A Polynomial Adaptive Controller for Nonlinearly Parameterized Systems

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Abstract—This paper extends the conventional linear-in-parameter adaptive control theory to general nonlinearly parameterized (NLP) systems. We propose a polynomial adaptive controller (PAC) which deals with piece-wise linearly parameterized systems as powerful as the traditional adaptive controller for linearly parameterized ones. Since most of the commonly encountered NLP systems can be piece-wise linearly approximated over the unknown parameter region through local linearization, the PAC serves as a general tool for them. Stability of the PAC with bounded disturbance and approximation error is also established. Controllers for several typical classes of NLP systems are provided to demonstrate the applications of the PAC.

I. INTRODUCTION

Adaptive control is an important area in the control theory which deals with the partial known system and the structured uncertainty is usually expressed by unknown parameters. Nonlinear parameterizations are inevitable in any realistic dynamic model of practical problems which possess intricate and complex dynamics. Friction dynamics [1], dynamics of magnetic bearing [2], and bio-chemical processes [3] are some examples where more than one physical parameter occurs nonlinearly in the underlying dynamic model. One of the assumptions that pervades almost all results in adaptive estimation and control is linearity in the unknown parameters. The structure of the dynamic system along with the choice of the estimator and controller have all been selected by and large to preserve a linear parametrization. The major stumbling block in the extension of the approaches that have been developed for linearly parameterized (LP) systems to NP systems is the inadequacy of a gradient algorithm for the control of a NP system since the underlying cost function is not available. In [4] and [5], for instance, a gradient algorithm is employed, but is shown to be sufficient only for stabilization and when the nonlinearity is concave. In [3], a special class of first-order NP systems is considered for which the gradient algorithm is shown to be adequate for stabilization. In [6] it was shown that an alternative to gradient algorithm can be derived for nonlinear systems, where the parametrization is concave or convex and the states are accessible. A globally stable adaptive controller was derived and shown that asymptotic tracking and regulation to within a desired precision $\epsilon$ can be carried out. In [7], it was shown that when the nonlinear parameterization is general, for a class of nonlinear systems, a min-max algorithm can be derived and shown to result in global stability. In [8]-[9], further extensions to this class have been derived. Despite the above results, to-date, an adaptive control algorithm for general nonlinearly parameterized (NLP) systems that is as powerful in its performance as the one for the LP systems is not available.

The stability of the linear adaptive controller is established via the Lyapunov analysis. The commonly chosen Lyapunov function is the quadratic function of the tracking error and parameter errors, while the associated adaptive law guarantees the time derivative of this Lyapunov function non-positive. In [10], a polynomial adaptive estimator was introduced which modifies the Lyapunov function and associated adaptive laws for nonlinearly parameterized functions, and allows parameter estimation under the standard conditions of persistent excitation. In this paper, a Polynomial Adaptive Controller (PAC) is proposed which uses a similar idea, and consists of higher-order polynomial of the parameter errors to construct the Lyapunov function instead of quadratic forms. New adaptive laws are constructed to guarantee the stability. Using the additional degrees of freedom, the PAC is capable of dealing with nonlinearly parameterized systems, by treating them as piece-wise linearly parameterized ones and treating the error between the piece-wise linear function and the original nonlinear function over the unknown parameter region as a disturbance. The resulting controller is shown to result in a closed-loop adaptive system with a bounded disturbance. We will also discuss the difference between the PAC and other comparable adaptive [7] and neural approaches.

The paper is organized as follows. In section II, the problem formulation is proposed and the approximation of a nonlinear function by a piece-wise linear one is discussed. The complete PAC for the system in section II is proposed in Section III. The stability and convergence properties of the PAC is stated in section IV. The situation with the existence of of bounded residual approximation error or disturbance is also considered. Section V provides simulation results of the PAC and section VI concludes the chapter.

II. PROBLEM FORMULATION

The system to be controlled is of the form

$$\dot{x}(t) = Ax(t) + bu(t) + bf(y(t), \omega), \quad x(0) = x_0,$$

$$y(t) = c^\top x(t), \quad t \geq 0, \quad \omega \in \Omega \subset \mathbb{R},$$

where $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the input and output respectively, $x \in \mathbb{R}^n$ is the state vector which is not measured directly, $\omega \in \mathbb{R}$ is an unknown scalar parameter.
which belongs to a given compact set \( \Omega = [\Omega_l, \Omega_u] \subset \mathbb{R}^c \), \( f \) is a given nonlinear map \( f : \mathbb{R}^n \times \Omega \to \mathbb{R}^n \), and \( A \in \mathbb{R}^{n \times n} \), \( b \in \mathbb{R}^n \), \( c \in \mathbb{R}^n \) are known, and \((A,b)\) is assumed to be controllable. The goal is to design control signal \( u(t) \) to make \( x(t) \) track some desired LTI reference model with time-varying reference input. For simplicity, we assume that \( A \) and \( b \) are known. If they are unknown, the standard linear-in-the-parameter methods in [11] can be added to the PAC proposed in this paper. We choose the desired LTI reference model to be:

\[
\dot{x}(t) = Ax(t) + bu(t), \quad y(t) = c^T x(t),
\]

where \( A \) is Hurwitz, and \( r(t) \) is a time-varying reference input chosen to yield a desired system output \( y(t) \). We make the following assumption of the system in (1):

**Assumption 1:** \( f(y, \omega) \) is second-order differentiable w.r.t. \( \omega \) and its second order derivative is bounded, i.e.

\[
\frac{\partial^2 f(y, \omega)}{\partial \omega^2} \leq q.
\]

In what follows, we demonstrate that Assumption 1 ensures \( f(y(t), \omega) \) to be approximated by a piecewise linear function over \( \Omega \) for any \( y \), i.e. there exist a constant \( d_{\text{max}} > 0 \), \( N \) intervals defined as \( \Omega_i = [\Omega_{li}, \Omega_{ui}] \) for \( i = 1, \ldots, N \), and known maps \( \psi_i(y(t)) : \Omega \to \mathbb{R} \), \( \psi_i(y(t)) : \Omega \to \mathbb{R} \), \( i = 1, \ldots, N \), such that

\[
\Omega \subseteq \bigcup_{i=1}^N \Omega_i, \quad |d(t)| \leq d_{\text{max}}, \quad \forall t \geq 0,
\]

\[
d(t) = f(y(t), \omega) - P(y(t), \omega),
\]

where \( P(y(t), \omega) \) is a piece-wise linear function:

\[
P(y(t), \omega) = \psi_i(y(t)) + \psi_i(y(t)) \omega - \bar{\omega}_i,
\]

\[
\bar{\omega}_i = (\Omega_{ui} + \Omega_{li})/2, \quad \text{if } \omega \in \Omega_i.
\]

We now divide \( \Omega \) uniformly into \( N \) regions:

\[
\Omega_i = [\Omega_{li}, \Omega_{ui}] = [\Omega_i + (i-1)(\Omega_{ui} - \Omega_{li})/N, \Omega_i + i(\Omega_{ui} - \Omega_{li})/N], \quad i = 1, \ldots, N,
\]

and let

\[
\psi_i(y) = f(y(t), \omega), \quad \bar{\omega} = (\Omega_{ui} + \Omega_{li})/2.
\]

\[
\psi_i(y, u) = \frac{f(y(t), \Omega_{ui}) - f(y(t), \Omega_{li})}{\Omega_{ui} - \Omega_{li}}.
\]

It can be proved easily, using Assumption 1, that

\[
|\mathcal{P}(y(t), \omega) - f(y(t), \omega)| \leq q(\Omega_{ui} - \Omega_{li})^3/(48N^3)
\]

where \( \mathcal{P}(y(t), \omega) \) are defined in (3), (4), (5), \( q \) is defined in (2), and hence \( d_{\text{max}} = \frac{q(\Omega_{li} - \Omega_{ui})^3}{48N^3} \). It is noted that we can obtain any desired \( d_{\text{max}} \), the upper-bound of the approximation error, by increasing \( N \).

We note that the piece-wise linear function \( \mathcal{P}(y(t), \omega) \) defined in (3) is nonlinear with respect to its arguments \( y \) and \( \omega \), and hence the adaptive controller design cannot be carried out using the currently available linear-in-parameter adaptive control architecture.

We now map the unknown parameter \( \omega \in \Omega \) into a new pair of unknown parameters \( [\theta, \zeta] \) as:

\[
\theta = \theta_i, \quad \zeta = \omega - (\Omega_{ui} + \Omega_{li})/2, \quad \text{if } \omega \in \Omega_i,
\]

\[
\Theta = \{ \theta_i, \ldots, \theta_{N}, \zeta \}, \quad \zeta \in [-\zeta_{m}, \zeta_{m}],
\]

\[
\theta_i = \Theta_i + (i-1)(\Theta_{ui} - \Theta_{li})/(N-1), \quad i = 1, \ldots, N,
\]

where \( \zeta_{m} = \max_{i=1,\ldots,N}(\Omega_{ui} - \Omega_{li})/2 \), and \( \Theta, \Theta_{ui} \) are arbitrary positive constants with \( \Theta_{ui} > \Theta_{li} \). We note here that \( \theta \) is a discrete unknown parameter indicating to which region the true unknown parameter \( \omega \) belongs, while \( \zeta \) is a continuous unknown one indicating the relative offset of \( \omega \) in that specific region. Since \( \omega \) is unknown, \( \theta \) and \( \zeta \) are known. Since \( \Omega \) is known, the compact sets \( \Theta \) and \( [-\zeta_{m}, \zeta_{m}] \) are known. Therefore, \( f(y(t), \omega) \) can be replaced by the sum of the piece-wise linear function \( \mathcal{P}(y(t), \omega) \), which is linear in every region \( \Omega_i \), and a bounded residual approximation error \( d(t) \).

Using the above discussions, we now reformulate the problem under consideration as follows: The problem is to find a stabilizing control input \( u \) in the system described by

\[
\dot{x}(t) = Ax(t) + bu(t) + b(\psi(y(t), \theta) + \psi(y(t), \theta) \zeta + d(t)),
\]

\[
y(t) = c^T x(t), \quad x(0) = x_0, \quad t \geq 0,
\]

\[
|d(t)| \leq d_{\text{max}}, \quad \theta, \zeta \in [-\zeta_{m}, \zeta_{m}],
\]

where \( u, x, y, A, b, c \) are the same as in (1), \( \theta \) (discrete) and \( \zeta \) are unknown parameters which belongs to known sets \( \Theta \) (discrete) and \( [-\zeta_{m}, \zeta_{m}] \) defined in (6), and \( \psi(y, \theta) : \mathbb{R}^n \to \mathbb{R}^n \), \( \psi(y, \theta) : \mathbb{R}^n \to \mathbb{R}^n \) are defined as: \( \psi(y, \theta) \) = \( \psi_i(y, \theta) \) if \( \theta = \theta_i \), and \( \psi(y, \theta) \) = \( \psi_i(y, \theta) \) if \( \theta = \theta_i \). We note here that while \( \psi(y, \theta) \) (discrete) and \( \psi(y, \theta) \) are unknown since \( \theta \) is unknown, while \( \psi_i(\cdot) \) and \( \psi_i(\cdot) \) are known for given \( \theta_i \).

### III. POLYNOMIAL ADAPTIVE CONTROLLER (PAC)

In section III, we propose a polynomial adaptive controller (PAC) for the system in (7). This PAC consists of three components: a companion adaptive system, an adaptive law and a control law.

**The companion model:** The companion model is:

\[
\dot{x}(t) = A\dot{x}(t) - k_c \dot{x}(t) + bu(t) + b\phi_0(\hat{\theta}(t), \hat{\zeta}(t), y(t)),
\]

\[
\dot{y}(t) = c^T \dot{x}(t), \quad \dot{y}(t) = \dot{y}(t) - y(t)
\]

where \( \dot{x} \in \mathbb{R}^n \), \( y \in \mathbb{R}^n \), \( k_c \) is such that \( A_c = A - k_c c^T \) is a Hurwitz matrix, and \( \hat{\theta}(t) = [\hat{\theta}_1(t), \ldots, \hat{\theta}_{N-1}(t)] \) and \( \hat{\zeta}(t) = [\hat{\zeta}_0(t), \ldots, \hat{\zeta}_{N-1}(t)] \) are auxiliary estimates.

**The adaptive controller:** The control input is chosen as

\[
u(t) = -k_c^T \dot{x}(t) - \phi_0(\hat{\theta}(t), \hat{\zeta}(t), y(t)) + r(t),
\]

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where $k_i$ is such that $A_m = A - bk_i^T$. The auxiliary estimates $\hat{\theta}$ and $\hat{\zeta}$ are updated according to the adaptive laws

$$
\dot{\hat{\theta}}(t) = \left\{ \begin{array}{ll}
0, & \text{if } \tilde{y}(t)\phi_i(\hat{\theta}(t), \hat{\zeta}(t), y(t)) > 0, \hat{\theta} \geq \Theta_u \\
0, & \text{if } \tilde{y}(t)\phi_i(\hat{\theta}(t), \hat{\zeta}(t), y(t)) < 0, \hat{\theta} \leq \Theta_l \\
\Gamma \tilde{y}(t)\phi_i(\hat{\theta}(t), \hat{\zeta}(t), y(t)), & \text{otherwise}
\end{array} \right.
$$

$$
\dot{\hat{\zeta}}(t) = \left\{ \begin{array}{ll}
0, & \text{if } \tilde{y}(t)\eta_i(y(t)) > 0 \text{ and } \hat{\zeta} \geq \zeta_m \\
0, & \text{if } \tilde{y}(t)\eta_i(y(t)) < 0 \text{ and } \hat{\zeta} \leq \zeta_m \\
\Gamma \tilde{y}(t)\eta_i(y(t)), & \text{otherwise}
\end{array} \right.
\forall i = 1, ..., N - 1, (10)

where $\Gamma \in \mathbb{R}^+$ is adaptive gain. The static maps $\phi_i(\hat{\theta}(t), \hat{\zeta}(t), y(t)), i = 0, ..., N - 1$ and $\eta_i(y(t)), i = 1, ..., N - 1$ in (8) and (10) are defined as follows:

$$
\Psi(y(t)) = [\psi(y(t), \theta_1), ..., \psi(y(t), \theta_N)]^T,
$$

$$
\eta_i(y(t)) = [\eta_0, ..., \eta_{N-1}] = -M^{-1}\Psi(y(t)),
$$

$$
\Phi(\hat{\theta}(t), \hat{\zeta}(t), y(t)) = [\phi_0, ..., \phi_{N-1}] - W(\hat{\theta}(t))^{-1}(\Phi(y(t)) - M\mu(\hat{\zeta}(t), y(t))),
$$

where $M$ is an $N$ by $N$ matrix with its $i$th row $j$th column element $M_{ij}$ as

$$
M_{ij} = \theta_j^{-1},
$$

and $W(\hat{\theta}(t))$ is an $N$ by $N$ matrix with its $i$th row $j$th column element $W_{ij}$ as

$$
W_{ij} = \left\{ \begin{array}{ll}
1, & j = 1, 1 \leq i \leq N \\
g_{j-1}(\hat{\theta}_{j-1}(t) - \theta_i), & 2 \leq j \leq N, 1 \leq i \leq N,
\end{array} \right.
$$

where the map $g_i(\cdot)$ is defined as

$$
g_i(x) = \left\{ \begin{array}{ll}
x^i, & \text{if } i \text{ is odd}; \\
x^{i-1} + k_ix^i, & \text{if } i \text{ is even}.
\end{array} \right.
\tag{14}
$$

$$
0 < k < 1/\Theta_u, \tag{15}
$$

and $\Theta_{\max}$ has been introduced in (6). We will show in the next section that the PAC controller specified by Eqs. (8)-(9) guarantees that the system in (7) can be stably controlled and shown to track $r(t)$ asymptotically.

\section{IV. Stability and Performance of the PAC}

Incorporating the control law (9) into the companion model (8), the closed loop of the companion model is:

$$
\dot{x}(t) = A_m \dot{x}(t) + b_m \dot{y} + br(t),
$$

$$
\dot{y}(t) = c^T \dot{x}(t). \tag{16}
$$

We divide our stability analysis into two parts. In the first part, discussed in section 4.1, we assume that $d(t)$ in (7) is identically zero, that is, the underlying nonlinear parameterization $f$ can be expressed precisely as a piecewise-linear parameterization. In the second part, we remove this assumption and show that the closed-loop system still leads to bounded solutions. This is presented in section 4.2.

\subsection{A. Case (i): $d(t) \equiv 0$}

We first establish a few useful lemmas. To avoid the singularity in the PAC, we have to show that the matrices $M$ and $W(\hat{\theta}(t))$ are non-singular.

\textbf{Lemma 1:} Matrix $M$ and $W(\hat{\theta}(t))$, $t \geq 0$, defined in (12) and (13), are full rank.

\textbf{Remark 1:} Although $W(\hat{\theta}(t))$ is a matrix dependent upon $k$, $\Theta$ and $\hat{\theta}(t), i = 1, ..., N - 1$, we can check easily that $\det(W(\hat{\theta}(t)))$ only depends on $k$ and $\Theta$ and not on the time-varying auxiliary estimates $\hat{\theta}(t), i = 1, ..., N - 1$. Since $k$ and $\Theta$ are pre-defined PAC parameters, $\det(W(\hat{\theta}(t)))$ is constant during the operation of the PAC, similar to $M$.

The following Lemma 2 states that the auxiliary estimates $\hat{\theta}(t), i = 1, ..., N - 1$ and $\hat{\zeta}(t), i = 0, ..., N - 1$ are bounded.

\textbf{Lemma 2:} For the system in (7) and the PAC in (8)-(11), we have

$$
\Theta_l < \hat{\theta}_i(t) < \Theta_u, \quad i = 1, ..., N - 1, \quad \forall t \geq 0; \tag{17}
$$

$$
-\zeta_m < \hat{\zeta}_i(t) < \zeta_m, \quad i = 0, ..., N - 1, \quad \forall t \geq 0. \tag{18}
$$

Using Lemmas 1 and 2 we now prove the stability of the overall adaptive system with the PAC. Our next step is to derive a Lyapunov function candidate, which we choose to be nonquadratic to accommodate the nonlinearity in the parameterization.

The adaptive laws in (10) can be rewritten as:

$$
\dot{\hat{\theta}}(t) = \Gamma \tilde{y} \phi_i(\hat{\theta}(t), \hat{\zeta}(t), y(t)) + w_i(t), \quad i = 1, ..., N - 1,
$$

$$
\dot{\hat{\zeta}}(t) = \Gamma \tilde{y} \eta_i(y(t)) + w_i(t), \quad i = 0, ..., N - 1, \tag{19}
$$

where

$$
\left\{ \begin{array}{ll}
v_i(t) = 0, & \text{if } \hat{\theta}_i(t) \in (\Theta_l, \Theta_u) \\
v_i(t) \leq 0, & \text{if } \hat{\theta}_i(t) \geq \Theta_u \\
v_i(t) \geq 0, & \text{if } \hat{\theta}_i(t) \leq \Theta_l, \quad i = 1, ..., N - 1 \\
 w_i(t) = 0, & \text{if } \hat{\zeta}_i(t) \in (-\zeta_m, \zeta_m) \\
w_i(t) \leq 0, & \text{if } \hat{\zeta}_i(t) \geq \zeta_m \\
w_i(t) \geq 0, & \text{if } \hat{\zeta}_i(t) \leq -\zeta_m, \quad i = 0, ..., N - 1,
\end{array} \right.
$$

and the other algebraic relationships in the adaptive laws are the same as in (10). Let the tracking error be $\tilde{x}(t) = \dot{x}(t) - x(t)$, it follows from (7) and (8) that the tracking error dynamics is reduced to

$$
\dot{\tilde{x}}(t) = A_m \tilde{x}(t) + b(\phi_0(\hat{\theta}(t), \hat{\zeta}(t), y(t)) - v(y(t), \theta) - \psi(y(t), \theta) \hat{\zeta}). \tag{21}
$$

Since $c^T(sI - A_m)^{-1}b$ is strictly positive real, it follows from the Kalman-Yakubovich Lemma that there exists $P = P^T > 0$ and $Q > 0$ such that $PB = c$ and $A_m^T P + PA_m = -Q$. Define $\tilde{\theta}(t) = \tilde{x}(t) - \theta$ for $i = 1, ..., N - 1$, and $\tilde{\zeta}(t) = \tilde{\zeta}(t) - \zeta$ for $i = 0, ..., N - 1$, we construct a Lyapunov function candidate

$$
V(t) = \tilde{x}^T(t) P \tilde{x}(t) + \sum_{i=1}^{N-1} \left( \frac{2p_i(\tilde{\theta}(t))}{\Gamma} + \frac{\theta_i^2 \tilde{\zeta}_i(t)^2}{\Gamma} \right), \tag{22}
$$

$$
p_i(x) = \left\{ \begin{array}{ll}
x^{1+i}/(i + 1), & \text{if } i \text{ is odd}; \\
kx^{1+i}/(i + 1) + x^i/i, & \text{if } i \text{ is even}.
\end{array} \right. \tag{23}
$$
Combining (14) and (23), it can be verified easily that
\[
\frac{dp_i(\hat{\theta}_i)}{d\theta_i} = g_i(\hat{\theta}_i), \forall i = 1, \ldots, N - 1.
\]

(24)

It follows from the definition of \( k \) in (15) that
\[
p_i(0) = 0, \\
g_i(x) = \frac{dp_i(x)}{dx} \leq 0, x \in [-\Theta_u, 0), \\
g_i(x) = \frac{dp_i(x)}{dx} \geq 0, x \in (0, \Theta_u],
\]
and hence the Lyapunov function candidate (22) is not globally positive definite but only locally positive definite in a set defined as:
\[
|\hat{\theta}_i| \leq \Theta_u, \quad \forall i = 1, \ldots, N.
\]

(26)

Lemma 2 ensures that \( |\hat{\theta}_i(t)| \leq \Theta_u \) for any \( t \geq 0 \), hence such a choice of a Lyapunov function candidate in (22) is reasonable.

The main stability theorem for case (i) is stated below:

**Theorem 1:** For the system in (7) and the PAC in (8)-(11), the following properties hold:

(i) \( \dot{V}(t) \leq -\tilde{x}^T(t)Q\tilde{x}(t), \)

(ii) \( \int_0^\infty \tilde{y}^2(t)dt \leq \beta V(0), \)

(iii) \( \lim_{t \to \infty} \tilde{y}(t) = 0, \)

(iv) \( \lim_{t \to \infty} (x(t) - x_m(t)) = 0, \)

where \( V \) is defined in (22), and
\[
\beta = (\max_{i=1,\ldots,N} c_i^2) / \lambda_{\min}(Q).
\]

(31)

**Remark 2:** The adaptive law in (10) is in fact
\[
\hat{\theta}_i(t) = \Gamma \hat{y}(t)\phi_i(\hat{\theta}(t), \hat{\zeta}(t), y(t)), \quad \forall i = 1, \ldots, N - 1,
\]
\[
\hat{\zeta}_i(t) = \Gamma \hat{y}(t)\eta_i(y(t)), \quad \forall i = 0, \ldots, N - 1,
\]
with additional projection algorithm as in [12], which keeps auxiliary estimates \( \hat{\theta}_i(t) \in [0, \Theta_u] \) and \( \hat{\zeta}_i(t) \in [-\zeta_m, \zeta_m] \) for any \( t \geq 0 \). Same as in [12], we can apply projection algorithm which leads to continuous \( \hat{\theta}_i(t) \) and \( \hat{\zeta}_i(t) \) while Lemma 1 still holds.

**B. Case (ii) \( d(t) \neq 0 \)**

In section IV-B, we consider the stability of the PAC with nonzero residual error due to a piece-wise linear approximation of the original nonlinear function \( f \) by \( P \), which leads to a disturbance \( d(t) \) in (7). The stability result in this case is stated in the following theorem:

**Theorem 2:** For the system in (7) and the PAC in (8), (10), (11), (9), the following properties hold:

(i) \( \dot{V}(t) \leq 0, \quad \text{for all} \quad V(t) \geq V_b, \)

where \( \beta \) is defined as in (31), \( \beta_1 = \lambda_{\max}(P) / \lambda_{\min}(Q) \), and
\[
V_b = 4\beta_1\beta d_{\max}^2 + 2 \sum_{i=1}^{N-1} p_i(\Theta_u) / \Gamma + 4 \sum_{i=0}^{N-1} (\Theta_u)^2(\zeta_m)^2 / \Gamma,
\]

(32)

\[
\lim_{t \to \infty} \int_0^T z(t)^2 dt \leq \beta^2 d_{\max}^2,
\]

where \( z(t) = \hat{y}(t) + \beta d(t) \).

Theorem 2 establishes that the Lyapunov function \( V(t) \) is ultimately bounded even with bounded disturbance. Since \( \tilde{x}^T(t)Q\tilde{x}(t) \leq V(t) \) for any \( t \geq 0 \), the tracking error \( \tilde{x}(t) \) is also ultimately bounded, same as output tracking error \( \hat{y}(t) \). With a nonzero disturbance, we cannot ensure the asymptotic convergence of \( \hat{y}(t) \) to zero anymore. Instead, (ii) in Theorem 2 characterizes the average property of this bounded error.

**Remark 3:** In the min-max algorithm [7], since the adaptive law still adopts a linear approximation over the unknown parameter region, the non-trivial approximation error between a nonlinear function and a linear one has to be considered explicitly. This is solved by adding some switching terms in the control signal which requires the knowledge of the direction of the control effectiveness. In the PAC, there is no such restrictions anymore since the adaptive law can provide a nonlinear matching over the unknown parameter region as shown in (42). As a general tool for nonlinear parameters, PAC can be extended into higher dimensional unknown parameters and other classes of NLP systems easily. For the sake of space, they are not elaborated here.

**V. SIMULATION RESULTS**

Consider system in (1) where \( f(y, \omega) = |y|^\omega, \omega = [1, 5] \), and
\[
A = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

The goal is to regulate or track the system output \( y(t) \) to given reference input \( r(t) = 1.5 \cos(0.4t) \). We choose \( N = 7 \) and divide \( \Omega \) evenly into \( N \) regions \( \Omega_i = [\Omega_{li}, \Omega_{ui}] \). Let \( \Theta_l = 0.1, \Theta_u = 2 \) and \( k_u = [6 \ 3]^T, k_c = [2.6 \ 0.8]^T, k_g = 2.4 \), which ensures \( A_c = A - b k_c^T \) and \( A_r = A - k_g^T \) Hurwitz and guarantees the existence of \( P = \begin{bmatrix} 0.5802 & -0.1604 \\ -0.1604 & 0.3208 \end{bmatrix} > 0 \), which satisfies \( Q = PA_c + A_c^T P < 0 \) and \( Pb = c \).

With the existence of bounded approximation error, output tracking error \( \hat{y}(t) - y(t) \) will not converge to zero, but bounded with its performance stated in Theorem 2. Hence, \( y(t) \) and \( \hat{y}(t) \) tracks the reference input \( r(t) \) with some bounded error as shown in Figure 1.
VI. CONCLUSION

The adaptive law in the PAC to deal with nonlinear parameters opens the door for the adaptive control of general nonlinearly parameterized systems. Stability of the PAC with bounded residual approximation error or disturbance is also established. For several classes of NLP systems, the control laws are given and the complete CPACs are constructed. We note that the PAC is not restricted to these classes and more applications are expected.

REFERENCES


APPENDIX

Proof of Lemma 1: It can be verified directly that $\mathcal{M}$ is a Vandermonde’s matrix and therefore full rank.

We denote the $i^{th}$ column of $\mathcal{W}(\hat{\theta}(t))$ as $\mathcal{W}_i$. First, let us consider $\mathcal{W}_2$ which is $[(\hat{\theta}_1 - \theta_1) \ldots (\hat{\theta}_j - \theta_j)]^\top$. Subtracting $\hat{\theta}_j \mathcal{W}_j$ from $\mathcal{W}_2$ and multiply by $-1$, we have $\mathcal{W}_2 = [\theta_1 \ldots \theta_i \ldots \theta_j]^\top$. We can continue this process through columns 3 to $N$. For $(j + 1)^{th}$ column, we assume that the columns 1 to $j$ have already been transformed into

$$[\mathcal{W}_1 \ldots \mathcal{W}_j] = \begin{bmatrix} 1 & \theta_1 & \ldots & \theta_j^{N-1} \\
\vdots & \vdots & & \vdots \\
1 & \theta_i & \ldots & \theta_i^{N-1} \\
1 & \theta_N & \ldots & \theta_N^{N-1} \end{bmatrix}.$$

Subtracting weighted $\mathcal{W}_j$ through $\mathcal{W}_i$ from $\mathcal{W}_{i+1}$, the new $(i+1)^{th}$ column can be scaled to be $\mathcal{W}_{i+1} = [\theta_1^i \ldots \theta_i^i \ldots \theta_N^i]^\top$. Repeating this process until the last column, the new matrix takes the form:

$$[\check{\mathcal{W}}_1 \ldots \check{\mathcal{W}}_N] = \begin{bmatrix} 1 & \theta_1 \ldots \theta_1^{N-1} \\
\vdots & \vdots & & \vdots \\
1 & \theta_i \ldots \theta_i^{N-1} \\
1 & \theta_N \ldots \theta_N^{N-1} \end{bmatrix}.$$

This matrix is Vandermonde’s matrix and therefore full rank. Since these linear transformations do not affect the rank of the matrix, $\mathcal{W}(\hat{\theta}(t))$ must be full rank too.

Proof of Lemma 2: For any $i = 1, \ldots, N - 1, (10)$ implies

$$\hat{\theta}_i(t) \leq 0, \text{ if } \hat{\theta}_i(t) \geq \Theta_a; \quad \hat{\theta}_i(t) \geq 0, \text{ if } \hat{\theta}_i(t) \leq \Theta_i,$$

and (17) is established. Using same method, (18) is proved.

Proof of Theorem 1: We prove (i) – (iii) respectively. Proof of (i) It follows from (19), (21), and (24) that the derivative of $V(t)$ in (22) is:

$$\dot{V}(t) = -\tilde{x}(t)Q\tilde{x}(t) + \frac{2}{\Gamma} \sum_{i=1}^{N-1} g_i(\hat{\theta}_i(t))\mathcal{W}_i(t)$$

$$+ \frac{2}{\Gamma} \sum_{i=0}^{N-1} \theta_i(\hat{\zeta}_i(t) - \zeta)w_i(t) + 2\tilde{g}(t)\left(\phi_0 - v(y(t), \theta) - \psi(y(t), \theta)\zeta + \sum_{i=1}^{N-1} g_i(\hat{\theta}_i(t))\phi_i(t) + \sum_{i=0}^{N-1} \theta_i(\hat{\zeta}_i(t) - \zeta)\eta_i(t)\right).$$

For a well-posed Lyapunov function, it follows from (25) that $g_i(\hat{\theta}_i(t)) \geq 0$, when $\theta_i(t) = \Theta_i$. Therefore, equation (20) implies that $g_i(\hat{\theta}_i(t))\mathcal{W}_i(t) \leq 0$, if $\hat{\theta}_i(t) = \Theta_i$.

Using the similar methodology, it can be shown that $g_i(\hat{\theta}_i(t))\mathcal{W}_i(t) \leq 0$, if $\hat{\theta}_i(t) = 0$.

It can be verified easily that

$$g_i(\hat{\theta}_i(t))\mathcal{W}_i(t) = 0, \quad \text{if } \hat{\theta}_i(t) \in (0, \Theta_i)$$

since $\mathcal{W}_i(t) = 0$ when $0 < \hat{\theta}_i(t) < \Theta_i$. Lemma 2 implies that $\hat{\theta}_i(t) \in [0, \Theta_i], \forall i = 1, \ldots, N - 1, \forall t \geq 0$.

Combining (34), (35), (36) and (37), we have

$$g_i(\hat{\theta}_i(t))\mathcal{W}_i(t) \leq 0, \quad \forall i = 1, \ldots, N - 1, \forall t \geq 0,$$

and hence

$$\sum_{i=1}^{N-1} g_i(\hat{\theta}_i(t))\mathcal{W}_i(t) \leq 0, \quad \forall t \geq 0.$$
In what follows, we show that the choice of $\Phi$ and $\eta$ ensures
\[
\phi_0(t) - v(y(t), \theta) - \psi(y(t), \theta) + \sum_{i=1}^{N-1} g_i(\tilde{\theta}_i(t))\phi_i(t) + \sum_{i=1}^{N-1} \theta_i^t(\zeta_i(t) - \zeta_i)\eta_i(t) = 0, \quad \forall t \geq 0.
\] (42)

The equation $\eta = -\mathcal{M}^{-1}\Psi$ in (11) implies $-\mathcal{M}\eta = \Psi$, and
\[
-\sum_{i=1}^{N-1} (\theta_j)^t \eta_j(y(t)) = \psi(y(t), \theta_j), \quad \forall j = 1, \ldots, N.
\] (43)

It follows from (43) that
\[
-\psi(y(t), \theta) - \zeta = -\sum_{i=1}^{N-1} \theta_i^t \eta_i(y(t))\zeta_i = 0,
\] (44)
for any $\theta \in \Theta, \zeta \in [-\zeta_m, \zeta_m]$. Similarly, $\Phi = \mathcal{W}^{-1}(\mathcal{Y} - \mathcal{M}\mu)$ in (11) implies that $\mathcal{W}\Phi = \mathcal{Y} - \mathcal{M}\mu$, which along with (13) ensures
\[
\sum_{i=1}^{N-1} g_i(\tilde{\theta}_i(t) - \theta_j)\phi_i + \phi_0 = v(y(t), \theta_j) - \sum_{i=1}^{N-1} (\theta_j)^t \eta_i(y(t))\tilde{\zeta}_i(t),
\] (45)
for any $j = 1, \ldots, N - 1$. It follows from (45) that
\[
\sum_{i=1}^{N-1} g_i(\tilde{\theta}_i(t) - \theta)\phi_i + \phi_0 = v(y(t), \theta) - \sum_{i=1}^{N-1} (\theta)^t \eta_i(y(t))\tilde{\zeta}_i(t),
\] (46)

Combining (44) and (46), we have
\[
\phi_0 - v(y, \theta) - \psi(y, \theta)\zeta + \sum_{i=1}^{N-1} g_i(\tilde{\theta}_i)\phi_i + \sum_{i=1}^{N-1} \theta_i^t (\zeta_i - \eta_i) = 0, \quad \forall \theta \in \Theta, \zeta \in [-\zeta_m, \zeta_m]
\] (47)
and hence (27) follows from (41).

Proof of (ii) It follows from (i) that $\int_0^\infty \dot{V}(t)dt \leq \int_0^\infty -\tilde{x}^T(t)Q\tilde{x}(t)dt$, and therefore
\[
V(\infty) - V(0) \leq \int_0^\infty -\tilde{x}^T(t)Q\tilde{x}(t)dt.
\] (48)

\[
\hat{x}^T(t)Q\hat{x}(t) \geq \lambda_{\min}(Q)||\tilde{x}(t)||^2, \quad \tilde{y}^T(t) \leq \lambda_{\max}(cc^T)||\tilde{x}(t)||^2
\]
implies that
\[
\hat{y}^T(t) \leq \beta \hat{x}^T(t)Q\tilde{x}(t),
\] (49)

which along with (48) leads to
\[
\int_0^\infty \frac{\hat{y}^T(t)}{\beta}dt \leq V(0) - V(\infty).
\] (50)

Since $V(t) \geq 0$ for any $t \geq 0$, equation (50) implies (28).

Proof of (iii) (ii) implies that $V(t)$ is bounded by $V(0)$ and hence $\hat{x}(t)$ is bounded and hence $\hat{y}(t)$ is bounded. Since the parameter estimates are bounded and $r(t)$ is bounded, we have that $\tilde{x}(t)$ is bounded, which further implies bounded $x(t)$. Since $\Phi(t)$ and its element $\phi_0(t)$ are calculated through non-singular static relationships using these bounded signals, $\phi_0(t)$ is bounded. Therefore, it follows from (9) and (21) that $\hat{x}$ is bounded. Thus, combing the fact $\hat{x} \in L_2$ from Lemma 1, Barbalat’s lemma ensures
\[
\lim_{t \to \infty} \hat{x}(t) = 0.
\] (51)

Since $\hat{y}(t) = c^T\hat{x}(t)$, (51) implies (29) and this proves (iii) in Theorem 1.

Proof of (iv) (ii) implies $\lim_{t \to \infty} \hat{x}(t) - x_m(t) = 0$, which along with (51) proves (iv) in Theorem 1.

Proof of Theorem 2: We prove (i) - (ii) respectively.

Proof of (i) It follows from Lemma 2 that
\[
\tilde{\theta}_i(t) - \theta \in [-\Theta_u, \Theta_u], \quad i = 1, \ldots, N - 1, \forall t \geq 0,
\]
\[
|\tilde{\zeta}_i(t) - \zeta| \leq 2\zeta_m, \quad i = 0, \ldots, N - 1, \forall t \geq 0.
\] (52)

The definition of $p(\cdot)$ in (23) implies that
\[
p_i(\Theta_u) = \max_{x \in [-\Theta_u, \Theta_u]} p_i(x), \quad i = 1, \ldots, N - 1.
\] (53)

Combining (52) and (53), we have
\[
2 \sum_{i=1}^{N-1} p_i(\tilde{\theta}_i(t) - \theta_i)/\Gamma + \sum_{i=0}^{N-1} \theta_i^t (\zeta_i - \zeta)^2 / \Gamma \leq 2 \sum_{i=1}^{N-1} p_i(\Theta_u)/\Gamma + 4 \sum_{i=0}^{N-1} (\theta_i)^4 (\zeta_m)^2 / \Gamma.
\] (54)

From the system equation (7) and the PAC, it follows that the derivative of $V(t)$ in (22) is given by
\[
\dot{V} = -\tilde{x}^T(t)Q\tilde{x}(t) + 2 \sum_{i=1}^{N-1} g_i(\tilde{\theta}_i)\nu_i/\Gamma + 2 \sum_{i=0}^{N-1} \theta_i^t (\zeta_i - \zeta) \nu_i/\Gamma + 2\tilde{y}(t)d(t) + 2\tilde{y}(t)d(t).
\]

It follows from (49) and the fact $|d(t)| \leq d_{max}$ that
\[
\dot{V} \leq -\tilde{x}^T(t)Q\tilde{x}(t) + 2\sqrt{T\beta\hat{y}^T(t)Q\tilde{x}(t)d_{max}}.
\] (55)

If $V(t) \geq V_0$, where $V_0$ is defined in (32), it follows from (22) and (54) that $\tilde{x}^T(t)P\tilde{x}(t) \geq 4\beta^2 d_{max}$. Since $\tilde{x}^T(t)Q\tilde{x}(t) \geq \tilde{x}^T(t)P\tilde{x}(t)$, we have $\dot{V}(t) \geq 4\beta^2 d_{max}$ and hence equation (55) implies that $V(t) \leq 0$, which proves (i) in Theorem 2.

Proof of (ii) It follows from (i) in Theorem 2 and equation (49) that $V(t) \leq -\frac{1}{\beta}\hat{y}^2(t) - 2\hat{y}(t)d(t)$, which can be rewritten as:
\[
\dot{V} \leq -\dot{y}^2(t) + \sqrt{3\beta d_{max}^2 + \beta d(t)^2}.
\]

Since $|d(t)| \leq d_{max}$, we have $\int_0^T \dot{V}(t)dt \leq -\frac{1}{\beta} \int_0^T (\hat{y}(t) + \beta d(t))^2 dt + \beta d_{max}^2 T$, and thus
\[
\frac{\beta(V(T) - V(0))}{T} \leq -\frac{1}{\beta} \int_0^T (\hat{y}(t) + \beta d(t))^2 dt / T + \frac{1}{\beta} d_{max}^2 T.
\] (56)

It follows from Lemma 2 that both $V(0)$ and $V(T)$ are finite, which implies that
\[
\lim_{T \to \infty} \frac{\beta(V(T) - V(0))}{T} = 0.
\] (57)

It follows from (56) and (57) that $\lim_{T \to \infty} \int_0^T (\hat{y}(t) + \beta d(t))^2 dt / T \leq \beta^2 d_{max}^2$, which proves (ii) in Theorem 2.