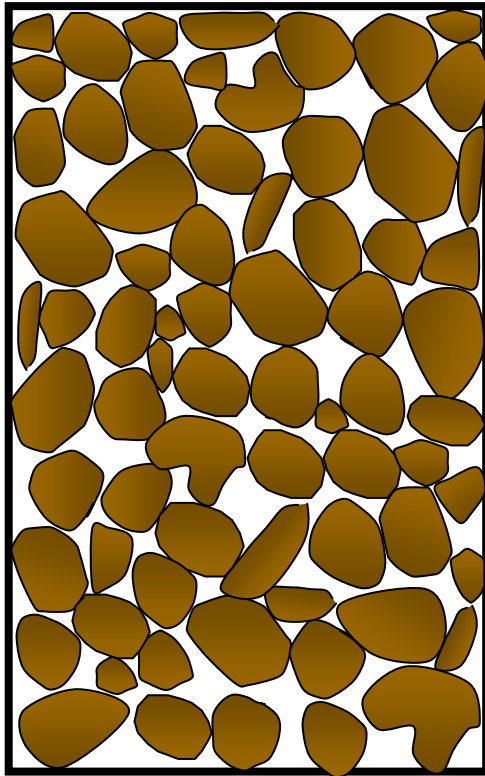


Mineral Skeleton



Fully Saturated

# Dry Soils (**Two** phase) [Oven Dried]

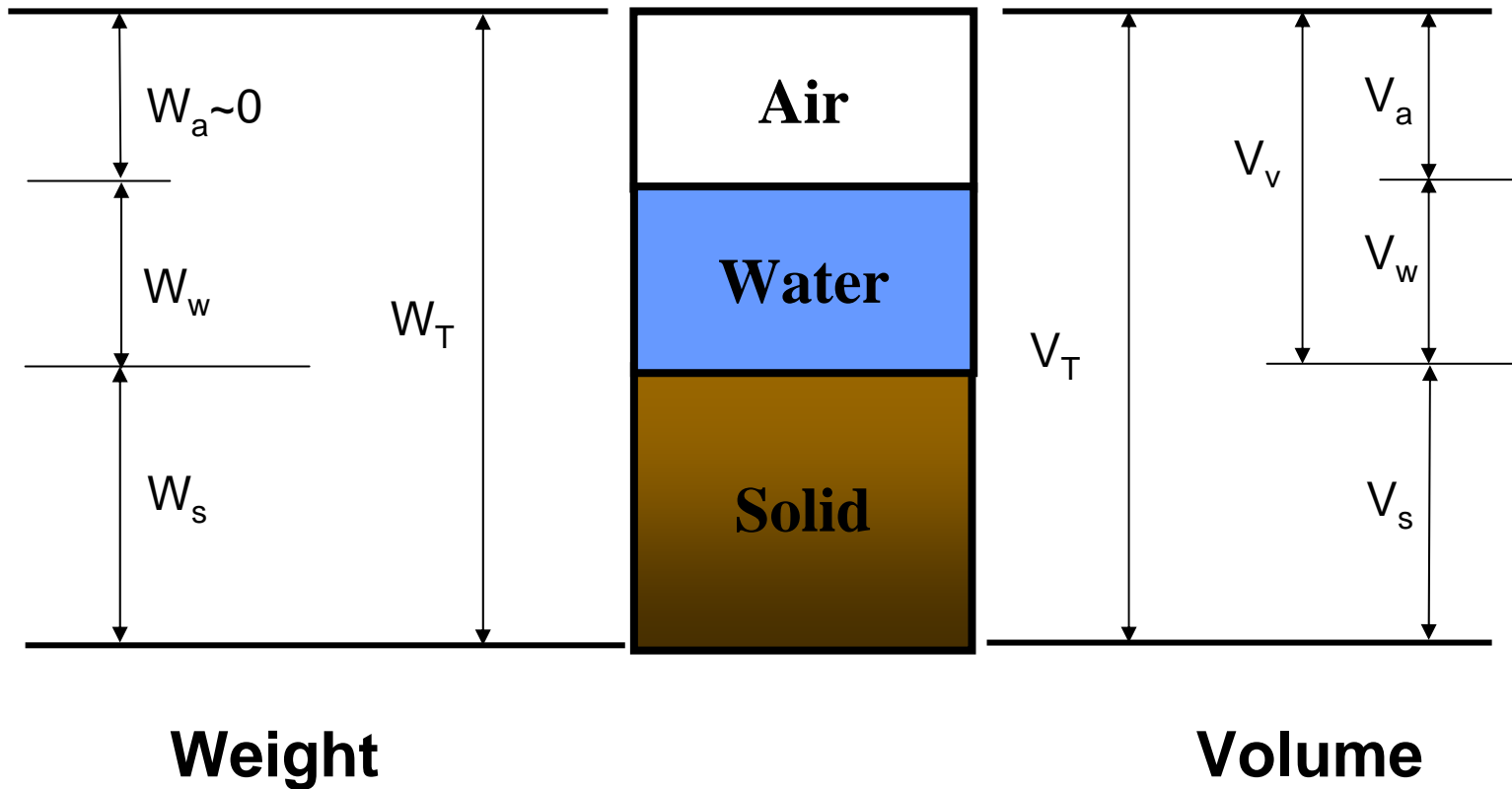


Mineral Skeleton

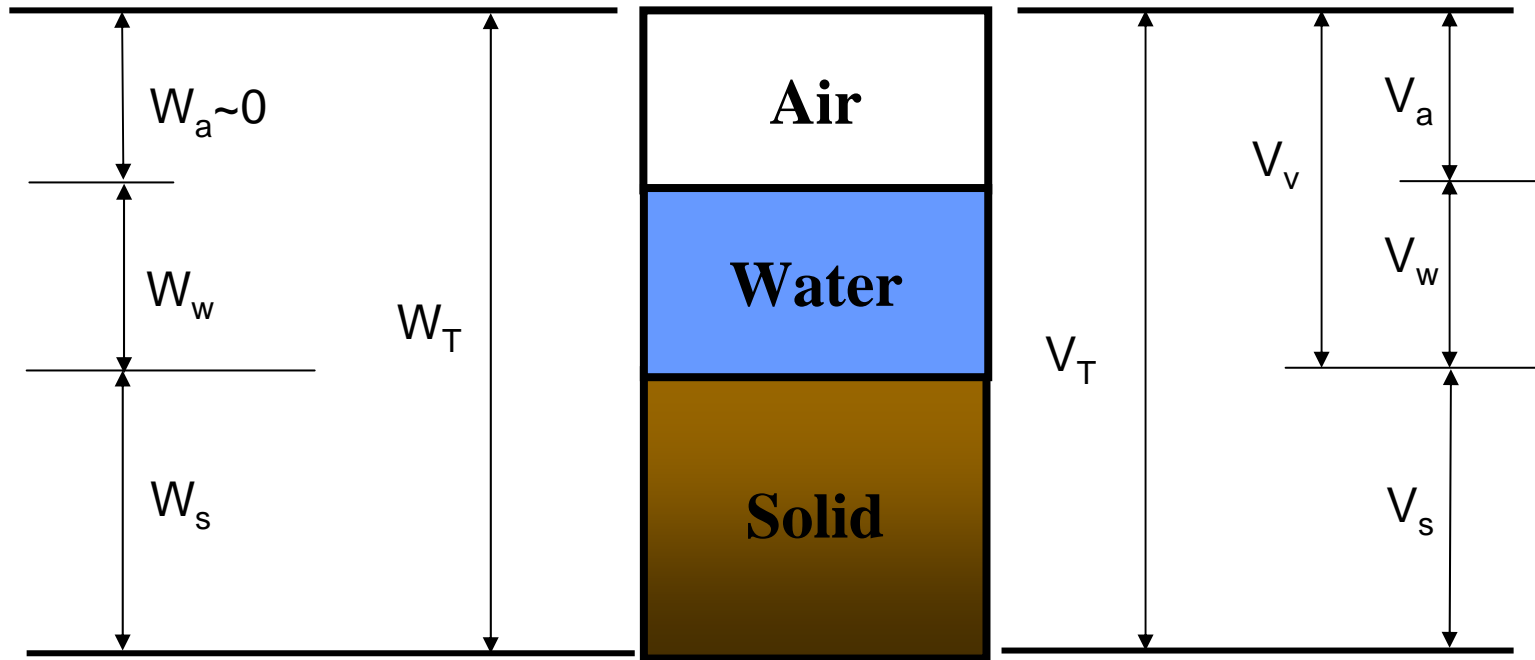


Dry Soil

# Three Phase Diagram

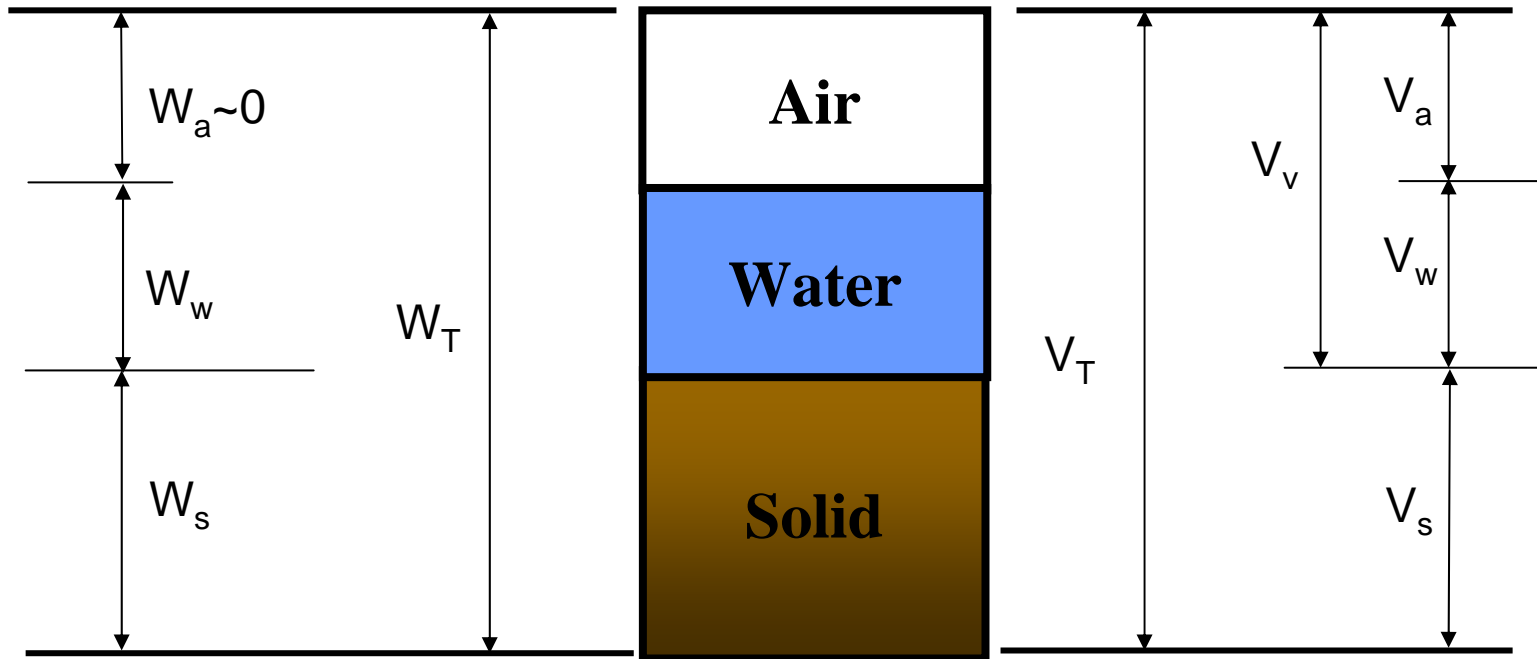


## Phase relationship: the phase diagram



**$W_t$ : total weight**  
 **$W_s$ : weight of solid**  
 **$W_w$ : weight of water**  
 **$W_a$ : weight of air = 0**

**$V_t$ : total volume**  
 **$V_s$ : volume of solid**  
 **$V_w$ : volume of water**  
 **$V_v$ : volume of the void**



$$V_t = V_s + V_v = V_s + V_w + V_a;$$

**It is convenient to assume the volume of the solid phase is unity (1) without lose generality.**

$$M_t = M_s + M_w; \text{ and}$$

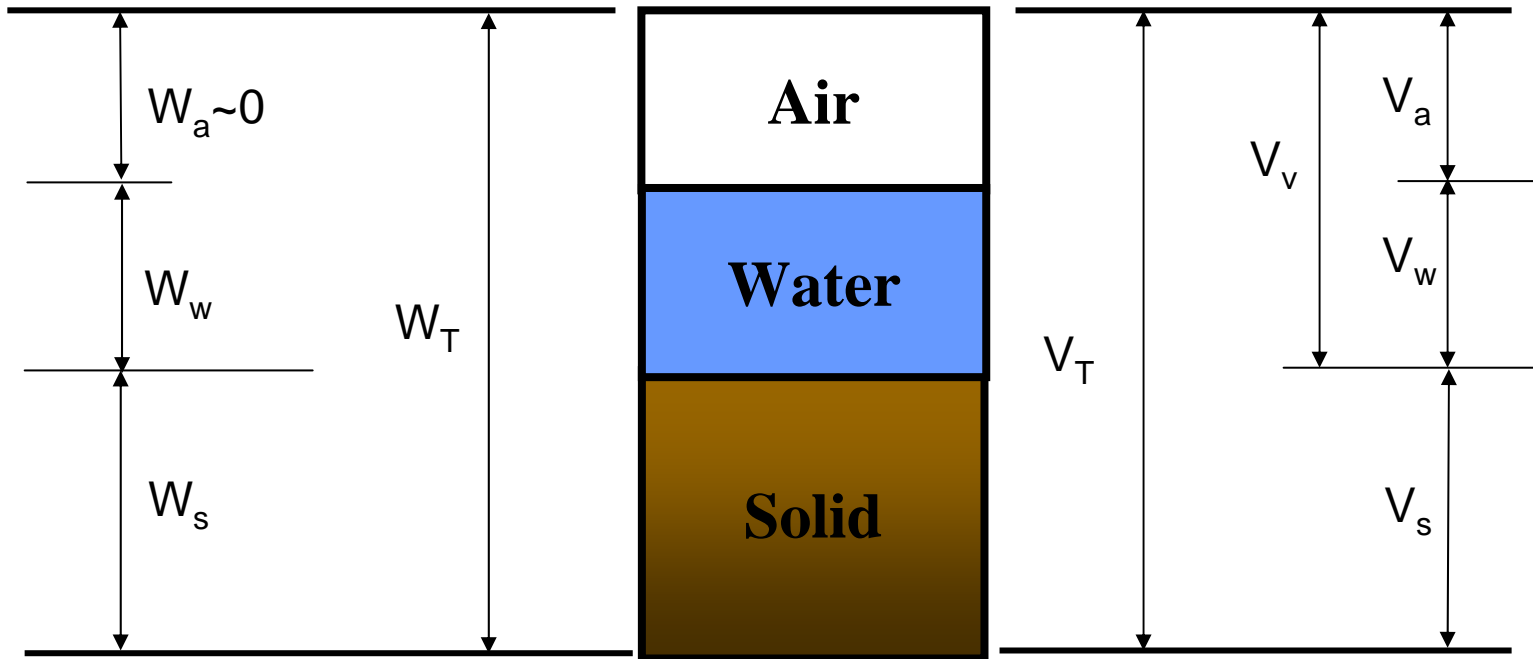
$$W_t = W_s + W_w, \text{ since } W=Mg$$

**Void ratio:  $e = V_v/V_s$ ; Porosity  $n = V_v/V_t$**

$$e = \frac{V_v}{V_s} = \frac{V_v}{V_t - V_v} = \frac{V_v / V_t}{1 - V_v / V_t} = \frac{n}{1 - n}$$

$$n = \frac{V_v}{V_t} = \frac{V_v}{V_s + V_v} = \frac{V_v / V_s}{1 + V_v / V_s} = \frac{e}{1 + e}$$

**Apparently, for the same material we always have  $e > n$ . For example, when the porosity is 0.5 (50%), the void ratio is 1.0 already.**



**Degree of saturation:  $S = V_w/V_v \times 100\%$**

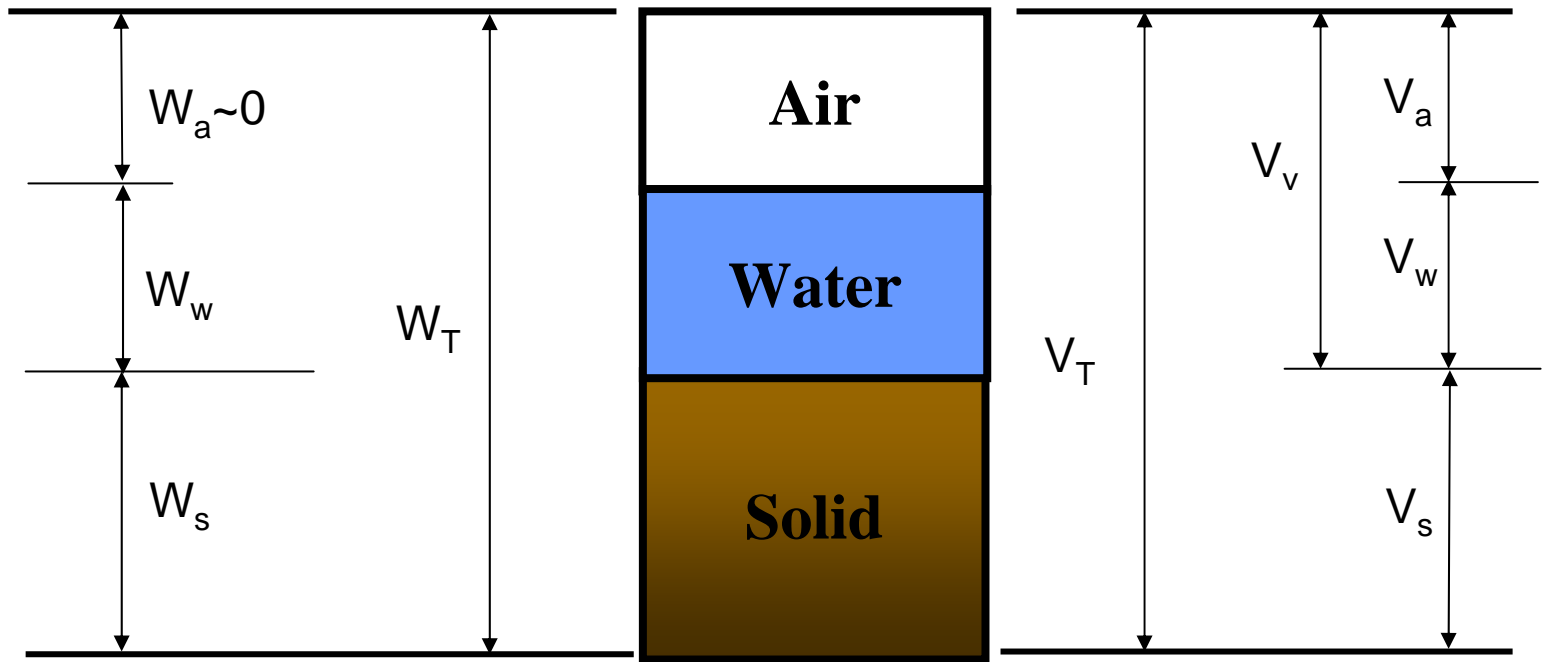
**Saturation is measured by the ratio of volume.**

**Moisture content (Water content):  $w = W_w/W_s$ ,**

**$W_w$  – weight of water,  $W_s$  – weight of solid**

**Water content is measured by the ratio of weight.**

**So that  $w$  can be greater than 100%.**



**Degree of saturation:  $S = V_w/V_v \times 100\%$**

**Saturation is measured by the ratio of volume.**

**Moisture content:  $w = W_w/W_s$ ,  $W_w = V_w \gamma_w$**

**$W_w$  – weight of water,  $W_s$  – weight of solid**

**Water content is measured by the ratio of weight.**

# Definition of 3 types of unit weight

Total unit weight (moisture unit weight, wet unit weight)  $\gamma$  :

$$\gamma = \frac{W_t}{V_t} = \frac{W_s + W_w}{V_t}$$

Dry unit weight  $\gamma_d$  :

$$\gamma_d = \frac{W_s}{V_t}, \quad \because V_t > V_s \quad \therefore \gamma_d < \gamma_s$$

Saturated unit weight (when saturation  $S=1$ )  $\gamma_{sat}$  :

$$\gamma_{sat} = \frac{W_t}{V_t}$$

**Moisture unit weight  $\gamma$ :**

$$\gamma = \frac{W_t}{V_t} = \frac{W_s + W_w}{V_t}$$

**Solid unit weight  $\gamma_s$**

$$\gamma_s = \frac{W_s}{V_s}$$

**dry unit weight  $\gamma_d$**

$$\gamma_d = \frac{W_s}{V_t}, \quad \because V_t > V_s \quad \therefore \gamma_d < \gamma_s$$

**Since**

$$\gamma_d = \frac{W_s}{V_t} = \frac{W_t - W_w}{V_t} = \frac{W_t}{V_t} - \frac{W_w}{V_t} = \gamma - \left( \frac{W_w}{W_s} \frac{W_s}{V_t} \right) = \gamma - w\gamma_d$$

so that  $\gamma_d + w\gamma_d = \gamma$  and  $\gamma_d = \frac{\gamma}{1 + w}$

**From the original form of the dry unit weight**

$$\gamma_d = \frac{\gamma}{1 + w}$$

**By taking the Taylor expansion and truncated at the first order term:**

$$\gamma_d = \frac{\gamma}{1 + w} = \gamma(1 - w + w^2 - w^3 + w^4 - w^5 + \dots) \approx \gamma(1 - w)$$

**Because the moisture content  $w$  is a number always smaller than one, i.e.,  $w < 1$ .**

**Thus, the dry unit weight  $\gamma_d$  can be approximated as:**

$$\gamma_d = (1 - w)\gamma$$

## Relationships among $S$ , $e$ , $w$ , and $G_s$

$$S = \frac{V_w}{V_v}, \text{ then } V_w = SV_v = Se, \quad \text{given } V_s = 1$$

**A simple way to get Das, Equation 3.18**

$$w = \frac{W_w}{W_s} = \frac{\gamma_w V_w}{\gamma_s V_s} = \frac{\gamma_w eS}{G_s \gamma_w} = \frac{eS}{G_s}, \quad \because V_s = 1, \quad \therefore V_v = 1$$

*thus*  $Se = wG_s$

**When the soil is 100% saturated ( $S=1$ )  
we have, Equation 3.20**

$$e = wG_s$$

## Relationships among $\gamma$ , $n$ , $w$ , and $G_s$

$$W_s = \gamma_s V_s = G_s \gamma_w (1-n), \quad \text{given } V_s = 1-n$$

$$W_w = w W_s = w G_s \gamma_w (1-n)$$

So that the dry unit weight  $\gamma_d$  is

$$\gamma_d = \frac{W_s}{V_t} = \frac{G_s \gamma_w (1-n)}{1-n+n} = G_s \gamma_w (1-n)$$

And the moist unit weight  $\gamma$  is

$$\begin{aligned} \gamma &= \frac{W_t}{V_t} = \frac{W_s + W_w}{V_t} = \frac{G_s \gamma_w (1-n) + w G_s \gamma_w (1-n)}{1} \\ &= (1+w) G_s \gamma_w (1-n) = G_s \gamma_w (1-n)(1+w) \end{aligned}$$

# Relationships among $\gamma$ , $n$ , $w$ , and $G_s$ (cont.)

When  $S=1$  (fully saturated soil)

$$\gamma_{sat} = \frac{W_s + W_w}{V_t} = \frac{G_s \gamma_w (1-n) + n \gamma_w}{1} = [G_s (1-n) + n] \gamma_w$$

the moisture content  $w$  when  $S=1$  can be expressed as

$$w = \frac{W_w}{W_s} = \frac{n \gamma_w}{G_s \gamma_w (1-n)} = \frac{n}{G_s (1-n)} = \frac{e}{G_s}$$

*recall*  $Se = wG_s$ , thus  $e = wG_s \quad \because S = 1$

# Weight-Volume Relationships (Table 3.1)

**Table 3.1** Various Forms of Relationships for  $\gamma$ ,  $\gamma_d$ , and  $\gamma_{sat}$

<u>Moist unit weight (<math>\gamma</math>)</u>		<u>Dry unit weight (<math>\gamma_d</math>)</u>		<u>Saturated unit weight (<math>\gamma_{sat}</math>)</u>	
Given	Relationship	Given	Relationship	Given	Relationship
$w, G_s, e$	$\frac{(1+w)G_s\gamma_w}{1+e}$	$\gamma, w$	$\frac{\gamma}{1+w}$	$G_s, e$	$\frac{(G_s+e)\gamma_w}{1+e}$
$S, G_s, e$	$\frac{(G_s+Se)\gamma_w}{1+e}$	$G_s, e$	$\frac{G_s\gamma_w}{1+e}$	$G_s, n$	$[(1-n)G_s+n]\gamma_w$
$w, G_s, S$	$\frac{(1+w)G_s\gamma_w}{1+\frac{wG_s}{S}}$	$G_s, n$	$G_s\gamma_w(1-n)$	$G_s, w_{sat}$	$\left(\frac{1+w_{sat}}{1+w_{sat}G_s}\right)G_s\gamma_w$
$w, G_s, n$	$G_s\gamma_w(1-n)(1+w)$	$G_s, w, S$	$\frac{G_s\gamma_w}{1+\left(\frac{wG_s}{S}\right)}$	$e, w_{sat}$	$\left(\frac{e}{w_{sat}}\right)\left(\frac{1+w_{sat}}{1+e}\right)\gamma_w$
$S, G_s, n$	$G_s\gamma_w(1-n)+nS\gamma_w$	$e, w, S$	$\frac{eS\gamma_w}{(1+e)w}$	$n, w_{sat}$	$n\left(\frac{1+w_{sat}}{w_{sat}}\right)\gamma_w$
		$\gamma_{sat}, e$	$\gamma_{sat}-\frac{e\gamma_w}{1+e}$	$\gamma_d, e$	$\gamma_d+\left(\frac{e}{1+e}\right)\gamma_w$
		$\gamma_{sat}, n$	$\gamma_{sat}-n\gamma_w$	$\gamma_d, n$	$\gamma_d+n\gamma_w$
		$\gamma_{sat}, G_s$	$\frac{(\gamma_{sat}-\gamma_w)G_s}{(G_s-1)}$	$\gamma_d, S$	$\left(1-\frac{1}{G_s}\right)\gamma_d+\gamma_w$
				$\gamma_d, w_{sat}$	$\gamma_d(1+w_{sat})$

The 3<sup>rd</sup> column is a special case of the 1st column when  $S = 1$ .

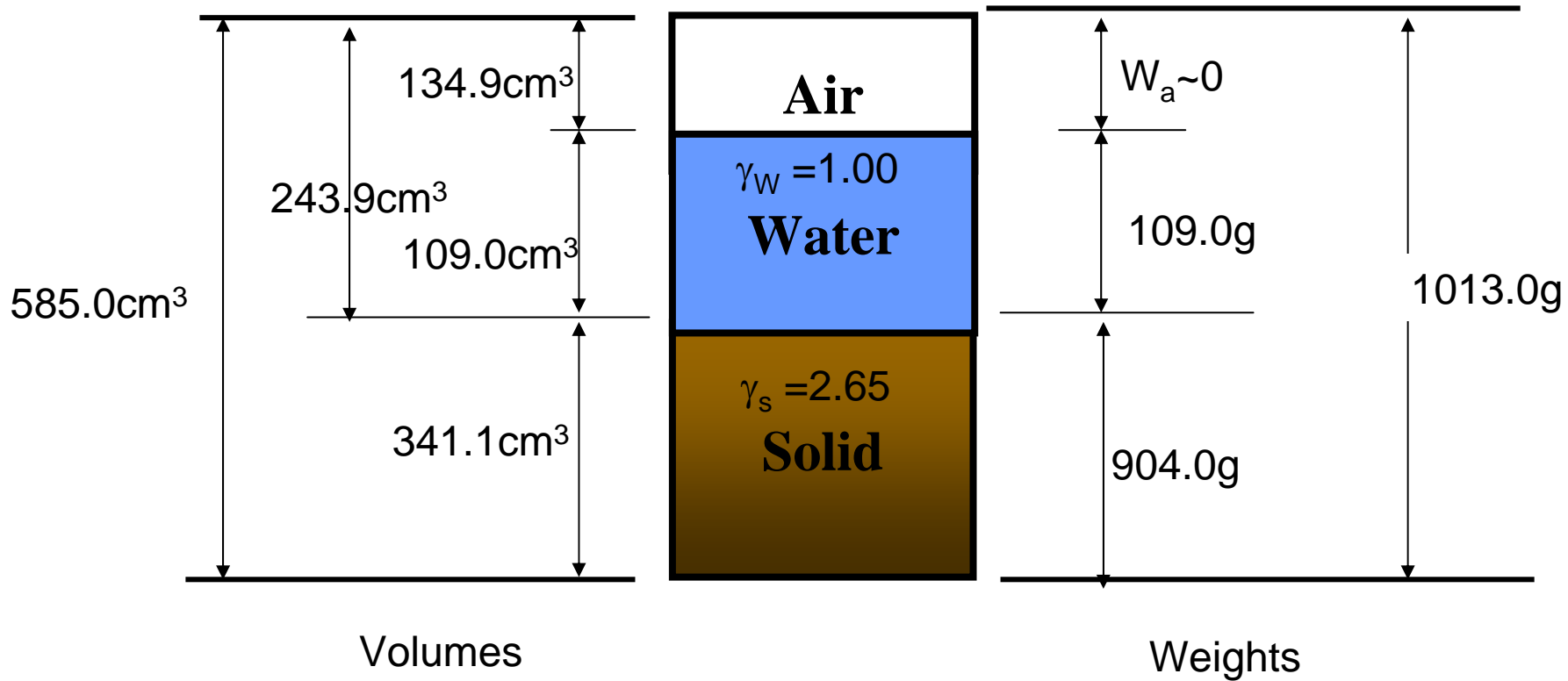
# Example:

- Determine moisture content, void ratio, porosity and degree of saturation of a soil core sample. Also determine the dry unit weight,  $\gamma_d$

## Data:

- Weight of soil sample = 1013g
- Vol. of soil sample = 585.0cm<sup>3</sup>
- Specific Gravity,  $G_s = 2.65$
- Dry weight of soil = 904.0g

# Example



# Results

- From the three phase diagram we can find:

- Moisture content,  $w$  
$$w = \frac{W_w}{W_s} = \frac{109 (g)}{904 (g)} \times 100 = 12.1\%$$

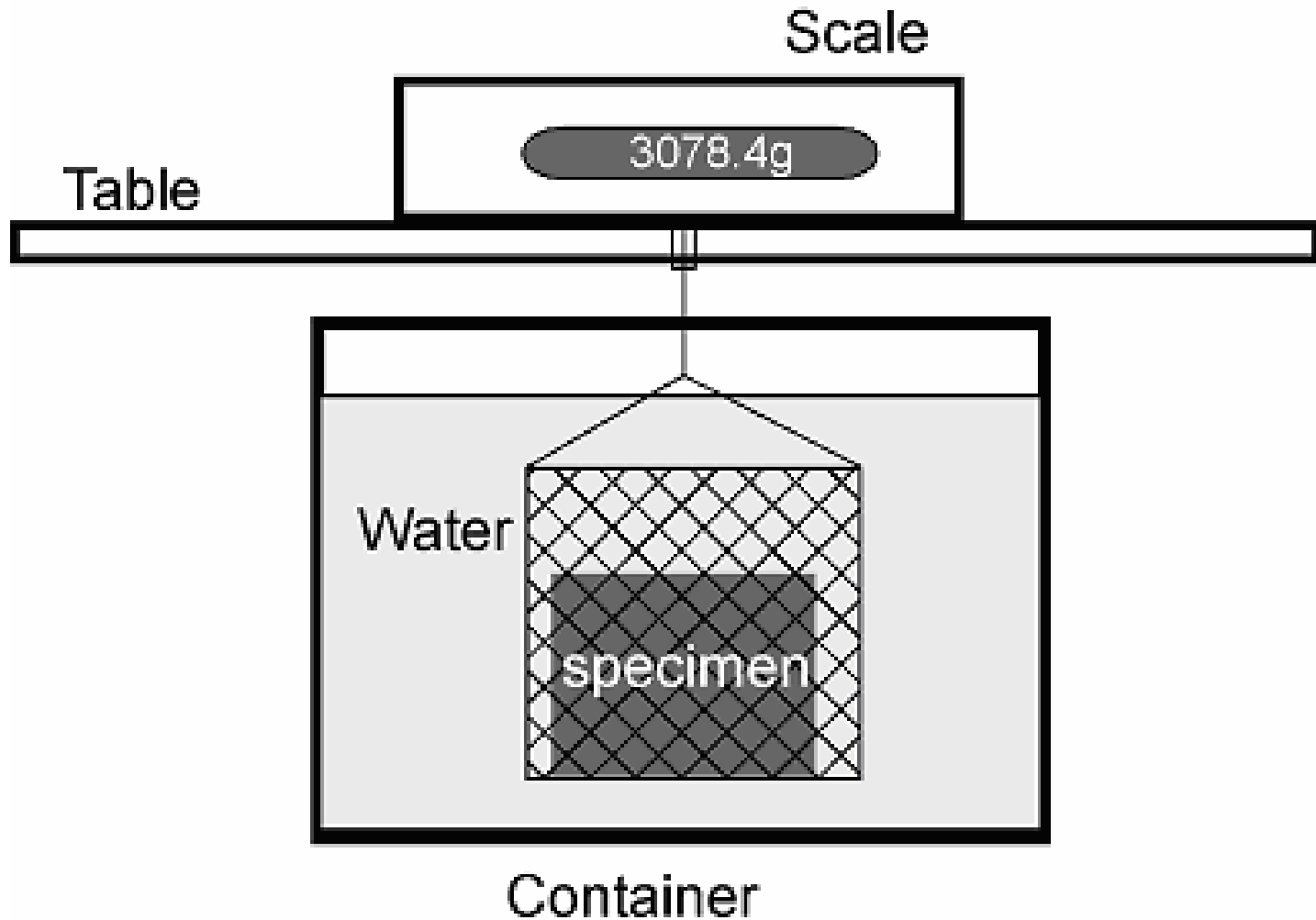
- Void ratio,  $e$  
$$e = \frac{V_v}{V_s} = \frac{243.9 \text{ cm}^3}{341.1 \text{ cm}^3} = 0.715$$

- Porosity,  $n$  
$$n = \frac{V_v}{V_T} = \frac{243.9 (cm^3)}{585.0 (cm^3)} \times 100 = 41.7\%$$

- Degree of saturation,  $S$  
$$S = \frac{V_w}{V_v} = \frac{109}{243.9} \times 100 = 44.7\%$$

- Dry unit weight,  $\gamma_d$  
$$\gamma_d = \frac{W_s}{V_T} = \frac{904}{585} = 1.55 \frac{g}{cm^3}$$

# Measurement of the submerged density (or unit weight)



**Now consider the submerged case, i.e., the two-phase system has been put into the water:**

$$M_w - \text{buoyancy} = e\rho_w - e\rho_w = 0 \quad \text{and}$$

$$M_s - \text{buoyancy} = G_s\rho_w - 1\rho_w = (G_s - 1)\rho_w$$

**Thus, the submerged density is**

$$\rho_{\text{submerg}} = \rho' = \frac{M'_t}{V_t} = \frac{M_w + M_s}{V_t} = \frac{0 + (G_s - 1)\rho_w}{1 + e} = \frac{(G_s - 1)\rho_w}{1 + e}$$

**In a two-phase system, i.e., if  $S=100\%$ , and we let  $V_s=1$ , we then have:**

$$V_t = 1 + e, \quad \text{since} \quad V_w = V_v = e$$

**Consider the not-submerged case, i.e., the two-phase system has been just put in the air:**

$$M_w = \rho_w V_w = e \rho_w \quad \text{and}$$

$$M_s = \rho_s V_s = G_s \rho_w V_s = G_s \rho_w$$

**Thus, the saturated density is**

$$\rho_{sat} = \frac{M_w + M_s}{V_t} = \frac{e \rho_w + G_s \rho_w}{1 + e} = \frac{\rho_w (e + G_s)}{1 + e}$$

**and the dry density is**

$$\rho_d = \frac{M_s}{V_t} = \frac{G_s \rho_w}{1 + e}$$

**Recall that the saturated density is**

$$\rho_{sat} = \frac{M_w + M_s}{V_t} = \frac{\rho_w (e + G_s)}{1 + e}$$

**If we do the following**

$$\begin{aligned} \rho_{sat} - \rho_w &= \frac{(e + G_s)\rho_w}{1 + e} - \rho_w = \frac{(e + G_s)\rho_w - (1 + e)\rho_w}{1 + e} \\ &= \frac{(e + G_s - 1 - e)\rho_w}{1 + e} = \frac{(G_s - 1)\rho_w}{1 + e} = \rho' \end{aligned}$$

$$i.e., \quad \rho' = \rho_{sat} - \rho_w$$

If you have got the submerged density  $\rho'$  and sure you know water density  $\rho_w$

$$\rho_{sat} = \frac{M_w + M_s}{V_t} = \frac{\rho_w (e + G_s)}{1 + e}$$

You can calculate the saturated density  $\rho_{sat}$ . If you know  $\rho_w$  then you can calculate the void ratio  $e$ . if you think you can know  $e$  from the dry density  $\rho_d$

$$\rho_d = \frac{G_s \rho_w}{1 + e}$$

You can also calculate the submerged density  $\rho'$  when the sample is not 100% saturated.

$$\rho' = \frac{(G_s - 1)\rho_w + e(S - 1)}{1 + e}$$

**Reading Assignment:**

**Das, Ch. 3**

**Homework:**

**3.4, 3.5**