SPARSE CHANNEL ESTIMATION FOR OFDM: OVER-COMPLETE DICTIONARIES AND SUPER-RESOLUTION

Christian R. Berger, Shengli Zhou, Weian Chen, and Peter Willett

Department of Electrical and Computer Engineering, University of Connecticut, Storrs, CT, USA

ABSTRACT

Wireless multipath channels can often be characterized as sparse, i.e., the number of significant paths is small even when the channel delay spread is large. This can be taken advantage of when estimating the unknown channel frequency response using pilot assisted modulation. Other work has largely focused on the greedy orthogonal matching pursuit (OMP) algorithm, using a dictionary based on an equivalent finite impulse response filter to model the channel. This is not necessarily realistic, as the physical nature of the channel is continuous in time, while the equivalent filter taps are based on baseband sampling. In this paper, we consider sparse channel estimation using a continuous time path-based channel model. This can be linked to the direction finding problem from the array processing literature and solved using the well-known root-MUSIC and ESPRIT algorithms, which have no formal time resolution. In addition, we show that a dictionary with finer time resolution considerably improves the performance of OMP and the related Basis Pursuit (BP) algorithm.

1. INTRODUCTION

Wireless transmission is typically characterized by multipath propagation, leading to fading (narrowband) or inter symbol interference (wideband) at the receiver. The effect of multipath can completely be characterized by an equivalent finite impulse response (FIR) filter model, where it is advantageous to know the average delay spread to choose an appropriate number of channel taps.

With the advent of compressed sensing or sparse estimation [1, 2], focus has come to the sparse nature of physical wireless multipath, as it can be leveraged to improve channel estimation to either work with less pilot symbols or achieve better noise suppression. In a nutshell, in a simple least-squares (LS) formulation the noise in the channel estimates depends on the ratio of unknown channel taps versus the number of observations. If there are fewer physical paths than channel taps and we would know the correct delays of each path, the least squares problem could be reformulated to estimate a smaller number of unknowns.

This work was supported the ONR YIP grant N00014-07-1-0805, the NSF grant ECCS-0725562, the NSF grant CNS-0721834, and the ONR grant N00014-07-1-0429.

In practice such "sparse" channels are encountered, but we do not know the delays of each path. Still, this problem can be exactly formulated by representing the received waveform using an over-complete dictionary, where each entry corresponds to a possible delay. By additionally enforcing a sparsity constraint, we try to explain the observed waveform with as few entries of the dictionary as possible. We focus on orthogonal frequency division multiplex (OFDM), as the channel effect can be represented in the most convenient form, but our results can be easily generalized to single carrier scenarios.

This formulation has been used in other work, see e.g. [3–6], but other work has largely focused on the greedy orthogonal matching pursuit (OMP) algorithm to take advantage of the sparse channel nature. The time resolution is determined by the equivalent FIR channel model and results are supported mostly only by computer simulations. The question of practical feasibility is open so far.

To approach the problems outlined above, we do the following: i) we adopt a continuous time path-based channel model, to take better advantage of the sparse multipath nature; ii) we link this formulation to the direction finding problem in the array processing literature and apply root-MUSIC and ESPRIT to solve it; iii) this prompts us to include overcomplete dictionaries for channel estimation based on a finer delay resolution; iv) we also include basis pursuit (BP), which has been reported as better at handling not exactly sparse problems; and v) we apply all algorithms to experimental data recorded for underwater acoustic communication.

We find in simulation studies, where the underlying path delays are generated from a continuous time distribution, that both array processing algorithms outperform the previously studied OMP approach when the dictionary is based on an FIR filter model using only baseband sampling. In contrast, when using a higher time resolution dictionary, both OMP and BP can match and even outperform root-MUSIC and ESPRIT. All four algorithms significantly outperform the LS estimator that does not account for channel sparsity.

In experimental results for underwater acoustic communications, where the data was recorded at the SPACE 2008 experiment, we find that the results match the simulation results for recorded channels that are truly sparse. On channels where path dispersion dominates, the model is less representative, and root-MUSIC and ESPRIT degrade in performance. Both BP and OMP perform uniformly well in both cases, but the performance gain over the LS esimator is more pronounced in the former case.

The rest of this paper is organized as follows. Section 2 introduces the OFDM system model, Section 3 explains how the problem can be reformulated as direction finding, Section 4 gives a short explanation of sparse channel estimation using OMP or BP, in Section 5 we present numerical and experimental results, and we conclude in Section 6.

2. SYSTEM MODEL

For data transmission, we use zero-padded (ZP) orthogonal frequency division multiplexing (OFDM). Let T denote the OFDM symbol duration and T_g the guard interval for the ZP. The total OFDM block duration is $T' = T + T_g$ and the subcarrier spacing is $1/T$. The kth subcarrier is at frequency

$$f_k = f_c + k/T, \quad k = -K/2, \dots, K/2-1, \tag{1}$$

where f_c is the carrier frequency and K subcarriers are used so that the bandwidth is $B = K/T$. Let $s[k]$ denote the information symbol to be transmitted on the kth subcarrier. The non-overlapping sets of active subcarriers $\mathcal{S}_A$ and null subcarriers $\mathcal{S}_N$ satisfy $\mathcal{S}_A \cup \mathcal{S}_N = \{-K/2, \dots, K/2-1\}$; the null subcarriers are used to facilitate Doppler compensation at the receiver [7]. The transmitted signal is given by

$$\tilde{x}(t) = \mathrm{Re}\left\{\left[\sum_{k\in\mathcal{S}_\mathrm{A}} s[k] e^{j2\pi\frac{k}{T}t} q(t)\right] e^{j2\pi f_c t}\right\}, \quad t \in [0, T+T_g], \tag{2}$$

where $q(t)$ describes the zero-padding operation, i.e.,

$$q(t) = \begin{cases} 1 & t \in [0,T], \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

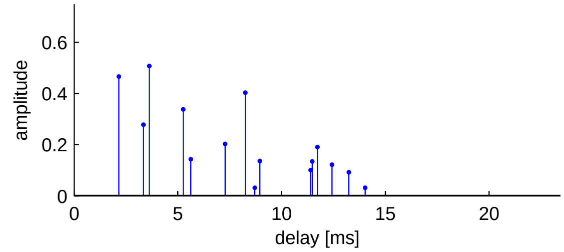
A stationary multipath channel can be described by

$$c(\tau) = \sum_{p=1}^{N_p} A_p \delta\left(\tau - \tau_p\right). \tag{4}$$

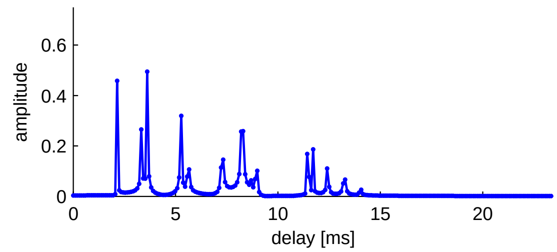
The number N_p describes the dominant physical multipath components and not the resultant number of baseband channel taps. As has been pointed out before even one physical multipath component can lead to many non-zero baseband channel taps, due to filter effects and the continuous time delay τ_p.

The received signal, after down conversion, is then

$$z(t) = \sum_{k\in\mathcal{S}_A} s[k] e^{j2\pi\frac{k}{T}t} \sum_{p=1}^{N_p} A_p e^{-j2\pi f_k \tau_p} q\left(t - \tau_p\right) + n(t), \tag{5}$$



(a) physical multipath



(b) equivalent baseband

Fig. 1. Even if physical multipath is sparse, conversion to equivalent baseband channel taps leads to a significantly larger number of non-zero components.

where $n(t)$ is an additive Gaussian noise process of power N_0. We purposely neglect carrier frequency offset (CFO) at this point, as it can be removed independently. The receiver applies an overlap-and-add (OLA) Fourier transform, as

$$z_m = \frac{1}{T}\int_0^{T'} z(t) e^{-j2\pi\frac{m}{T}t}\, dt \tag{6}$$

$$= H_m s[m] + n_m, \tag{7}$$

where n_m is the noise component and

$$H_m = \sum_{p=1}^{N_p} A_p e^{-j2\pi f_m \tau_p}. \tag{8}$$

Taking the inverse Fourier transform, we get the equivalent FIR baseband model,

$$h_l = \sum_{m=-K/2}^{K/2-1} H_m e^{j2\pi\frac{m}{K}l}, \tag{9}$$

$$= \sum_{p=1}^{N_p} A_p e^{-j2\pi f_c \tau_p} \frac{\sin\left(l - \frac{\tau_p}{T/K}\right)}{\sin\left(\frac{l}{K} - \frac{\tau_p}{T}\right)}. \tag{10}$$

We notice that due to the finite bandwidth of the signal, the discrete physical multipath components have been low-pass filtered. An example can be seen in Fig. 1; we notice that the number of non-zero equivalent baseband taps is much larger than the number of physical multipath components.

As discussed, due to the finite bandwidth, the resolution to estimate the τ_p is finite as well. On the other hand we only need to estimate them in a sufficient accuracy to reconstruct H_m for equalization. Without loss of generality we will assume that a quarter of the symbols are used as pilots, spaced evenly, denoted as $\mathcal{S}_P$, where obviously $\mathcal{S}_P \subset \mathcal{S}_A$. This will introduce an ambiguity in the time domain, but it can usually be assumed that the length of an OFDM symbol is larger than the delay spread, in this case $T \approx 4T_g$.

Writing (7) in vector form, we have

$$\mathbf{z} = \mathbf{D_s}\tilde{\mathbf{h}} + \mathbf{n}, \tag{11}$$

where $\mathbf{z}$, $\mathbf{s}$, $\tilde{\mathbf{h}}$, and $\mathbf{n}$ are column vectors containing the z_m, $s[m]$, H_m, and n_m for all m in $\mathcal{S}_P$; $\mathbf{D_s}$ is a diagonal matrix with the elements of vector $\mathbf{s}$ on its main diagonal. We rewrite the vector $\tilde{\mathbf{h}}$ as,

$$\tilde{\mathbf{h}} = \sum_{p=1}^{N_p} A_p \mathbf{w}(\tau_p), \tag{12}$$

where the vector $[\mathbf{w}(\tau_p)]_m = e^{-j2\pi f_m \tau_p}$ can be interpreted as either a column of a discrete Fourier transform matrix or a steering vector of a uniform linear array (ULA).

3. MUSIC AND ESPRIT

As mentioned before, the observations can be denoted as a superposition of steering vectors, see (12). This makes this formulation amenable to subspace based approaches, typically employed in array processing, see e.g. [9]. If we choose the pilots from a unit-amplitude constellation, we can easily reformulate our observations to

$$\hat{\mathbf{h}} = \mathbf{D_s}^H \mathbf{z} = \tilde{\mathbf{h}} + \tilde{\mathbf{n}}, \tag{13}$$

where $\tilde{\mathbf{n}}$ is still white Gaussian. Subspace based approaches have been used before in OFDM [10], but have usually assumed that many OFDM symbols are available to approximate a covariance matrix $\mathbf{R} = E\{\hat{\mathbf{h}}\hat{\mathbf{h}}^H\}$. Instead we will employ spatial smoothing (see [9]), which divides a large array into multiple observations of an equivalent smaller array. A tradeoff has to be found between array size – therefore resolution – and the number of observations availble to approximate $\mathbf{R}$.

Since we chose uniformly spaced pilots, specialized solutions for ULAs like root-MUSIC and ESPRIT can be used to find closed-form estimates for the τ_p. We choose to implement the unitary versions of root-MUIC and ESPRIT (see [9] for details). Our implementation uses overlapping equivalent arrays of length N_a, where each observations is shifted by one pilot symbol. When determining the signal subspace, we choose all eigen-values of $\mathbf{R}$ that are at least twice the noise variance N_0. Once the τ_p's are estimated, we formulate a reduced size least-squares problem to determine the A_p.

4. COMPRESSED SENSING

To formulate the compressed sensing problem, we need to use a large, but finite, dictionary. Accordingly we cover the interval of assumed channel delay spread $[0, T_g]$ with a set of uniformly spaced delays, where the spacing is determined using a multiple of the baseband symbol rate B as,

$$\tau_p \in \left\{0, \frac{1}{\alpha B}, \frac{2}{\alpha B}, \ldots, T_g\right\}, \tag{14}$$

which will lead to a dictionary of $N = T_g/(\alpha B) + 1$ entries. With this we construct a partial DFT matrix as

$$\mathbf{W} = \begin{bmatrix}\mathbf{w}(0) & \mathbf{w}\left(\frac{1}{\alpha T}\right) & \cdots & \mathbf{w}(T_g)\end{bmatrix}, \tag{15}$$

and rewrite (12) as

$$\tilde{\mathbf{h}} = \mathbf{W}\mathbf{a}, \tag{16}$$

where $\mathbf{a}$ contains the N possible A_n corresponding to the dictionary columns.

We now have arrived at the standard compressed sensing problem formulation,

$$\mathbf{z} = (\mathbf{D_s}\mathbf{W})\,\mathbf{a} + \mathbf{n}, \tag{17}$$

which is linear in the set of unknowns $\mathbf{a}$ and uses an over-complete dictionary $\mathbf{D_s}\mathbf{W}$ to explain the observations $\mathbf{z}$. The sparsity will be enforced on $\mathbf{a}$ to make the solution well defined.

Possible algorithms to solve this problem formulation are orthogonal matching pursuit (OMP), see [3–6], or basis pursuit (BP). While OMP iteratively picks elements of $\mathbf{a}$ and solves a constrained least-squares problem to optimally fit the observations, BP uses the l_1-norm to regularize the problem,

$$\min_{\mathbf{a}} |\mathbf{z} - (\mathbf{D_s}\mathbf{W})\,\mathbf{a}|^2 + \lambda|\mathbf{a}|_1. \tag{18}$$

Since generally we have to consider complex values for the elements of $\mathbf{a}$, the l_1-norm is defined as,

$$|\mathbf{a}|_1 = \sum_{n=1}^{N} \sqrt{|\mathrm{Re}\,(A_n)|^2 + |\mathrm{Im}\,(A_n)|^2}. \tag{19}$$

An efficient implementation for the complex valued version of BP has been suggested in the appendix of [8]. For the OMP implementation we terminate the algorithm if the residual fitting error or its improvement drops below a threshold based on the noise variance.

5. RESULTS

In this section we will present results based on numerical simulation and recorded experimental data. Both cases use the same OFDM system with the following specifications: carrier frequency $f_c = 13$ kHz, $K = 1024$ subcarriers, symbol

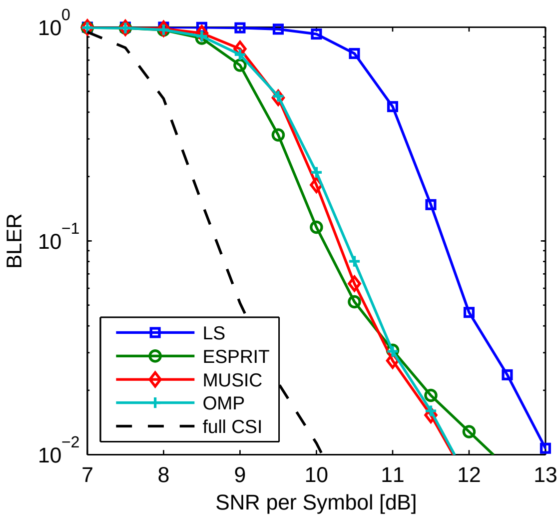


Fig. 2. The ESPRIT and root-MUSIC algorithms outperform the OMP implementation that uses an equivalent FIR channel model based on baseband sampling rate ($\alpha = 1$).

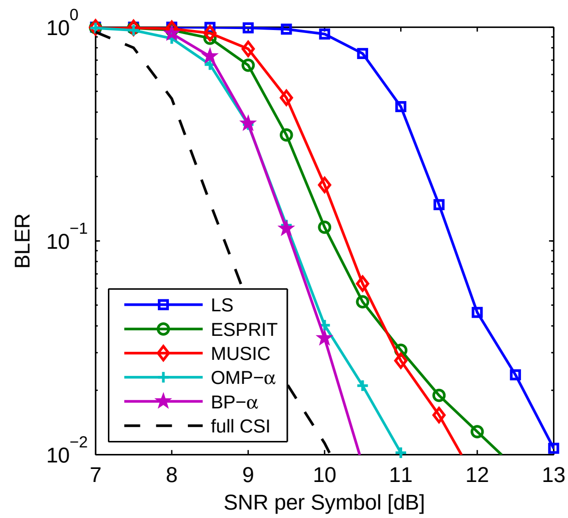


Fig. 3. Both OMP and BP improve significantly when working with a finer delay resolution dictionary ($\alpha = 2$ for BP, $\alpha = 4$ for OMP).

duration $T = 104.86$ ms, and the guard time is $T_g = 24.6$ ms. It follows that the bandwidth is $B = 9.7656$ kHz. There are $K/4 = 256$ pilot tones and 96 null subcarriers for edge protection and Doppler estimation, leaving 672 data subcarriers. The data within each OFDM symbol is encoded using a rate $1/2$ LDPC code, see [11], and 16-QAM, leading to 1344 bits per $T + T_g = 129.4$ ms or about 10.4 kbit/s. As a bottom line comparison, we will compare the sparse channel estimators against a common least-squares (LS) scheme using two FFT operations to interpolate the pilots (see [7] for details).

5.1. Simulation Study

For the simulation scenario we generate $N_p = 15$ discrete paths, the inter-arrival times are exponentially distributed with 1 ms mean (same parameters as in Fig. 1). The amplitudes of each path are Rayleigh distributed, with decreasing mean with increasing delay. As each OFDM symbol is encoded separately, we will use block-error-rate (BLER) as figure of merit. For each OFDM symbol, the channel is estimated based on the pilot tones. Each OFDM symbol experiences an independently generated channel.

We plot a comparison between the continuous resolution direction finding formulation and the LS and OMP algorithms based on the baseband sampling rate FIR channel model in Fig. 2. We see that all sparse channel estimation schemes significantly outperform the simple LS estimator, but that OMP is trailing root-MUSIC and ESPRIT. As reference we also include the performance based on full channel state information (CSI), only affected by the fading and additive noise. We next include BP, and use higher delay resolution dictionaries for both BP and OMP. The results in Fig. 3 show that both BP and OMP benefit from the larger dictionaries, outperforming root-MUSIC and ESPRIT.

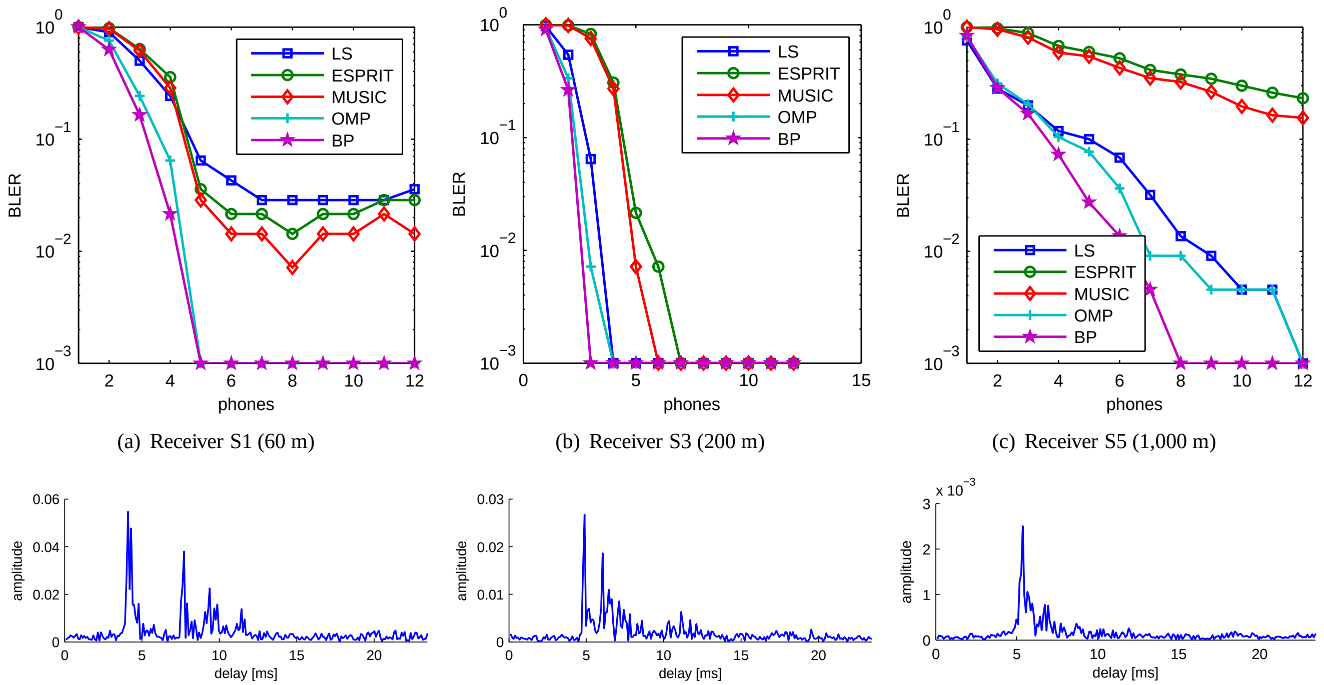
5.2. Experimental Data

The experimental data was recorded as part of the SPACE'08 experiment off the coast of Martha's Vineyard, MA, from Oct. 14 to Nov. 1, 2008. The water depth was about 15 meters leading to significant multipath. There are three different receivers, S1, S3, and S5, at 60 m, 200 m, and 1,000 m respectively. We focus on results recorded on Julian date 297.

We first note that the observed channels vary significantly with the receiver distance, see Fig. 4; while at a short distance there are well distinguishable paths, at the farthest distance the delay spread is reduced and most multipath is diffuse. At the shortest distance the performance matches the simulation results well, although root-MUSIC is now ahead of ESPRIT and BP outperforms OMP more clearly. At the other receivers, the performance of root-MUSIC and ESPRIT degrades, while for BP and OMP the gain over LS becomes smaller.

6. CONCLUSION

In this paper, we compared sparse channel estimation algorithms for OFDM. On the one hand Basis Pursuit and Orthogonal Matching Pursuit as implementations of compressed sensing, relying on over-complete, but finite, dictionaries; on the other hand subspace based super-resolution in form of root-MUSIC and ESPRIT. We find in simulation results – the channel is generated from sparse multipath components – that all algorithms outperform the commonly used least-squares



(a) Receiver S1 (60 m) (b) Receiver S3 (200 m) (c) Receiver S5 (1,000 m)

Fig. 4. Example channel impulse responses and BLER performance for the SPACE'08 experiment; we see that on truly sparse channels root-MUSIC and ESPRIT outperform the common LS approach, while BP and OMP are uniformly superior.

approach that does not take advantage of the sparse nature of the channel. To achieve this, both Basis Pursuit and Orthogonal Matching Pursuit need to choose dictionaries with a higher time resolution than the baseband rate, while root-MUSIC and ESPRIT do this naturally. When using experimentally recorded data, we find that root-MUSIC and ESPRIT outperform the LS estimator only on truly sparse channels, while BP and OMP are uniformly superior.

Acknowledgements

We thank Dr. James C. Preisig and his team for carrying out the SPACE'08 experiment.

7. REFERENCES

[1] R. Baraniuk, "Compressive sensing," *IEEE Signal Processing Magazine*, vol. 24, no. 4, pp. 118–121, Jul. 2007.

[2] E. J. Candes and M. B. Wakin, "An introduction to compressive sampling," *IEEE Signal Processing Magazine*, vol. 25, no. 2, pp. 21–30, Mar. 2008.

[3] C.-J. Wu and D. W. Lin, "A group matching pursuit algorithm for sparse channel estimation for OFDM transmission," in *Proc. of Intl. Conf. on Acoustics, Speech and Signal Proc.*, Toulouse, France, May 2006.

[4] W. Li and J. C. Preisig, "Estimation of rapidly time-varying sparse channels," *IEEE J. Ocean. Eng.*, vol. 32, no. 4, pp. 927–939, Oct. 2007.

[5] M. Sharp and A. Scaglione, "Application of sparse signal revocery to pilot-assisted channel estimation," in *Proc. of Intl. Conf. on Acoustics, Speech and Signal Proc.*, Las Vegas, NV, Apr. 2008.

[6] T. Kang and R. A. Iltis, "Iterative carrier frequency offset and channel estimation for underwater acoustic OFDM systems," *IEEE J. Select. Areas Commun.*, vol. 26, no. 9, pp. 1650–1661, Dec. 2008.

[7] B. Li, S. Zhou, M. Stojanovic, L. Freitag, and P. Willett, "Multicarrier communication over underwater acoustic channels with nonuniform Doppler shifts," *IEEE J. Ocean. Eng.*, vol. 33, no. 2, Apr. 2008.

[8] S.-J. Kim, K. Koh, M. Lustig, S. Boyd, and D. Gorinevsky, "An interior-point method for large-scale l_1-regularized least squares," *IEEE J. Select. Topics Signal Proc.*, vol. 1, no. 4, pp. 606–617.

[9] H. Van Trees, *Optimum Array Processing*, 1st ed., ser. Detection, Estimation, and Modulation Theory (Part IV). New York: John Wiley & Sons, Inc., 2002.

[10] C.-J. Wu and D. W. Lin, "Sparse channel estimation for OFDM transmission based on representative subspace fitting," in *Proc. of Vehicular Technology Conference*, Stockholm, Sweden, May 2005.

[11] J. Huang, S. Zhou, and P. Willett, "Nonbinary LDPC coding for multicarrier underwater acoustic communication," *IEEE J. Select. Areas Commun.*, vol. 26, no. 9, pp. 1684–1696, Dec. 2008.