This paper presents an extension of the $L_1$ adaptive output feedback controller to systems of unknown relative degree in the presence of time-varying uncertainties without restricting the rate of their variation. As compared to earlier results in this direction, a new piece-wise continuous adaptive law is introduced along with the low-pass filtered control signal that allows for achieving arbitrarily close tracking of the input and the output signals of the reference system, the transfer function of which is not required to be strictly positive real (SPR). Stability of this reference system is proved using small-gain type argument. The performance bounds between the closed-loop reference system and the closed-loop $L_1$ adaptive system can be rendered arbitrarily small by reducing the step size of integration. Simulations verify the theoretical findings.

I. Introduction

This paper extends the results of Ref.\textsuperscript{1} by considering reference systems that do not verify the SPR condition for their input-output transfer function. Similar to Ref.\textsuperscript{1}, the $L_\infty$-norms of both input/output error signals between the closed-loop adaptive system and the reference system can be rendered arbitrarily small by reducing the step-size of integration. The key difference from the results in Ref.\textsuperscript{1} is the new piece-wise continuous adaptive law. The adaptive control is defined as output of a low-pass filter, resulting in a continuous signal despite the discontinuity of the adaptive law.

We notice that adaptive algorithms achieving arbitrarily improved transient performance for system’s output were reported in Refs.\textsuperscript{2–16} References\textsuperscript{1,17} presented the opportunity to regulate also the performance bound for system’s input signal, by rendering it arbitrarily close to the corresponding signal of a bounded reference LTI system. This paper presents the adaptive output feedback counterpart of the results in Refs.\textsuperscript{1,17} without enforcing the SPR condition on the input/output transfer function of the desired reference system, which typically appears in conventional adaptive output feedback schemes. Application of the methodology presented in this paper can be found in Refs.\textsuperscript{18,19}

The paper is organized as follows. Section II gives the problem formulation. In Section III, the closed-loop reference system is introduced. In Section IV, some preliminary results are developed towards the definition of the $L_1$ adaptive controller. In Section V, the novel $L_1$ adaptive control architecture is presented. Stability and uniform performance bounds are presented in Section VI. In Section VII, simulation results are presented and two aerospace applications are discussed, while Section VIII concludes the paper. The small-gain theorem and some basic definitions from linear systems theory used throughout the paper are given in Appendix. Unless otherwise mentioned, $\| \cdot \|$ will be used for the 2-norm of the vector.

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II. Problem Formulation

Consider the following single–input single–output (SISO) system:

\[ y(s) = A(s)(u(s) + d(s)) , \quad y(0) = 0 , \]  

where \( u(t) \in \mathbb{R} \) is the input, \( y(t) \in \mathbb{R} \) is the system output, \( A(s) \) is a strictly proper unknown transfer function of unknown relative degree \( n_r \), for which only a known lower bound \( 1 < d_r \leq n_r \) is available, \( d(s) \) is the Laplace transform of the time-varying uncertainties and disturbances \( d(t) = f(t, y(t)) \), while \( f \) is an unknown map, subject to the following assumptions.

Assumption 1 There exist constants \( L > 0 \) and \( L_0 > 0 \) such that the following inequalities hold uniformly in \( t \geq 0 \):

\[ |f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2| , \quad |f(t, y)| \leq L|y| + L_0 . \]

Assumption 2 There exist constants \( L_1 > 0 \), \( L_2 > 0 \) and \( L_3 > 0 \) such that for all \( t \geq 0 \):

\[ |\dot{d}(t)| \leq L_1|\dot{y}(t)| + L_2|y(t)| + L_3 , \]

where the numbers \( L, L_0, L_1, L_2, L_3 \) can be arbitrarily large. Let \( r(t) \) be a given bounded continuous reference input signal. The control objective is to design an adaptive output feedback controller \( u(t) \) such that the system output \( y(t) \) tracks the reference input \( r(t) \) following a desired reference model, i.e. \( y(s) \approx M(s)r(s) \), where \( M(s) \) is a minimum-phase stable transfer function of relative degree \( d_r > 1 \). We rewrite the system in (1) as:

\[ y(s) = M(s)(u(s) + \sigma(s)) , \]  

\[ \sigma(s) = \left( (A(s) - M(s))u(s) + A(s)d(s) \right) / M(s) . \]

Let \((A_m, b_m, c_m)\) be the minimal realization of \( M(s) \), i.e. it is controllable and observable and \( A_m \) is Hurwitz. The system in (3) can be rewritten as:

\[ \dot{x}(t) = A_m x(t) + b_m (u(t) + \sigma(t)) , \]

\[ y(t) = c_m^T x(t) , \quad x(0) = x_0 = 0 . \]

III. Closed-loop Reference System

Consider the following closed-loop reference system:

\[ y_{\text{ref}}(s) = M(s)(u_{\text{ref}}(s) + \sigma_{\text{ref}}(s)) \]  

\[ \sigma_{\text{ref}}(s) = \frac{(A(s) - M(s))u_{\text{ref}}(s) + A(s)d_{\text{ref}}(s)}{M(s)} \]  

\[ u_{\text{ref}}(s) = C(s)(r(s) - \sigma_{\text{ref}}(s)) , \]

where \( d_{\text{ref}}(t) = f(t, y_{\text{ref}}(t)) \), and \( C(s) \) is a strictly proper system of relative order \( d_r \), with its DC gain \( C(0) = 1 \). We notice that there is no algebraic loop involved in the definition of \( \sigma(s) \), \( u(s) \) and \( \sigma_{\text{ref}}(s) \), \( u_{\text{ref}}(s) \).

For the proof of stability of the reference system in (6)-(8), the selection of \( C(s) \) and \( M(s) \) must ensure that

\[ H(s) = A(s)M(s)/(C(s)A(s) + (1 - C(s))M(s)) \]

is stable and

\[ \|G(s)\|_{\mathcal{L}_1} L < 1 , \]

where \( G(s) = H(s)(1 - C(s)) \). This in its turn restricts the class of systems \( A(s) \) in (1), for which the proposed approach in this paper can achieve stabilization. However, as discussed later in Remark 3, the class of such systems is not empty. Letting

\[ A(s) = \frac{A_n(s)}{A_d(s)} , \quad C(s) = \frac{C_n(s)}{C_d(s)} , \quad M(s) = \frac{M_n(s)}{M_d(s)} . \]
it follows from (9) that
\[
H(s) = \frac{C_d(s)M_n(s)A_n(s)}{H_d(s)},
\]
where
\[
H_d(s) = C_n(s)A_n(s)M_d(s) + M_n(s)A_d(s)(C_d(s) - C_n(s)).
\]
A strictly proper \( C(s) \) implies that the order of \( C_d(s) - C_n(s) \) and \( C_d(s) \) is the same. Since the order of \( A_d(s) \) is higher than that of \( A_n(s) \), the transfer function \( H(s) \) is strictly proper.

The next Lemma establishes the stability of the closed-loop system in (6)-(8).

**Lemma 1** If \( C(s) \) and \( M(s) \) verify the condition in (10), the closed-loop reference system in (6)-(8) is stable.

**Proof.** It follows from (7)-(8) that
\[
u_{ref}(s) = C(s)r(s) - C(s)((A(s) - M(s))u_{ref}(s) + A(s)d_{ref}(s))/M(s),
\]
and hence
\[
u_{ref}(s) = \frac{C(s)M(s)r(s) - C(s)A(s)d_{ref}(s)}{C(s)A(s) + (1 - C(s))M(s)}.
\]
From (6)-(7) one can derive \( y_{ref}(s) = A(s)(u_{ref}(s) + d_{ref}(s)) \), which upon substitution of (14), leads to
\[
y_{ref}(s) = H(s)\left(C(s)r(s) + (1 - C(s))d_{ref}(s)\right).
\]
Since \( H(s) \) is strictly proper and stable, \( G(s) \) is also strictly proper and stable and therefore \( \|y_{ref}\|_{\infty} \leq \|H(s)C(s)\|_{\infty}1\|r\|_{\infty} + \|G(s)\|_{\infty}1\|y_{ref}\|_{\infty} + L_0 \). Using the condition in (10), one can write
\[
\|y_{ref}\|_{\infty} \leq \rho,
\]
where
\[
\rho = \frac{\|H(s)C(s)\|_{\infty}1\|r\|_{\infty} + \|G(s)\|_{\infty}1L_0}{1 - \|G(s)\|_{\infty}L} < \infty.
\]
Hence, \( \|y_{ref}\|_{\infty} \) is bounded, and the proof is complete. \( \square \)

### IV. Preliminaries for the Main Result

Let
\[
H_0(s) = \frac{A(s)}{(C(s)A(s) + (1 - C(s))M(s))}
\]
\[
H_1(s) = \frac{(A(s) - M(s))C(s)}{C(s)A(s) + (1 - C(s))M(s)}
\]
\[
H_2(s) = \frac{M(s)C(s)}{(C(s)A(s) + (1 - C(s))M(s))}
\]
\[
H_3(s) = \frac{H(s)C(s)}{M(s)}.
\]
Using the expressions from (11) and (13), \( H_0(s) \) and \( H_1(s) \) can be rewritten as:
\[
H_0(s) = C_d(s)A_n(s)M_d(s)/H_d(s),
\]
\[
H_1(s) = \left( C_n(s)A_n(s)M_d(s) - C_n(s)A_d(s)M_n(s) \right)/H_d(s).
\]
Since the degree of \( C_d(s) - C_n(s) \) is larger of \( C_n(s) \) by \( d_r \), the degree of \( M_n(s)A_d(s)(C_d(s) - C_n(s)) \) is larger of \( C_n(s)A_d(s)M_n(s) \) by \( d_r \). Since the degree of \( A_d(s) \) is larger of \( A_n(s) \) by \( \geq d_r \), while the degree of \( M_n(s) \) is larger of \( M_d(s) \) by \( d_r \), the degree of \( M_n(s)A_d(s)(C_d(s) - C_n(s)) \) is larger than that of \( C_n(s)A_n(s)M_d(s) \). Therefore, \( H_1(s) \) is strictly proper with relative degree \( d_r \). We notice from (12) and (22) that \( H_1(s) \) has the same denominator as \( H(s) \), and therefore it follows from (10) that \( H_1(s) \) is stable. Using similar arguments, it can be verified that \( H_0(s) \) is proper and stable. Similarly, \( H_3(s) \) is strictly proper and stable.
Further, let
\[ \Delta = \|H_1(s)\|_{\mathcal{L}_1} r \|\mathcal{L}_\infty + \|H_0(s)\|_{\mathcal{L}_1}(L\rho + L_0) + \bar{\gamma} \left( \|H_1(s)/M(s)\|_{\mathcal{L}_1} + L\|H_0(s)\|_{\mathcal{L}_1} \frac{\|H_3(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1} L} \right), \]
where \( \bar{\gamma} > 0 \) is an arbitrary constant. Since both \( H_1(s) \) and \( M(s) \) are stable and strictly proper with relative degree \( d_r, \) and \( M(s) \) is minimum-phase, \( H_1(s)/M(s) \) is stable and proper. Hence, \( \|H_1(s)/M(s)\|_{\mathcal{L}_1} \) exists. Therefore \( \Delta \) is bounded.

Further, since \( A_m \) is Hurwitz, there exists \( P = P^T > 0 \) that satisfies the algebraic Lyapunov equation \( A_m^T P + PA_m = -Q, \) \( Q > 0. \) From the properties of \( P \) it follows that there exits non-singular \( \sqrt{P} \) such that \( P = (\sqrt{P})^T \sqrt{P}. \)

Given the vector \( e_m(\sqrt{P})^{-1}, \) let \( D \) be a \((n-1) \times n\) matrix that contains the null space of \( e_m(\sqrt{P})^{-1}, \) i.e.
\[ D(e_m(\sqrt{P})^{-1})^T = 0, \] (23)
and further let \( \Lambda = \begin{bmatrix} c_m^T \\ D\sqrt{P} \end{bmatrix}. \)

**Lemma 2** For any \( \xi = \begin{bmatrix} y \\ z \end{bmatrix} \in \mathbb{R}^n, \) where \( y \in \mathbb{R} \) and \( z \in \mathbb{R}^{n-1}, \) there exist \( p_1 > 0 \) and positive definite \( P_2 \in \mathbb{R}^{(n-1) \times (n-1)} \) such that \( \xi^T(\Lambda^{-1})^T P \Lambda^{-1} \xi = p_1 y^2 + z^T P_2 z. \)

**Proof.** Using \( P = (\sqrt{P})^T \sqrt{P}, \) one can write
\[ \xi^T(\Lambda^{-1})^T P \Lambda^{-1} \xi = \xi^T(\sqrt{P} \Lambda^{-1})^T(\sqrt{P} \Lambda^{-1}) \xi. \] (24)

We notice that
\[ \Lambda(\sqrt{P})^{-1} = \begin{bmatrix} c_m^T(\sqrt{P})^{-1} \\ D \end{bmatrix}. \]

Letting \( q_1 = (c_m^T(\sqrt{P})^{-1})^T c_m(\sqrt{P})^{-1}, Q_2 = DD^T, \) it follows from (23) that
\[ (\Lambda(\sqrt{P})^{-1})(\Lambda(\sqrt{P})^{-1})^T = \begin{bmatrix} q_1 & 0 \\ 0 & Q_2 \end{bmatrix}. \] (25)

Non-singularity of \( \Lambda \) and \( \sqrt{P} \) implies that \( (\Lambda(\sqrt{P})^{-1})(\Lambda(\sqrt{P})^{-1})^T \) is non-singular, and therefore \( Q_2 \) is also non-singular. Hence,
\[ (\sqrt{P} \Lambda^{-1})^T (\sqrt{P} \Lambda^{-1}) = (\Lambda(\sqrt{P})^{-1})(\Lambda(\sqrt{P})^{-1})^{-1} = (\Lambda(\sqrt{P})^{-1})^{-T} (\sqrt{P} \Lambda^{-1}) = \begin{bmatrix} q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{bmatrix}. \] (26)

Letting \( p_1 = q_1^{-1} \) and \( P_2 = Q_2^{-1}, \) it follows from (24) that
\[ \xi^T(\sqrt{P} \Lambda^{-1})^T(\sqrt{P} \Lambda^{-1}) \xi = p_1 y^2 + z^T P_2 z, \] (27)
which completes the proof. \( \Box \)

Let \( T \) be any positive constant and \( 1_1 \in \mathbb{R}^n \) be the basis vector with first element 1 and all other elements zero. Let \( \phi(T) \in \mathbb{R}^{n-1} \) be a vector which consists of 2 to \( n \) elements of \( 1_1^T e^{\Lambda A_m T} \) and let
\[ \kappa(T) = \int_0^T [1_1^T e^{\Lambda A_m (T-\tau)} \Lambda b_m] d\tau. \] (28)

Further, let
\[ \zeta(T) = \|\phi(T)\| \sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} + \kappa(T) \Delta, \] (29)
\[ \alpha = \lambda_{\max}(\Lambda^{-T} P \Lambda^{-1}) \left( \frac{2\Delta \|\Lambda^{-T} P b_m\|}{\lambda_{\min}(\Lambda^{-1} Q \Lambda^{-1})} \right)^2. \]
Finally, let \( \alpha \) which proves (40). Boundedness of 
\[
\lim_{T \to 0} \eta_3(t) = 0
\]
where \( \eta_1(t) \in \mathbb{R} \) and \( \eta_2(t) \in \mathbb{R}^{n-1} \) contain the first and 2 to \( n \) elements of the row vector \( \mathbf{1}_1 e^{A\mathbf{1}_m A^{-1}T} \), respectively, we introduce the following definitions

\[
\beta_1(T) = \max_{t \in [0, T]} |\eta_1(t)| ,
\]
\[
\beta_2(T) = \max_{t \in [0, T]} \| \eta_2(t) \| .
\]

Further, let \( \Phi(T) \) be the \( n \times n \) matrix

\[
\Phi(T) = \int_0^T e^{A\Lambda A^{-1} (T-\tau)} \Lambda d\tau ,
\]
and let

\[
\beta_3(T) = \max_{t \in [0, T]} \eta_3(t) ,
\]
\[
\beta_4(T) = \max_{t \in [0, T]} \eta_4(t) ,
\]
where

\[
\eta_3(t) = \int_0^t [\mathbf{1}_1 e^{A\Lambda A^{-1} (t-\tau)} \Lambda \Phi^{-1}(T) e^{A\Lambda A^{-1}T} \mathbf{1}_1] d\tau ,
\]
\[
\eta_4(t) = \int_0^t [\mathbf{1}_1 e^{A\Lambda A^{-1} (t-\tau)} \Lambda \nu_m] d\tau .
\]

Finally, let

\[
\gamma_0(T) = \beta_1(T) \varsigma(T) + \beta_2(T) \sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} + \beta_3(T) \varsigma(T) + \beta_4(T) \Delta .
\]

**Lemma 3**

\[
\lim_{T \to 0} \gamma_0(T) = 0 .
\]

**Proof.** Notice that since \( \beta_1(T), \beta_3(T), \alpha \text{ and } \Delta \) are bounded, it is sufficient to prove that

\[
\lim_{T \to 0} \varsigma(T) = 0 ,
\]
\[
\lim_{T \to 0} \beta_2(T) = 0 ,
\]
\[
\lim_{T \to 0} \beta_4(T) = 0 .
\]

Since \( \lim_{T \to 0} [\mathbf{1}_1 e^{A\Lambda A^{-1}T} = \mathbf{1}_1^T] \), then \( \lim_{T \to 0} \phi(T) = 0_{n-1} \), which implies \( \lim_{T \to 0} \| \phi(T) \| = 0 \). Further, it follows from the definition of \( \kappa(T) \) in (28) that \( \lim_{T \to 0} \kappa(T) = 0 \). Since \( \alpha \text{ and } \Delta \) are bounded, then \( \lim_{T \to 0} \varsigma(T) = 0 \), which proves (38). Since \( \eta_2(t) \) is continuous, it follows from (32) that \( \lim_{T \to 0} \beta_2(T) = \lim_{t \to 0} [\lim_{T \to 0} \| \eta_2(t) \| ] \). Hence, the upper bound in (39) follows from \( \lim_{t \to 0} [\mathbf{1}_1 e^{A\Lambda A^{-1}t} = \mathbf{1}_1^T] \). Thus, \( \lim_{t \to 0} \| \eta_2(t) \| = 0 \), which proves (39). Similarly \( \lim_{t \to 0} \beta_4(T) = \lim_{t \to 0} [\lim_{T \to 0} \| \eta_4(t) \| = 0 \), which proves (40). Boundedness of \( \alpha, \beta_3(T) \text{ and } \Delta \) implies that \( \lim_{T \to 0} [\beta_1(T) \varsigma(T) + \beta_2(T) \sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} + \beta_3(T) \varsigma(T) + \beta_4(T) \Delta ] = 0 \), which completes the proof. \( \square \)

**V. \( L_1 \) Adaptive Output Feedback Controller**

We consider the following state predictor (or passive identifier):

\[
\dot{x}(t) = A_m \dot{x}(t) + b_m u(t) + \bar{\sigma}(t) \quad (41)
\]
\[
\dot{\gamma}(t) = c_m^T \dot{x}(t), \quad \dot{x}(0) = x_0 = 0 .
\]
where \( \hat{\sigma}(t) \in \mathbb{R}^n \) is the vector of adaptive parameters. Notice that while \( \sigma(t) \in \mathbb{R} \) in (5), i.e. the unknown disturbance is matched, the uncertainty estimation in (41) is \( \hat{\sigma}(t) \in \mathbb{R}^n \), i.e. the estimation of it is unmatched. This is the key step of the solution and the subsequent analysis.

Taking the Laplace transform of (41), gives

\[
y(s) = M(s)u(s) + C^T_m(sI - A_m)^{-1}\hat{\sigma}(s) .
\]

(42)

Letting \( \hat{y}(t) = \hat{y}(t) - y(t) \), the update law for \( \hat{\sigma}(t) \) is given by

\[
\hat{\sigma}(t) = \hat{\sigma}(iT), \quad t \in [iT, (i+1)T)
\]

\[
\hat{\sigma}(iT) = -\Phi^{-1}(T)\mu(iT), \quad i = 0, 1, 2, \ldots ,
\]

(43)

(44)

where \( \Phi(T) \) is defined in (33), and

\[
\mu(iT) = e^{AA_m}A^{-1}T 1_i \hat{y}(iT), \quad i = 0, 1, 2, 3, \ldots .
\]

(45)

The control signal is the output of the low-pass filter:

\[
u(s) = C(s)r(s) - \frac{C(s)}{c^T_m(sI - A_m)^{-1}b_m}c^T_m(sI - A_m)^{-1}\hat{\sigma}(s) .
\]

(46)

The complete \( L_1 \) adaptive controller consists of (41), (44) and (46), subject to the \( L_1 \)-gain upper bound in (10).

Let \( \hat{x}(t) = \hat{x}(t) - x(t) \). The error dynamics between (5) and (41) are

\[
\dot{x}(t) = A_\hat{x}(t) + \hat{\sigma}(t) \bar{\sigma}(t) - b_m \sigma(t)
\]

\[
\dot{y}(t) = c^T_m \bar{x}(t), \quad \hat{x}(0) = 0 .
\]

(47)

**Lemma 4**

Let \( e(t) = y(t) - y_{ref}(t) \). Then

\[
\| e(t) \|_{L_\infty} \leq \| H_3(s) \|_{L_1 L} \| \hat{y}(t) \|_{L_\infty} .
\]

(48)

**Proof.** Let

\[
\hat{\sigma}(s) = \frac{C(s)}{c^T_m(sI - A_m)^{-1}b_m}c^T_m(sI - A_m)^{-1}\hat{\sigma}(s) - C(s)\sigma(s) .
\]

(49)

It follows from (46) that

\[
u(s) = C(s)r(s) - C(s)\sigma(s) - \hat{\sigma}(s) ,
\]

(50)

and the system in (3) consequently takes the form:

\[
y(s) = M(s) \left( C(s)r(s) + (1 - C(s))\sigma(s) - \hat{\sigma}(s) \right) .
\]

(51)

Substituting \( u(s) \) from (50) into (4), we have

\[
\sigma(s) = \left( (A(s) - M(s)) \left( C(s)r(s) - C(s)\sigma(s) - \hat{\sigma}(s) \right) + A(s)\hat{d}(s) \right) / M(s) ,
\]

(52)

and hence

\[
\sigma(s) = \left( A(s) - M(s) \right) \left( C(s)r(s) - \hat{\sigma}(s) \right) + A(s)\hat{d}(s) / M(s) + C(s)(A(s) - M(s)) .
\]

(53)

Using the definitions of \( H_0(s) \) and \( H_1(s) \) in (18) and (19), we can rewrite \( \sigma(s) \) into the following form:

\[
\sigma(s) = H_1(s)r(s) - \frac{H_1(s)}{C(s)}\hat{\sigma}(s) + H_0(s)\hat{d}(s) .
\]

(54)

Substitution into (51) leads to

\[
y(s) = M(s) \left( C(s) + H_1(s)(1 - C(s)) \right) \left( r(s) - \frac{\hat{\sigma}(s)}{C(s)} \right) + H_0(s)M(s) \left( 1 - C(s) \right) d(s) .
\]

Recalling the definition of \( H(s) \) from (9), one can verify that

\[
M(s)(C(s) + H_1(s)(1 - C(s))) = H(s)C(s) ,
\]

where \( H(s) \) is the \( L_1 \)-gain upper bound in (10).
\( \tilde{H}(s) = H_0(s)M(s) \), which consequently implies \( y(s) = \tilde{H}(s) \left( C(s)r(s) - \tilde{\sigma}(s) \right) + H(s)(1 - C(s))d(s) \). Using the expression for \( y_{\text{ref}}(s) \) from (15), one can derive \( e(s) = \tilde{H}(s) \left( (1 - C(s))d_e(s) - \tilde{\sigma}(s) \right) \), where \( d_e(s) \) is introduced to denote the Laplace transform of \( d_e(t) = f(t, y(t)) - f(t, y_{\text{ref}}(t)) \). Lemma 5 and Assumption 1 give the following upper bound:

\[
\|e_t\|_{\mathcal{L}_\infty} \leq L \|H(s)(1 - C(s))\|_{\mathcal{L}_1} \|e_t\|_{\mathcal{L}_\infty} + \|r_1\|_{\mathcal{L}_\infty}, \tag{55}
\]

where \( r_1(t) \) is the signal with its Laplace transform being

\[
r_1(s) = \tilde{H}(s)\tilde{\sigma}(s). \tag{56}
\]

Using the expression for \( \tilde{\sigma}(s) \) from (49) along with the expressions for \( y(s) \) from (3) and \( \hat{y}(s) \) from (42), one can derive

\[
\hat{y}(s) = \frac{1}{\tilde{C}(s)\tilde{M}(s)} e_m^T (s\mathbb{I} - A_m)^{-1} \tilde{\sigma}(s) - \frac{M(s)}{C(s)} \tilde{\sigma}(s) + \frac{M(s)}{C(s)} C(s) \sigma(s) = \frac{M(s)}{C(s)} \tilde{\sigma}(s) + \frac{M(s)}{C(s)} C(s) \sigma(s) \tag{57}
\]

This implies that \( r_1(s) \) can be rewritten as \( r_1(s) = \frac{C(s)H(s)}{M(s)} C(s) \sigma(s) = H_3(s)\hat{y}(s) \), and hence \( \|r_1\|_{\mathcal{L}_\infty} \leq \|H_3(s)\|_{\mathcal{L}_1} \|\hat{y}_t\|_{\mathcal{L}_\infty} \). Substituting this back into (55) completes the proof.

**VI. Analysis of \( \mathcal{L}_1 \) Adaptive Controller**

In this section, we analyze the stability and the performance of the \( \mathcal{L}_1 \) adaptive controller. Using the definitions from (11), \( H_2(s) \) in (20) can be rewritten as

\[
H_2(s) = \frac{-C_n(s)A_d(s)M_n(s)}{H_d(s)}, \tag{58}
\]

where \( H_d(s) \) is defined in (13). Since \( \deg(C_d(s) - C_n(s)) = \deg C_n(s) = d_r \), it can be checked straightforwardly that \( H_2(s) \) is strictly proper. We notice from (12) and (58) that \( H_2(s) \) has the same denominator as \( H(s) \), and therefore it follows from (10) that \( H_2(s) \) is stable. Since \( H_2(s) \) is strictly proper and stable with relative degree \( d_r \), \( H_2(s)/M(s) \) is stable and proper and, therefore its \( \mathcal{L}_1 \) gain is finite.

Consider the state transformation:

\[
\ddot{\xi} = \Lambda \ddot{x}. \tag{59}
\]

It follows from (47) that

\[
\begin{align*}
\dot{\xi}(t) &= \Lambda A_m \Lambda^{-1} \dot{\xi}(t) + \Lambda \dot{\sigma}(t) - \Lambda b_m \sigma(t) \tag{60} \\
\hat{y}(t) &= \hat{\xi}_1(t), \tag{61}
\end{align*}
\]

where \( \xi_1(t) \) is the first element of \( \dot{\xi}(t) \).

**Theorem 1** Given the system in (1) and the \( \mathcal{L}_1 \) adaptive controller in (41), (44), (46) subject to (10), if we choose \( T \) to ensure

\[
\gamma_0(T) < \bar{\gamma}, \tag{62}
\]

where \( \bar{\gamma} \) is an arbitrary positive constant introduced in (23), then

\[
\begin{align*}
\|\ddot{y}\|_{\mathcal{L}_\infty} &< \bar{\gamma} \tag{63} \\
\|y - y_{\text{ref}}\|_{\mathcal{L}_\infty} &\leq \gamma_1 \tag{64} \\
\|u - u_{\text{ref}}\|_{\mathcal{L}_\infty} &\leq \gamma_2 \tag{65}
\end{align*}
\]

with \( \gamma_1 = \|H_3(s)\|_{\mathcal{L}_1} \bar{\gamma} / (1 - \|G(s)\|_{\mathcal{L}_1} L) \), \( \gamma_2 = L \|H_3(s)\|_{\mathcal{L}_1} \gamma_1 + \|H_2(s)/M(s)\|_{\mathcal{L}_1} \bar{\gamma} \).
Proof. First we prove the bound in (63) by a contradiction argument. Since \( \dot{y}(0) = 0 \) and \( \ddot{y}(t) \) is continuous, then assuming the opposite implies that there exists \( t' \) such that
\[
\| \dot{y}(t) \| < \tilde{\gamma}, \quad \forall \ 0 \leq t < t',
\]
(66)
\[
\| \ddot{y}(t') \| = \tilde{\gamma},
\]
(67)
which leads to
\[
\| \ddot{y}' \|_{L_\infty} = \tilde{\gamma}.
\]
(68)
Since \( y(t) = y_{re}(t) + e(t) \), the upper bound in (16) can be used to arrive at
\[
\| y' \|_{L_\infty} \leq \| y_{ref} \|_{L_\infty} + \| e \|_{L_\infty} \leq \rho + \| C(s)H(s)/M(s) \|_{L_1} \gamma/(1 - \| G(s) \|_{L_1} L).
\]
It follows from (54) and (57) that \( \sigma(s) = H_1(s)r(s) - H_1(s)\dot{y}(s)/M(s) + H_0(s)d(s) \), and hence (68) implies that
\[
\| \sigma' \|_{L_\infty} \leq \| H_1(s) \|_{L_1} \| r \|_{L_\infty} + \| H_1(s)/M(s) \|_{L_1} \gamma + \| H_0(s) \|_{L_1} \| y \|_{L_\infty} + L_0 \),
\]
which along with (69) leads to
\[
\| \sigma' \|_{L_\infty} \leq \Delta.
\]
(69)
It follows from (60) that
\[
\tilde{\xi}(iT + t) = e^{iA_s(iT + t)}\tilde{\xi}(iT) + \int_{iT}^{iT+t} e^{iA_s(iT + \tau)}\Delta\tilde{\sigma}(iT)d\tau - \int_{iT}^{iT+t} e^{iA_s(iT + \tau)}\Delta b_m\sigma(\tau)d\tau
\]
(70)
\[
= e^{iA_s(iT + t)}\tilde{\xi}(iT) + \int_0^t e^{iA_s(iT + \tau)}\Delta\tilde{\sigma}(iT)d\tau - \int_0^t e^{iA_s(iT + \tau)}\Delta b_m\sigma(iT + \tau)d\tau.
\]
Since \( \tilde{\xi}(iT) = \begin{bmatrix} \tilde{y}(iT) \\ 0 \end{bmatrix} \), it follows from (70) that
\[
\tilde{\xi}(iT + t) = \chi(iT + t) + \zeta(iT + t),
\]
(71)
where
\[
\chi(iT + t) = e^{iA_s(iT + t)}\begin{bmatrix} \tilde{y}(iT) \\ 0 \end{bmatrix} + \int_0^t e^{iA_s(iT + \tau)}\Delta\tilde{\sigma}(iT)d\tau,
\]
(72)
\[
\zeta(iT + t) = e^{iA_s(iT + t)}\begin{bmatrix} 0 \\ \tilde{z}(iT) \end{bmatrix} - \int_0^t e^{iA_s(iT + \tau)}\Delta b_m\sigma(iT + \tau)d\tau.
\]
(73)
In what follows, we prove that for all \( iT \leq t' \) one has
\[
|\tilde{y}(iT)| \leq \varsigma(T),
\]
(74)
\[
\tilde{z}^T(iT)P_2\tilde{z}(iT) \leq \alpha,
\]
(75)
where \( \varsigma(T) \) and \( \alpha \) are defined in (29). Since \( \tilde{\xi}(0) = 0 \), it is straightforward that \( |\dot{y}(i)\| \leq \varsigma(T) \), \( \tilde{z}^T(0)P_2\tilde{z}(0) \leq \alpha \).
For any \((j + 1)T \leq t'\), we will prove that if
\[
|\tilde{y}(jT)| \leq \varsigma(T),
\]
(76)
\[
\tilde{z}^T(jT)P_2\tilde{z}(jT) \leq \alpha,
\]
(77)
then (76)-(77) hold for \( j + 1 \) too. Hence, (74)-(75) hold for all \( iT \leq t' \).
Assume (76)-(77) hold for \( j \), and in addition,
\[
(j + 1)T \leq t'.
\]
(78)
It follows from (71) that
\[
\tilde{\xi}((j + 1)T) = \chi((j + 1)T) + \zeta((j + 1)T),
\]
(79)
where
\[ \chi((j+1)T) = e^{\Lambda_{m}\Lambda^{-1}T} \begin{bmatrix} \hat{y}(jT) \\ 0 \end{bmatrix} + \int_{0}^{T} e^{\Lambda_{m}\Lambda^{-1}(T-\tau)\Lambda_{b}m\sigma(jT+\tau)}d\tau, \] (80)
\[ \zeta((j+1)T) = e^{\Lambda_{m}\Lambda^{-1}T} \begin{bmatrix} 0 \\ \bar{z}(jT) \end{bmatrix} - \int_{0}^{T} e^{\Lambda_{m}\Lambda^{-1}(T-\tau)\Lambda_{b}m\sigma(jT+\tau)}d\tau. \] (81)

Substituting the adaptive law from (44) in (80), we have
\[ \chi((j+1)T) = 0. \] (82)

It follows from (73) that \( \zeta(t) \) is the solution of the following dynamics:
\[ \dot{\zeta}(t) = \Lambda_{m}\Lambda^{-1}\zeta(t) - \Lambda_{b}m\sigma(t), \] (83)
\[ \zeta(jT) = \begin{bmatrix} 0 \\ \bar{z}(jT) \end{bmatrix}, \quad t \in [jT, (j+1)T]. \] (84)

Consider the following function:
\[ V(t) = \zeta^T(t)\Lambda^{-T}PA^{-1}\zeta(t), \] (85)
over \( t \in [jT, (j+1)T] \). Since \( \Lambda \) is non-singular and \( P \) is positive definite, \( \Lambda^{-T}PA^{-1} \) is positive definite and, hence, \( V(t) \) is a positive definite function. It follows from (83) that over \( t \in [jT, (j+1)T] \)
\[ \dot{V}(t) = \zeta^T(t)\Lambda^{-T}PA^{-1}\Lambda_{m}\Lambda^{-1}\zeta(t) + \zeta^T(t)\Lambda^{-T}A_{m}^{T}\Lambda^{-T}PA^{-1}\zeta(t) - 2\zeta^T(t)\Lambda^{-T}PA^{-1}\Lambda_{b}m\sigma(t) \]
\[ = \zeta^T(t)\Lambda^{-T}(PA_{m} + A_{m}^{T}P)\Lambda^{-1}\zeta(t) - 2\zeta^T(t)\Lambda^{-T}P\Lambda_{b}m\sigma(t) \]
\[ = -\zeta^T(t)\Lambda^{-T}QA^{-1}\zeta(t) - 2\zeta^T(t)\Lambda^{-T}P\Lambda_{b}m\sigma(t). \] (86)

Using the upper bound from (69) we can compute the following upper bound over \( t \in [jT, (j+1)T] \)
\[ \dot{V}(t) \leq -\lambda_{\min}(\Lambda^{-T}QA^{-1})\|\zeta(t)\|^2 + 2\|\zeta(t)\|\|\Lambda^{-T}P\Lambda_{b}m\|\Delta. \] (87)

Notice that for all \( t \in [jT, (j+1)T] \), if
\[ V(t) \geq \alpha, \] (88)
we have \( \|\zeta(t)\| \geq \alpha \sqrt{\frac{\alpha}{\lambda_{\max}(\Lambda^{-T}QA^{-1})}} = \frac{2\Delta\|\Lambda^{-T}P\Lambda_{b}m\|}{\lambda_{\min}(\Lambda^{-T}QA^{-1})} \), and hence the upper bound in (87) yields
\[ \dot{V}(t) \leq 0. \] (89)

It follows from Lemma 2 and the relationship in (84) that \( V(\zeta(jT)) = \bar{z}^T(jT)P_{2}\bar{z}(jT) \), which further along with the upper bound in (77) leads to the following
\[ V(\zeta(jT)) \leq \alpha. \] (90)

It follows from (88)-(89) and (90) that
\[ V(t) \leq \alpha, \quad \forall \ t \in [jT, (j+1)T], \] (91)
and therefore
\[ V((j+1)T) = \zeta^T((j+1)T)(\Lambda^{-T}PA^{-1})\zeta((j+1)T) \leq \alpha. \] (92)

Since
\[ \dot{\xi}((j+1)T) = \chi((j+1)T) + \zeta((j+1)T), \] (93)
then the equality in (82) and the upper bound in (92) lead to the following inequality
\[ \dot{\xi}^T((j+1)T)(\Lambda^{-T}PA^{-1})\dot{\xi}((j+1)T) \leq \alpha. \] Using the result of Lemma 2 one can derive that
\[ \bar{z}^T((i+1)T)P_{2}\bar{z}(i+1)T) \leq \xi^T((i+1)T)(\Lambda^{-T}PA^{-1})\xi((i+1)T) \leq \alpha, \] which implies that the upper bound in (77) holds for \( j + 1 \).
Taking into consideration (76)-(77) and recalling the definitions of $\eta$ and $\|\|$ along with our assumption on $\gamma$, which yields (74)-(75) hold for all $\phi$.

Theorem 2 Given the system in (1) and the $L_1$ adaptive controller in (41), (44), (46) subject to (10), we have:

\[
\lim_{T \to 0} (y(t) - y_{ref}(t)) = 0, \quad \forall t \geq 0, \tag{101}
\]
\[
\lim_{T \to 0} (u(t) - u_{ref}(t)) = 0, \quad \forall t \geq 0. \tag{102}
\]
where without re-tuning. The control signal and the system response are plotted in Figs. 2(a)-2(b). Further, we consider a in Figs 1(a)-1(b). Next, we consider the unknown nonlinearity and independent of unknown nonlinearities, it will not ensure uniform transient performance in the presence of unknown hardware requirements and implies that the performance limitations are consistent with the hardware limitations. This in turn is consistent with the results in Refs., where improvement of the transient performance was achieved by increasing the adaptation rate in the continuous adaptive laws.

Remark 2 We notice that the following ideal control signal \( u_{\text{ideal}}(t) = r(t) - \sigma_{\text{ref}}(t) \) is the one that leads to desired system response \( y_{\text{ideal}}(s) = M(s)r(s) \) by cancelling the uncertainties exactly. Thus, the reference system in (6)-(8) has a different response as compared to this one. It only cancels the uncertainties within the bandwidth of \( C(s) \), which can be selected comparable to the control channel bandwidth. This is exactly what one can hope to achieve with any feedback in the presence of uncertainties.

Remark 3 We notice that stability of \( H(s) \) is equivalent to stabilization of \( A(s) \) by

\[
\frac{C(s)}{M(s)(1 - C(s))}.
\]

Indeed, consider the closed-loop system, comprised of the system \( A(s) \) and negative feedback of (103). The closed-loop transfer function is:

\[
A(s)
1 + A(s)\frac{C(s)}{M(s)(1 - C(s))}.
\]

Incorporating (11), one can verify that the denominator of the system in (104) is exactly \( H_d(s) \). Hence, stability of \( H(s) \) is equivalent to the stability of the closed-loop system in (104). This implies that the class of systems \( A(s) \) that can be stabilized by the \( \mathcal{L}_1 \) adaptive output feedback controller (41), (44) and (46) is not empty.

Remark 4 Finally, we notice that while the feedback in (103) may stabilize the system in (1) for some classes of unknown nonlinearities, it will not ensure uniform transient performance in the presence of unknown \( A(s) \). On the contrary, the \( \mathcal{L}_1 \) adaptive controller ensures uniform transient performance for system’s both signals, independent of the unknown nonlinearity and independent of \( A(s) \).

VII. Simulations

**Numerical Example.**

As an illustrative example, consider the system in (1) with \( A(s) = (s + 1)/(s^3 - s^2 - 2s + 8) \). We notice that \( A(s) \) has poles in the right half complex plane. We consider \( \mathcal{L}_1 \) adaptive controller defined via (41), (44) and (46), where

\[
M(s) = \frac{1}{s^2 + 1.4s + 1}, \quad C(s) = \frac{100}{s^2 + 14s + 100}, \quad T = 10^{-4}.
\]

First, we consider the step response when \( d(t) = 0 \). The simulation results of \( \mathcal{L}_1 \) adaptive controller are given in Figs 1(a)-1(b). Next, we consider \( d(t) = f(t, y(t)) = \sin(0.1t)y^2(t) + \sin(0.4t) \), and apply the same controller without re-tuning. The control signal and the system response are plotted in Figs. 2(a)-2(b). Further, we consider a time-varying reference input \( r(t) = 0.5 \sin(0.3t) \) and notice that without any re-tuning of the controller the system response and the control signal behave as expected, Figs. 3(a)-3(b).

**Missile Longitudinal Autopilot Design.**

We consider the missile dynamics from Ref.\(^{20}\), which is given by the following state-space representation:

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} Z_a & 1 & Z_d & 0 \\ M_a & 0 & M_d & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega_n^2 & -2\zeta_n\omega_n \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \omega_n^2 \end{bmatrix} \left( u(t) + k_m^T x(t) \right), \\
y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t),
\end{align*}
\]
where \( x(t) = [A_z \quad q \quad \delta_a \quad \delta_a']^\top \) is the system state, in which \( A_z \) is the vertical acceleration, \( q \) is the pitch rate, \( \delta_a \) is the fin deflection, and \( \delta_a' \) is the fin deflection rate, \( k_m \) is the vector of matched parametric uncertainties, and \( \Delta A \) contains both unmatched and additional matched uncertainties. In simulations, the following numerical values have been used for the missile dynamics\(^\text{20}\):

\[
\begin{align*}
Z_a &= -1.3046, & Z_d &= -0.2142, & M_a &= 47.7109, & M_d &= -104.8346, \\
\Delta A &= \begin{bmatrix} -K & 0 & -K & 0 \\ K & 0 & -K & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \end{bmatrix}, & k_m &= 0.3 \begin{bmatrix} 0 \\ 0 \\ -1 \\ -2 \frac{Z_d}{Z_a} \end{bmatrix},
\end{align*}
\]

where \( K \) is the uncertainty scaling factor for unmatched uncertainties, and \( a_1, a_2, a_3, a_4 \) can assume arbitrary values, modeling additional matched uncertainties. For simulation of the \( L_1 \) adaptive output feedback controller, the following...
$M(s) = \frac{1}{\omega s^2 + 2\zeta \omega s + 1}$, $C(s) = \frac{1}{1/\omega_c s^2 + 2\zeta_c \omega_c s + 1}$.

where $\omega = 10 \text{ rad/s}$, $\zeta = 0.8$, $\omega_c = 150 \text{ rad/s}$, and $\zeta_c = 0.8$. Figure 9 shows scaled response for scaled uncertainties, with $K$ denoting the scaling factor for unmatched uncertainties. We notice that the system response does not change from variation of $a_1, a_2, a_3, a_4$.

**Flexible Wing Application**

We now apply the $L_1$ adaptive output feedback control architecture to a highly flexible semi-span wind tunnel model, considered earlier in the work of authors in Ref.\textsuperscript{21} The wind-tunnel model is instrumented with accelerometers along the spar, strain gauges at the root and mid-spar, a rate gyro at the wing tip, a gust sensor vane in front of the wing, and a rate gyro and accelerometers at the tunnel attachment point. The accelerometers, strain gauges and rate gyro allow the control system to sense the bending modes and the structural stresses. The test objectives are, first, to control the first and second bending modes of the wing, while executing altitude hold and controlling pitch moment at the pivot point. The second control objective is GLA/flutter suppression, particularly around the frequency of the first bending mode. In the following figures, we compare the performance of $L_1$ adaptive output feedback controller to a baseline linear design that was provided with the model. Control signals are generated by the leading and the trailing edge control surfaces. In Figs. 5(a)-5(b), we notice that the $L_1$ adaptive output feedback controller leads to the same performance as the baseline controller in the absence of turbulence. In the presence of turbulence, Figs. 6(a)-7(b) demonstrate that the $L_1$ adaptive output feedback controller improves its performance in the presence of turbulence. We further modify the original $A$ matrix of the system by adding 0.1 to its every element. Figs. 8(a)-9(b) demonstrate that the $L_1$ adaptive controller outperforms the baseline controller for this class of uncertainty as well.
Figure 5. Simulation results of baseline and $\mathcal{L}_1$ adaptive controller in the absence of turbulence

Figure 6. Simulation results of baseline controller in the presence of turbulence

Figure 7. Simulation results of $\mathcal{L}_1$ adaptive controller in the presence of turbulence
Figure 8. Simulation results of baseline controller in the presence of additive uncertainty in the $A$ matrix

Figure 9. Simulation results of $L_1$ adaptive controller in the presence of additive uncertainty in the $A$ matrix
VIII. Conclusions

We presented the $L_1$ adaptive output feedback controller for reference systems that do not verify the SPR condition for their input-output transfer function. The new piece-wise constant adaptive law along with low-pass filtered control signal ensures uniform performance bounds for system’s both input/output signals simultaneously. The performance bounds can be systematically improved by reducing the integration time-step.

References


IX. Appendix

Definition 1 22 For a signal $\xi(t)$, $t \geq 0$, $\xi \in \mathbb{R}^n$, its truncated $L_\infty$ and $L_\infty$ norms are $\|\xi_i\|_{L_\infty} = \max_{i=1, \ldots, n} (\sup_{0 \leq \tau \leq t} |\xi_i(\tau)|)$, $\|\xi\|_{L_\infty} = \max_{i=1, \ldots, n} (\sup_{\tau \geq 0} |\xi_i(\tau)|)$, where $\xi_i$ is the $i^{th}$ component of $\xi$.

Definition 2 22 The $L_1$ gain of a stable proper SISO system is defined $\|H(s)\|_{L_1} = \int_0^\infty |h(t)| dt$, where $h(t)$ is the impulse response of $H(s)$.

Lemma 5 22 For a stable proper multi-input multi-output (MIMO) system $H(s)$ with input $r(t) \in \mathbb{R}^m$ and output $x(t) \in \mathbb{R}^n$, we have $\|x_t\|_{L_\infty} \leq \|H(s)\|_{L_1} \|r_t\|_{L_\infty}$, $\forall t \geq 0$. American Institute of Aeronautics and Astronautics