Neighbor Discovery in Wireless Networks with Multipacket Reception

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ABSTRACT

Neighbor discovery is one of the first steps in configuring and managing a wireless network. Most existing studies on neighbor discovery assume a single-packet reception model where only a single packet can be received successfully at a receiver. In this paper, motivated by the increasing prevalence of multipacket reception (MPR) technologies such as CDMA and MIMO, we study neighbor discovery in MPR networks that allow multiple packets to be received successfully at a receiver. More specifically, we design and analyze a series of randomized algorithms for neighbor discovery in MPR networks. We start with a simple Aloha-like algorithm that assumes synchronous node transmissions and the number of neighbors, n, is known. We show that the time for all the nodes to discover their respective neighbors is $\Theta(\ln n)$ in an idealized MPR network that allows an arbitrary number of nodes to transmit simultaneously. In a more realistic scenario, in which no more than k nodes can transmit simultaneously, we show that the time to discover all neighbors is $\Theta(\frac{n \ln n}{k})$. When a node knows whether its transmission is successful or not (e.g., based on feedbacks from other nodes), we design an adaptive Aloha-like algorithm that dynamically determines the transmission probability for each node, and show that it yields a $\ln n$ improvement over the simple Aloha-like scheme. Last, we extend our schemes to take into account a number of practical considerations, such as lack of knowledge of the number of neighbors and asynchronous algorithm operation, while resulting in only a constant or $\log n$ factor slowdown in algorithm performance.

Categories and Subject Descriptors

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Keywords

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1. INTRODUCTION

Neighbor discovery is one of the first steps in configuring and managing wireless networks. The information obtained from neighbor discovery viz. the set of nodes that a wireless node can directly communicate with, is needed to support basic functionalities such as medium access and routing. Furthermore, this information is needed by topology control and clustering algorithms to improve the performance and efficiency of the network [16, 10]. Due to its critical importance, neighbor discovery has received a lot of attention, and a number of studies have been devoted to this topic (e.g., [25, 23, 24, 14, 7]). Most studies however assume a single packet reception (SPR) model, i.e., a transmission is successful if and only if there are no other simultaneous transmissions.

In this paper, in contrast to prior literature, we study neighbor discovery in multipacket reception (MPR) networks where packets from multiple transmitters can be received successfully at a receiver. This is motivated by the increasing prevalence of MPR technologies in wireless networks. For instance, CDMA (code division multiple access) and MIMO (multiple-input and multiple-output), two widely used technologies, both support multipacket reception. Neighbor discovery in MPR networks differs fundamentally from that in SPR networks in the following manner. In a SPR network, a node is discovered by each of its neighbors if it is the only node that transmits at a given time instant; while in an MPR network, a node can transmit simultaneously with several other neighbors, and each of these nodes may be discovered simultaneously by the receiving nodes, while none of them is guaranteed to be discovered by all of its neighbors.

Our focus is on randomized algorithms where each node makes independent decisions on whether to transmit or not at a given time instant. The reason for focusing on randomized algorithms is two-fold. First, they do not need a central controller to coordinate the actions of the various nodes, and are therefore easy to implement in large scale networks. Second, they are more flexible and often require less *a priori* knowledge of the network and the operating conditions as compared to deterministic algorithms.

To provide insights, we first consider the neighbor discovery problem assuming that: (i) nodes transmit in synchronized slots and, (ii) the number of neighbors is known. We later relax each of these assumptions to study neighbor discovery when the number of neighbors is not known beforehand, and when the nodes transmit asynchronously. For each randomized algorithm presented in this paper, we analyze their performance in terms of *neighbor discovery time*, i.e., the time for all nodes to discover theirs respective neighbors. This is a critical performance metric since faster neighbor discovery leads to less energy consumption and shorter delays to start other operations of the network.

Our main contributions are as follows:

- When node transmissions are synchronous and the number of neighbors, n, is known[1], we analyze the Aloha-like neighbor discovery algorithm, and show that the neighbor discovery time is $\Theta(\ln n)$ in an idealized MPR network that allows an arbitrary number of nodes to transmit simultaneously, and is $\Theta\left(\frac{n \ln n}{k}\right)$, when allowing up to k nodes to transmit simultaneously.

  As a special case, when $k = 1$, our MPR model reduces to a SPR model, and the discovery time is $\Theta(n \ln n)$, which coincides with the result in [24] for SPR models. Thus, the results in this paper can be considered to be a generalization of the results derived in [24] for SPR networks.

- When k nodes are allowed to transmit simultaneously, we show that the performance of the Aloha-like algorithm can be improved significantly if a node knows whether its transmission is successful or not (e.g., based on feedbacks from other nodes). In particular, we propose an adaptive Aloha-like algorithm in which each node adjusts its transmission probability dynamically, and show that it provides a $\ln n$ improvement over the simple Aloha-like scheme.

- We extend our schemes to the cases where the number of neighbors is not known beforehand and the nodes transmit asynchronously, and show that they result in a constant or $\log_2 n$ factor slowdown.

Our results demonstrate that, compared to SPR, MPR indeed leads to shorter neighbor discovery time, and in particular, allowing k simultaneous transmissions leads to k times speedup.

The rest of the paper is organized as follows. Section 2 reviews related work. Section 3 presents the problem setting. Section 4 describes an Aloha-like neighbor discovery algorithm and its analysis. Section 5 describes an adaptive Aloha-like neighbor discovery algorithm, and shows that it improves upon the basic scheme by $\ln n$. Section 6 extend our results to scenarios where a node has no estimation of the number of its neighbors. Section 7 extends our results to asynchronous systems. Last, Section 8 concludes the paper and presents future work.

[1] The actual number of neighbors is $n - 1$ (see Section 3). We state the asymptotic results in n for simplicity.

2. RELATED WORK

Our work is inspired by [24] that designs and analyzes several randomized neighbor discovery algorithms in SPR networks. As we will see in this paper, the design and analysis of neighbor discovery algorithms is substantially more challenging in the case of MPR networks as compared to the SPR networks studied in [24]. Furthermore, our study generalizes the study of [24], viz. our results are for the case where we allow up to $k \geq 1$ simultaneous transmissions, and reduce to those in [24] by simply letting $k = 1$.

Several other studies develop randomized/deterministic neighbor discovery algorithms in SPR networks [19, 5, 12, 7, 13]. Our study differs from them in that we consider neighbor discovery in MPR instead of SPR networks.

The studies of [17, 2, 3, 18] use multiuser-detection based approach for neighbor discovery. They require each node to possess a signature as well as know the signatures of all the other nodes in the network. Further, nodes are assumed to operate in a synchronous manner. When a node receives transmission from multiple neighbors, it determines which nodes are the transmitters based on the received signal (or energy) and the prior knowledge of the node signatures in the network. Although these studies also allow multiple transmitters to transmit simultaneously, their focus is on using coherent/noncoherent detection [17, 2, 3] or group testing [18] to identify neighbors with high detection ratio and low false positive ratio, while our focus is on designing randomized algorithms and analyzing their performance. Our approach does not need signatures and can operate in asynchronous systems.

There are numerous studies on neighbor discovery when nodes have directional antennas (e.g., [21, 11, 23, 25]). The focus in these works is on antenna scanning strategies for efficient neighbor discovery. There have been several recent proposals on neighbor discovery in cognitive radio networks (e.g., [15, 4]). They determine the set of neighbors for a node as well as the channels that can be used to communicate among neighbors. Our study is in settings different from the above, viz. we assume omni-directional antennas (or antenna arrays) and do not consider cognitive radios.

Last, our study also differs from existing studies on MPR networks: we focus on neighbor discovery while existing studies focus on other aspects, e.g., MAC protocol design [26, 6], stability [8] and capacity analysis [22].

3. PROBLEM SETTING

Consider a static network with n nodes indexed from 1 to n. Each node has a unique ID (e.g., its MAC address or geographic location). For ease of analysis, we assume that these nodes are within the transmission range of each other, i.e., they form a clique. A similar assumption is also made in [23, 24, 13]. In Section 4.3, we justify the clique assumption by validating it against simulations of our algorithms in a multi-hop network setting.

Each node embeds its ID in the messages it transmits to its neighbors. A node, i, is discovered by another node, j, if and only if j successfully receives a message from i. Each node has an omni-directional antenna (or an antenna array). The radio at each node is assumed to be half-duplex, i.e., a node can either transmit or receive packets, but not both at the same time. We assume that all nodes have multi-packet reception capabilities. That is, a node can correctly receive packets from multiple transmitters simultaneously. This MPR capability can be provided through smart antenna array techniques such as MIMO, or coding techniques such as CDMA.

We consider the following two MPR models in this paper:

- MPR-k: In this model, only up to k simultaneous packets can be decoded successfully at a receiver. The value of k is fixed and is known beforehand. In practice, it is determined by the number of orthogonal codes when using CDMA [26], and is determined by the number of antennas when using MIMO. Note that SPR is a special case of MPR-k (i.e., when $k = 1$).
- Idealized MPR: As a special case of the MPR-k model, we study the idealized MPR model, where $k = n - 1$. In other words, an arbitrary number of packets are decodable simultaneously at a receiver. The idealized MPR model is of practical interest in scenarios where the number of neighbors is close to the amount of diversity provided by MPR technologies (i.e., the number of orthogonal codes in CDMA or the number of antennas in MIMO).

Note that the MPR-k model studied in this paper is a simple generalization of the well-known collision channel model studied in the case of SPR networks. We recognize that the model represents an idealization of real-world conditions, but our motivation in employing this model is the same as in the case of the collision channel model, viz. it provides us with useful insights for designing algorithms and understanding (to a first-order approximation) their performance when deployed in the real world.

4. ALOHA-LIKE NEIGHBOR DISCOVERY ALGORITHM

In this section, we consider a simple Aloha-like neighbor discovery algorithm and analyze it under various MPR models. We start with the simplifying assumptions that all nodes know the clique size, n. Furthermore, time is divided into slots, and nodes are synchronized on slot boundaries. These assumptions will be relaxed later, in Sections 6 and 7.

The algorithm operates as follows. Each node transmits with probability p and listens with probability $1-p$ in each slot, and this transmission probability does not change over time. The case where the transmission probability is allowed to change will be studied in Section 5.

In the following, we first determine the optimal transmission probability and present an asymptotic analysis of the Aloha-like neighbor discovery algorithm. We then validate the clique assumption against simulation results of the algorithm in a multi-hop network setting.

4.1 Optimal transmission probability

Consider two arbitrary nodes, i and j. Let p_s denote the probability that i is discovered by j in a given time slot. Then p_s is the probability that i transmits, j listens, and there are at most $k-1$ other nodes transmitting ($k = n-1$ in an idealized MPR). Therefore,

$$p_s = p(1-p)\sum_{i=0}^{k-1}\binom{n-2}{i}p^i(1-p)^{n-2-i}. \tag{1}$$

To minimize the time to discover all neighbors, we need to choose a p that maximizes p_s. Let p^* denote this optimal transmission probability, and let p_s^* denote the corresponding optimal p_s. We obtain p^* and p_s^* in the following models.

- Idealized MPR: In this model, $p_s = p(1-p)$. Therefore, $p^* = 1/2$ and $p_s^* = 1/4$.
- SPR model: In this model, $p_s = p(1-p)^{n-1}$. Therefore, $p^* = 1/n$ and $p_s^* \approx \frac{1}{ne}$.
- MPR-k: For a given n and k, we can obtain p^* by solving (1) numerically. It is however non-trivial to analytically derive a closed-form solution for p^* from (1). In order to obtain analytical insights, we prove that $p^* = \alpha k/n$, where α is between two constants (see Theorem 2 in Appendix A). Our numerical results in Appendix B show that the optimal transmission probability is close to k/n for small k, and is close to $2k/(ne) \approx 0.73k/n$ for relatively large k.

4.2 Asymptotic analysis

Let T be a random variable that denotes the neighbor discovery time, i.e., the time until all n nodes have discovered their respective neighbors. We next present an asymptotic analysis of the neighbor discovery time in MPR networks. In particular, we show that $T = \Theta(\ln n)$ under the idealized MPR model, and $T = \Theta(\frac{n \ln n}{k})$ under the MPR-k model.

4.2.1 Idealized MPR

Under idealized MPR, $p^* = 1/2$ and $p_s^* = 1/4$. We show that we can find two constants, $c_1 > c_2 > 0$, such that $\Pr(T > c_1 \ln n) \to 0$ w.h.p., and $\Pr(T > c_2 \ln n) \to 1$ w.h.p., thus implying that $T = \Theta(\ln n)$.

We first show that we can find a positive constant c_1 such that $P(T > c_1 \ln n) \to 0$ w.h.p.. Consider an arbitrary node, i. Let T_{ij} denote the time until i discovers a given neighbor j, $j \neq i$. Then

$$\Pr(T_{ij} \le t) = 1 - (1-p_s^*)^t = 1 - \left(\frac{3}{4}\right)^t.$$

That is,

$$\Pr(T_{ij} > t) = (1-p_s^*)^t = \left(\frac{3}{4}\right)^t. \tag{2}$$

Let $\beta_1 > 2$ be a constant, and let $c_1 = \frac{-\beta_1}{\ln(3/4)}$. Then,

$$\Pr(T_{ij} > c_1 \ln n) = \left(\frac{3}{4}\right)^{c_1 \ln n} = \frac{1}{n^{\beta_1}}. \tag{3}$$

Let T_i denote the time until i discovers all its neighbors. Then,

$$\Pr(T_i > t) = \Pr(\max_j T_{ij} > t) \le \sum_{j=1, j\neq i}^{n} \Pr(T_{ij} > t). \tag{4}$$

Combining (3) and (4) yields

$$\Pr(T_i > c_1 \ln n) \le n \Pr(T_{ij} > c_1 \ln n) \le \frac{1}{n^{\beta_1 - 1}}. \tag{5}$$

Recalling that T is the time until all n nodes have discovered their respective neighbors, it follows that

$$\Pr(T > t) = \Pr(\max_i T_i > t) \le \sum_{i=1}^{n} \Pr(T_i > t) \tag{6}$$

Combining (5) and (6), we obtain

$$\Pr(T > c_1 \ln n) \leq n \Pr(T_i > c_1 \ln n) \leq \frac{1}{n^{\beta_1 - 2}}, \quad (7)$$

where the right hand side approaches 0 as $n \to \infty$, since $\beta_1 > 2$.

We now prove that we can find another positive constant c_2 such that $\Pr(T > c_2 \ln n) \to 1$ w.h.p.. Note that to finish node discovery by time t, each node must transmit at least once by time t. Let T' be a random variable that denotes the time when all the nodes have transmitted at least once. Then $T \geq T'$. The probability that each node has transmitted at least once by time t is

$$\left(1 - (1-p^*)^t\right)^n = \left(1 - \left(\frac{1}{2}\right)^t\right)^n.$$

Let $c_2 = \beta_2 / \ln 2$, $0 < \beta_2 < 1$. Then

$$\left(1 - \left(\frac{1}{2}\right)^{c_2 \ln n}\right)^n = (1 - n^{-\beta_2})^n \leq e^{-n^{1-\beta_2}},$$

where the right hand side approaches zero as $n \to \infty$ since $0 < \beta_2 < 1$. Therefore, $\Pr(T' \geq c_2 \ln n) \to 1$ as $n \to \infty$. Since $T \geq T'$, we have $\Pr(T \geq c_2 \ln n) \to 1$ as $n \to \infty$.

4.2.2 MPR-k

In MPR-k, as shown in Theorem 2 in Appendix A, we represent the optimal transmission probability, p^*, as $\alpha k/n$, where $\alpha \in (0, k/n]$ is a constant. Define ϕ as

$$\phi = \sum_{i=0}^{k-1} \binom{n-2}{i} p^i (1-p)^{n-2-i}.$$

That is, ϕ represents the binomial sum term in (1). Then

$$p_s^* = \frac{\alpha k}{n}(1 - \frac{\alpha k}{n})\phi.$$

Using a Poisson distribution with parameter λ to approximate the binomial distribution yields

$$\phi = e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!},$$

where $\lambda = \frac{\alpha k(n-2)}{n} \approx \alpha k$. Therefore, ϕ can be considered to be a constant, independent of n. In addition, for small k and large n, we have $1 - \frac{\alpha k}{n} \approx 1$. Therefore,

$$p_s^* \approx \frac{\alpha \phi k}{n}.$$

Let $\gamma = \alpha\phi$. As described earlier, we can regard γ as a constant, and we have

$$p_s^* \approx \frac{\gamma k}{n}. \quad (8)$$

We next show that we can find constants $d_1 > d_2$, so that $\Pr(T > \frac{d_1 n \ln n}{k}) \to 0$ w.h.p., and $\Pr(T > \frac{d_2 n \ln n}{k}) \to 1$ w.h.p.. Therefore, $T = \Theta(\frac{n \ln n}{k})$. We now prove the first statement. Following (2) and (8), we have

$$\Pr(T_{ij} > t) = \left(1 - \frac{\gamma k}{n}\right)^t.$$

Let $\beta_1 > 2$ be a constant, and let $d_1 = \frac{\beta_1}{\gamma}$. Then we have

$$\Pr\left(T_{ij} > \frac{d_1 n \ln n}{k}\right) = \left(1 - \frac{\gamma k}{n}\right)^{\frac{d_1 n \ln n}{k}} \leq \frac{1}{n^{\beta_1}}.$$

An analysis similar to the one for the idealized MPR case yields

$$\Pr(T > \frac{d_1 n \ln n}{k}) \leq \frac{1}{n^{\beta_1 - 2}}. \quad (9)$$

It is easy to see that since $\beta_1 > 2$, the right hand approaches 0 as $n \to \infty$.

We now prove the second statement that we can find a constant d_2 such that $\Pr(T > \frac{d_2 n \ln n}{k}) \to 1$ w.h.p. As in the idealized MPR setting, let T' represent the time when all the nodes have transmitted at least once. Then $T \geq T'$. The probability that each node has transmitted at least once by time t is

$$\left(1 - (1-p^*)^t\right)^n = \left(1 - \left(1 - \frac{\alpha k}{n}\right)^t\right)^n.$$

Let $d_2 = \beta_2/\alpha$, $0 < \beta_2 < 1$. Then

$$\left(1 - \left(1 - \frac{\alpha k}{n}\right)^{d_2 n \ln n / k}\right)^n = (1 - n^{-\beta_2})^n \leq e^{-n^{1-\beta_2}},$$

where since $\beta_2 < 1$, the right hand side approaches zero as $n \to \infty$. Therefore, $\Pr(T' \geq \frac{d_2 n \ln n}{k}) \to 1$ as $n \to \infty$. Since $T \geq T'$, we have $\Pr(T \geq \frac{d_2 n \ln n}{k}) \to 1$ as $n \to \infty$.

From the above, we conclude that $T = \Theta(\frac{n \ln n}{k})$ w.h.p.. For $k = 1$, i.e., the SPR case, we recover the $\Theta(n \ln n)$ result derived in [24]. For values of k approaching n, we recover the $\Theta(\ln n)$ result derived under the idealized MPR setting.

4.3 Validation of Clique Assumption

We now validate the clique assumption used in our analysis so far against results obtained using simulation of the Aloha-like algorithm in a wireless multi-hop network. In particular, we demonstrate that the clique assumption yields a very good approximation when using the expected time to discover neighbors as the metric of interest.

We employ a similar setting as that in [24] for our simulations. In particular, we consider a set of nodes uniformly distributed in a 2D plane of area 3km × 3km. Each node has a fixed transmission range of 150m and knows the number of its neighbors beforehand. We vary the average number of neighbors per node by varying the number of nodes inside the area. For instance, when there are 2000 nodes, the average number of neighbors per node is 16; and when there are 3056 nodes, the average number of neighbors per node is 24. Since the nodes in a network have different numbers of neighbors, a meaningful performance metric is the average time that a node discovers all its neighbors. For each setting, we obtain the empirical results on this metric from 20 simulation runs, and compare it with those from the analysis that are derived as follows. Let T_i be a random variable that denotes the time for an arbitrary node i to discover all its neighbors. Then

$$\begin{aligned} E(T_i) &= \sum_{t=1}^{\infty} \Pr(T_i \geq t) \\ &= \sum_{t=1}^{\infty} \left(1 - \Pr(T_i < t)\right). \end{aligned} \quad (10)$$

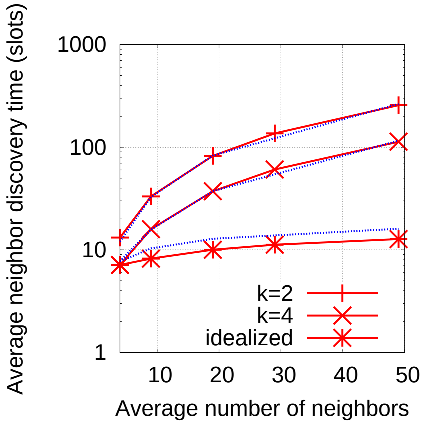

Figure 1: Validation of the clique assumption, where the dashed lines represent the analytical results. The average neighbor discovery time being plotted is the average time for an arbitrary node to discover all its neighbors.

Assume a clique where each node has $(n-1)$ neighbors. Assuming that node i discovers its $(n-1)$ neighbors independently (which is reasonable for small k), we have

$$\Pr(T_i < t) = \left[1 - (1-p_s)^{t-1}\right]^{n-1}. \tag{11}$$

Substituting (11) into (10) yields

$$\begin{aligned} E(T_i) &= \sum_{t=1}^{\infty} 1 - [1-(1-p_s)^{t-1}]^{n-1} \\ &= \sum_{t=0}^{\infty} 1 - \sum_{i=0}^{n-1} \binom{n-1}{i} \left[-(1-p_s)^t\right]^i \\ &= \sum_{i=1}^{n-1} \frac{(-1)^{i+1}\binom{n-1}{i}}{1-(1-p_s)^i}. \end{aligned} \tag{12}$$

Fig. 1 compares the analytical results from (12) (indicated using dashed lines) and simulation under the idealized MPR and MPR-k models (for $k = 2, 4$). In the simulation, the average number of neighbors is varied from 4 to 49. Correspondingly, in the analysis, for each node, the number of neighbors is varied from 4 to 49. In each setting, the results are obtained using the optimal transmission probability. The 95% confidence intervals for the simulation are very tight and hence omitted. We observe a close match between the analytical and simulation results for the MPR-2 and MPR-4 models, indicating that the clique assumption provides a good approximation and the independence assumption in (11) is reasonable. Under the idealized model, the error is larger (but below 28%), and the mismatch between the analysis and simulation is mainly due to the independence assumption in (11) since our simulation results under a clique network topology (not plotted) match those from the multi-hop wireless networks very well.

5. ADAPTIVE ALOHA-LIKE SCHEME

We now consider neighbor discovery under MPR-k models when a node knows whether its transmission is successful or not. This information can be obtained based on feedback as follows [24, 13]. We divide a slot into two sub-slots. Nodes either transmit or listen in the first sub-slot. If a node listens in the first sub-slot and can decode the received packets successfully, then it deterministically sends a signal in the second sub-slot; otherwise, it is silent. A node that transmits in the first sub-slot knows its transmission is successful if and only if it hears a signal (or sensed energy) in the second sub-slot.

Existing studies [24, 13] have shown that this feedback information can significantly reduce neighbor discovery time in SPR networks. Their main idea is that once a node knows its transmission is successful, it knows that the rest of the nodes have discovered it, and hence will not transmit any more in the future time slots; knowing this, the rest of the nodes increase their transmission probabilities correspondingly, which reduces the neighbor discovery time. This strategy is not applicable to MPR-k models since a node that transmits successfully is not guaranteed to have been discovered by the rest of the nodes.

In the following, we design an adaptive Aloha-like scheme that utilizes the feedback information for neighbor discovery under MPR-k models, and show that it improves the simple Aloha-like scheme by $\ln n$. The main idea of the scheme is as follows. We refer to a node that only listens as *passive*, and *active* otherwise. At the beginning, all the nodes are active. We divide time into phases. Phase i contains $W_i = \Theta(\frac{n_i}{k})$ slots, where n_i is the number of active nodes in phase i. In each phase, we use the simple Aloha-like scheme for neighbor discovery where each active node transmits with probability p_i in phase i. At the end of a phase, all *successful* nodes, i.e., nodes that have transmitted successfully at least three times in the phase, become passive (since the probability that a successful node has been discovered by all its neighbors is in the order of $1 - \frac{1}{n^2}$, close to 1 for relatively large n). We can show that when W_i and p_i are chosen properly, at least half of the active nodes in a phase will become passive at the end of the phase. Therefore, the next phase has at most half as many active nodes, and each of them transmits with a higher probability than that in the previous phase. We continue the phases until there are at most $\frac{n}{\ln n}$ active nodes left. At that point of time, we run the simple Aloha-like scheme where the remaining active nodes transmit and the passive nodes listen until neighbor discovery terminates.

In the above scheme, the number of active nodes in phase i, $n_i \le \frac{n}{2^{i-1}}$. Let m denote the number of phases. Since each phase reduces the number of active nodes by at least a half, and the phases stop when there are at most $\frac{n}{\ln n}$ active nodes, we have $m \le \log_2 \ln n$, and the total run time of the phases is

$$\sum_{i=1}^{m} O\left(\frac{n}{k2^{i-1}}\right) = O\left(\frac{2n}{k}\right).$$

Following our asymptotic results in Section 4.2.2, the run time of the last part (i.e., running the simple Aloha-like scheme until neighbor discovery terminates) is

$$\Theta\left(\frac{n}{k\ln n}\ln\frac{n}{\ln n}\right) = \Theta\left(\frac{n}{k}\right).$$

Therefore, the neighbor discovery time using this scheme is $T = O(\frac{2n}{k})$, yielding a $\ln n$ improvement over the simple Aloha-like scheme.

We now prove that in each phase i, when choosing W_i and p_i properly, at least half of the active nodes are successful at the end of the phase w.h.p., as stated in the following theorem.

THEOREM 1. *Let S_i denote the set of successful nodes at the end of phase i. When $k \geq 3$, let $W_i = \eta n_i/(k-2)$ with $\eta \geq 115$ and $p_i = (k-2)/n_i$; when $k = 2$, let $W_i = \eta n_i/2$ with $\eta \geq 85$ and $p_i = 1/n_i$. We then have for $\forall k \geq 2$,*

$$\Pr[|S_i| < n_i/2] < e^{-n_i/k}.$$

PROOF. Since all the phases are similar, it is sufficient to prove the above for phase 1; proof for the rest of the phases are similar. For ease of exposition, in the following, we drop the subscript that represents the index of a phase, and simply prove the following. Let S denote the set of successful nodes at the end of phase 1, and let W denote the number of slots in phase 1. When $k \geq 3$, let $W = \eta n/(k-2)$ with $\eta \geq 83$ and the transmission probability $p = (k-2)/n$; when $k = 2$, let $W = \eta n/2$ with $\eta \geq 85$ and $p = 1/n$. We then have $\forall k \geq 2$,

$$\Pr[|S| < n/2] < e^{-n/k}.$$

We first prove the results for $k \geq 3$. Let X_i^t, for $1 \leq t \leq W, 1 \leq i \leq n$, be a family of i.i.d. indicator random variables, each with expectation $E[X_i^t] = p$. We refer to the variables $X_1^t, \ldots, X_n^t$ as those in *slot* t, and say that t is a *good slot* if no more than k of the variables take the value 1. For each pair t, i, we define

$$Y_i^t \triangleq \begin{cases} X_i^t & \text{if } t \text{ is a good slot,} \\ 0 & \text{otherwise.} \end{cases}$$

Setting $p = (k-2)/n$ note that, for each t, $E[\sum_i X_i^t] = k-2$. The random variable $\sum_i X_i^t$ is binomially distributed; it follows that the mean and median are separated by no more than $\ln(2) \approx .69 < 1$ (see [9]) and thus, for each t,

$$\Pr\Big[\sum_i X_i^t \geq k-1\Big] \leq 1/2.$$

We conclude that

$$\Pr[t \text{ is a good slot}] \geq 1/2,$$

$$\Pr[Y_i^t = 1] = \Pr[X_i^t = 1] \cdot \Pr\Big[\sum_{j \neq i} X_j^t \leq k-1\Big] \geq \frac{p}{2}. \quad (13)$$

We now want to show that with high probability

$$\sum_t Y_i^t \geq 3$$

for at least half of the i. Now, for each i, t define

$$P_i^t = \min\Big(3, \sum_{t' \leq t} Y_i^{t'}\Big), \quad S^t = \{i \mid P_i^t = 3\},$$

and $P^t = \sum_i P_i^t$. For convenience, we additionally define $P^0 = 0$ and $S^W = S$. Reformulating our goal above, we wish to show that for sufficiently large W, $\Pr[|S^W| < n/2] \leq e^{-n/k}$.

Observe that $P^t \geq \frac{5}{2}n \Rightarrow |S^t| \geq n/2$ so we will focus on establishing that, for sufficiently large W,

$$\Pr[P^W < 5n/2] \leq e^{-n/k}.$$

To this end, define

$$Z_t = \begin{cases} P^t - P^{t-1} & \text{if } |S^{t-1}| \leq n/2, \\ k & \text{if } |S^{t-1}| > n/2. \end{cases}$$

Let $Z = \sum_{t=1}^W Z_t$. Hence $Z = P^W - P^0 = P^W$. Note that if $Z \geq 5n/2$ it follows that $|S^W| \geq n/2$, the event we wish to lower bound. Observe also that

$$E[Z_t \mid Y_i^s, s < t]$$

depends only on $|S^{t-1}|$, the number of "saturated" vertices (i.e., those that have already transmitted three times successfully) for $t-1$, and when $|S^{t-1}| \leq n/2$ we have

$$\begin{aligned} E[Z_t \mid Y_i^s, s < t] &= \sum_{i \notin S^{t-1}} Y_i^t \\ &\geq (n - |S^{t-1}|) \cdot p/2 \\ &\geq (k-2)/4; \end{aligned}$$

of course, when $|S^{t-1}| > n/2$ we have

$$E[Z_t \mid Y_i^s, s < t] = k.$$

In any case, $E[Z_t \mid Y_i^s, s < t] \geq (k-2)/4$. In preparation for application of Azuma's inequality (Theorem 4.16, [20]), define

$$\tilde{Z}_t = E[Z \mid X_i^s, s \leq t]$$

and observe that $E[\tilde{Z}_t \mid \tilde{Z}_{t-1}] = \tilde{Z}_{t-1}$ so the sequence forms a martingale. Additionally,

$$\tilde{Z}_0 = E[Z] \geq W(k-2)/4 \quad \text{and} \quad |\tilde{Z}_t - \tilde{Z}_{t-1}| \leq k, \text{for all } t.$$

(The second of these two claims follows from the fact that if two assignments to the X_i^t differ only among the $\{X_i^s\}$ for a particular s, the value of Z can change by no more than k. See [1, §7.4] for a detailed treatment.)

Applying Azuma's inequality to the random variables $\tilde{Z}_t$ with $W = \eta n/(k-2)$, we have

$$\Pr[|S^W| < n/2] \leq \Pr[Z < 5n/2] = \Pr[\tilde{Z}^W < 5n/2].$$

As $E[\tilde{Z^W}] = E[Z] \geq W(k-2)/4 = \eta n/4$, we have

$$\begin{aligned} \Pr[\tilde{Z}^W < 5n/2] &\leq \Pr\left[\tilde{Z}^W < E[\tilde{Z}^W] - \frac{n(\eta-10)}{4}\right] \\ &\leq \exp\left(-\frac{n^2(\eta-10)^2(k-2)}{32\eta n k^2}\right). \end{aligned}$$

When $k = 3$, this expression is no more than $e^{-n/k}$ when $\eta \geq 115$. When $k \geq 4$, we have $k/2 \leq (k-2)$ and this is no more than

$$\exp\left(-\frac{n(\eta-10)^2}{64\eta k}\right).$$

Setting $\eta \geq 83$, $(\eta-10)^2 > 64\eta$ and this expression is no more than $e^{-n/k}$.

We now prove the results for $k = 2$. The proof is similar as that for $k \geq 3$. However, instead of (13), we have

$$\begin{aligned} \Pr[Y_i^t = 1] &= p[(1-p)^{n-1} + \binom{n-1}{1} p(1-p)^{n-2}] \\ &= \frac{2}{n}(1-\frac{1}{n})^{n-1} \\ &\geq \frac{2}{ne} = \frac{2p}{e}. \end{aligned}$$

Following similar steps as those for $k \geq 3$, we have when $W = \eta n/2$ with $\eta \geq 85$,

$$\Pr[|S^W| < n/2] \leq \Pr[\tilde{Z}^W < 5n/2] \leq e^{-n/k}.$$

□

6. UNKNOWN NUMBER OF NEIGHBORS

So far, we have assumed that each node knows n, and hence the number of its neighbors. We now describe how the simple and adaptive Aloha-like schemes can be extended to handle the scenario where this information is not known *a priori*. Our schemes are similar in spirit to the algorithm proposed in [24] for the case of SPR networks. The main idea is to double the estimate for n until a stopping rule is satisfied. We next describe these schemes in more detail.

6.1 Aloha-like scheme

The algorithm runs in *stages*. In stage i, each node assumes that there are 2^i neighbors and runs the simple Aloha-like scheme for a duration of W_i slots with the optimal transmission probability determined by optimizing (1) assuming $n = 2^i$. We choose W_i so that all nodes discover their respective neighbors w.h.p. in stage i. Based on our asymptotic analysis in Section 4.2, we choose $W_i = c_1 \ln 2^i$ under idealized MPR and $W_i = d_1 2^i \ln 2^i / k$ under MPR-k, where c_1 and d_1 are constants, defined in Sections 4.2.1 and 4.2.2, respectively. Each node records the number of neighbors that it discovers in stage i, denoted as N_i, and decides to terminate the neighbor discovery process in stage $i + 1$ if the stopping rule

$$N_i \geq 2^{i-1} \wedge N_{i+1} \leq 2^i \tag{14}$$

is satisfied.

With our choice of W_i, a node can correctly terminate at the end of stage $\lceil \log_2 n \rceil + 1$ and discovers all its neighbors w.h.p. This is because under this choice of W_i, we have $N_{\lceil \log_2 n \rceil} = n \geq 2^{\lceil \log_2 n \rceil - 1}$ and $N_{\lceil \log_2 n \rceil + 1} = n \leq 2^{\lceil \log_2 n \rceil}$ w.h.p., which satisfies the stopping rule at the end of stage $\lceil \log_2 n \rceil + 1$.

We apply the above scheme to the idealized MPR and MPR-k ($k = 2$ and 8) models when the number of neighbors is 5, 10, 20, 50, 100 or 200. For each setting, we repeat the simulation 100 times. We find that indeed all nodes find their respective neighbors.

We next briefly describe the total neighbor discovery time of the above scheme. Under idealized MPR, the neighbor discovery time is

$$\sum_{i=1}^{\lceil \log_2 n \rceil} c_1 \ln 2^i = \Theta(\log_2 n \ln n).$$

Under MPR-k, the neighbor discovery time is

$$\sum_{i=1}^{\lceil \log_2 n \rceil} \frac{d_1 2^i \ln 2^i}{k} = 2\Theta\left(\frac{n \ln n}{k}\right).$$

From the above, we observe that not knowing n leads to at most a $\log_2 n$ factor slowdown under idealized MPR, and a factor of two slowdown under MPR-k.

6.2 Adaptive Aloha-like scheme

We now extend the adaptive Aloha-like scheme that we designed for MPR-k to the case where we do not have an estimate of n. Our extension is similar to that for the simple Aloha-like scheme. More specifically, the scheme runs in stages doubling the estimate for n after each stage. In the ith stage, we assume there are 2^i neighbors and run the adaptive Aloha-like scheme as described in Section 5. Therefore, each stage contains multiple phases with the total run time being $O(\frac{2 \cdot 2^i}{k})$. We continue the stages until the stopping rule (14) is satisfied. Similar to the extension to the simple Aloha-like scheme, the neighbor discovery stops in stage $\lceil \log_2 n \rceil + 1$. We then have a total run time that equals

$$\sum_{i=1}^{\lceil \log_2 n \rceil} O\left(\frac{2 \cdot 2^i}{k}\right) = O\left(\frac{4n}{k}\right).$$

Again, we observe not knowing n leads to a factor of two slowdown.

7. ASYNCHRONOUS TRANSMISSIONS

We now consider the scenarios where nodes transmit asynchronously. In particular, we consider the following asynchronous slotted model. We assume each node transmits with probability p at the beginning of a slot, the length of a slot equals transmission duration, τ, and the slots of the nodes are not aligned. Consider two arbitrary nodes, i and j. Suppose i transmits at time t. Let p_s denote the probability that i is discovered by j. Since the slots of the nodes are not synchronized, to hear from i, j cannot transmit in the two adjacent slots that overlap with the interval $[t, t + \tau]$. The probability of this event is $(1 - p)^2$. Consider another node l. It does not have a concurrent transmission with node i if it does not transmit in either of the two adjacent slots that overlap with the interval $[t, t+\tau]$. Therefore, the probability of no concurrent transmission with node i is $(1 - p)^2$, and the probability of concurrent transmission is $1 - (1 - p)^2$. For ease of notation, let $q = (1 - p)^2$. We then have (again $k = n - 1$ under idealized MPR)

$$p_s = pq \sum_{i=0}^{k-1} \binom{n-2}{i} (1 - q)^i q^{n-2-i}. \tag{15}$$

We obtain the optimal transmission probability, p^*, and optimal p_s^* in the following models.

- Idealized MPR: in this model, We have $p_s = (1-p)^2 p$. Hence $p^* = 1/3$, and $p_s^* = 4/27$.
- SPR model: in this model, we have $p_s = pq^{n-1} = p(1-p)^{2n-2}$. Therefore, we have $p^* = 1/(2n-1) \approx \frac{1}{2n}$, and $p_s^* \approx \frac{1}{2ne}$.
- MPR-k: In this model, we can obtain p^* by solving (15) numerically. To provide insights, similar to the synchronous transmission case in Section 4, we show that $p^* = \psi k/n$, where ψ is between two constants (detailed statement and proof are omitted). Our numerical results indicate that the optimal transmission probability is approximately $k/(ne)$, half as that of the synchronized case, for relatively large k, and larger than $k/(ne)$ for small k (figures omitted).

Since the optimal transmission probability under the asynchronous model is approximately half as that in the synchronous model, we have similar asymptotic bounds as those in the synchronous model. Extending the adaptive Aloha-like scheme to this asynchronous model is straightforward. We can further extend the schemes when n is unknown to this asynchronous model by requiring each node to embed its stage number in the messages. For all the cases, the asynchronous model leads to a factor of two slowdown compared to the synchronous model.

8. CONCLUSIONS AND FUTURE WORK

In this paper, we designed and analyzed randomized algorithms for neighbor discovery under various MPR models. We started with a simple Aloha-like algorithm that assumes synchronous node transmissions and *a priori* knowledge of the number of neighbors. We showed that the total neighbor discovery time for this algorithm is $\Theta(\ln n)$ under the idealized MPR model, and $\Theta(\frac{n \ln n}{k})$ under the MPR-k model. We further designed an adaptive Aloha-like algorithm for the case when a node knows whether its transmission is successful or not (e.g., based on feedbacks from other nodes), and showed that it provides a $\ln n$ improvement over the simple Aloha-like scheme. Last, we extended our schemes to accomodate a number of practical scenarios such as when the number of neighbors is not known beforehand and the nodes are allowed to transmit asynchronously, and analyze the performance of our algorithms in each of these cases.

As future work, we are pursuing two interesting directions: (1) extend our study to more generalized MPR models, and (2) investigate neighbor discovery in multi-hop MPR networks.

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APPENDIX

A. OPTIMAL TRANSMISSION PROBABILITY FOR MPR-K MODEL

We next state and prove a theorem on the optimal transmission probability for the MPR-k model in Section 4.

THEOREM 2. *Under the simple Aloha-like algorithm, assuming the number of neighbors is known beforehand and synchronous transmission, for the MPR-k model, the optimal transmission probability* $p^* = \alpha k/n$, *where (i)* $\alpha \in (0.09, 3.55)$ *for* $k \geq 4$, *(ii)* $\alpha \in (0.07, 6.38)$ *for* $k = 2$, *and (iii)* $\alpha \in (0.11, 4.17)$ *for* $k = 3$.

PROOF. We prove the above theorem by considering three cases: $k \geq 4$, $k = 2$, and $k = 3$. In each case, we first derive a lower bound on the optimal p_s^*, and then derive the constants stated in the theorem. Since we are considering the MPR-k case, we have $n \geq k+2$. Let $X = X_1 + \cdots + X_{n-2}$, where $X_i = 1$ when node i transmits, and $X_i = 0$ otherwise. Then X follows a Binomial distribution, and (1) can be rewritten as

$$p_s = p(1-p)\Pr(X < k). \tag{16}$$

- Case 1 ($k \geq 4$). We first derive a lower bound on p_s^*, denoted by $\underline{p_s^*}$, by taking $p = (k-3)/(n-2)$. The mean of the Binomial random variable, X, is $(n-2)p = k-3$. Since the mean and the median are at most $\ln 2$ apart [9], the median is in $[k-3-\ln 2, k-3+\ln 2]$. Since $k-1 > k-3+\ln 2$, we have $\Pr(X < k) \geq 1/2$. Since $n \geq k+2$, we have

$$p = \frac{k-3}{n-2} \leq \frac{k-3}{k} = 1 - \frac{3}{k}.$$

Therefore,

$$1 - p \geq \frac{3}{k}.$$

Summarizing the above, we obtain a lower bound

$$\underline{p_s^*} = \frac{k-3}{n-2}\cdot\frac{3}{k}\cdot\frac{1}{2} = \frac{3(k-3)}{2k(n-2)}. \tag{17}$$

Note that $p_s^* < p^* = \alpha k/n$. We have a contradiction, $p_s^* < \alpha k/n \leq \underline{p_s^*}$, when

$$\alpha \leq \frac{3(k-3)n}{2k(n-2)k}.$$

Since $k \geq 4$, we have

$$\frac{3(k-3)n}{2k(n-2)k} \geq \frac{3(k-3)}{2k^2} \geq \frac{3}{32} \approx 0.09.$$

Therefore, we need $\alpha > 0.09$.
On the other hand, note that

$$p_s^* < \frac{\alpha k}{n}\Pr(X < k),$$

and using the bound of Binomial distribution (Theorem A.1.13 in [20]), we have

$$\begin{aligned} \Pr(X < k) &< e^{-((n-2)p^*-k)^2/2(n-2)p^*} \\ &\approx e^{-(np^*-k)^2/2np^*} \\ &= e^{-\frac{(\alpha k-k)^2}{2\alpha k}} \\ &= e^{-\frac{k(\alpha-1)^2}{2\alpha}}. \end{aligned}$$

Hence,

$$p_s^* < \frac{\alpha k}{n}e^{-\frac{k(\alpha-1)^2}{2\alpha}}. \tag{18}$$

We next derive the condition under which

$$\frac{\alpha k}{n}e^{-\frac{k(\alpha-1)^2}{2\alpha}} \leq \frac{3(k-3)}{2k(n-2)},$$

that is, the condition under which

$$\alpha e^{-\frac{k(\alpha-1)^2}{2\alpha}} \leq \frac{3(k-3)n}{2k(n-2)k}. \tag{19}$$

In (19), the left hand side is a decreasing function of k, and hence the maximum value is $\alpha e^{-\frac{4(\alpha-1)^2}{2\alpha}}$. On the other hand, as described earlier, the right hand side is larger than 0.09. We have $\alpha e^{-\frac{4(\alpha-1)^2}{2\alpha}} < 0.09$ when $\alpha > 3.55$. That is, when $\alpha > 3.55$, the inequality in (19) is satisfied, leading to $p^* < \underline{p_s^*}$, a contradiction. Hence, we need $\alpha < 3.55$. In summary, we have shown that α must lie in $(0.09, 3.55)$ when $k \geq 4$.

- Case 2 ($k = 2$). Taking $p = 1/(n-2)$ yields

$$\begin{aligned} p_s &= \frac{1}{n-2}(1-\frac{1}{n-2})^{n-2}(1-\frac{1}{n-2}) \\ &+ (n-2)\frac{1}{(n-2)^2}(1-\frac{1}{n-2})^{n-2} \\ &\geq \frac{1}{n-2}(1-\frac{1}{n-2})e^{-2} + \frac{1}{n-2}e^{-2} \\ &= e^{-2}\left[\frac{1}{n-2}(1-\frac{1}{n-2}+1)\right] \\ &\geq e^{-2}\frac{1}{n-2} = \frac{1}{e^2(n-2)}. \end{aligned}$$

Therefore, we derive a lower bound $\underline{p_s^*}$ as

$$\underline{p_s^*} = \frac{1}{e^2(n-2)}. \tag{20}$$

Note that $p_s^* < p^* = \alpha k/n = 2\alpha/n$. We have a contradiction, $p_s^* < 2\alpha/n \leq \underline{p_s^*}$, when

$$\alpha \leq \frac{n}{n-2}\frac{1}{2e^2}.$$

Since

$$\frac{n}{n-2}\frac{1}{2e^2} \geq \frac{1}{2e^2} \approx 0.07,$$

we need $\alpha > 0.07$.

On the other hand, similar to the case where $k \geq 4$, we have (18). We next derive the condition under which

$$\frac{\alpha k}{n} e^{-\frac{k(\alpha-1)^2}{2\alpha}} = \frac{2\alpha}{n} e^{-\frac{(\alpha-1)^2}{\alpha}} \leq \frac{1}{e^2(n-2)},$$

that is, the condition under which

$$\alpha e^{-\frac{(\alpha-1)^2}{\alpha}} < \frac{1}{2e^2}\frac{n}{n-2}.$$

The right hand side is larger than 0.07, while the left hand side is less than 0.07 when $\alpha \geq 6.38$. Therefore, $\alpha \geq 6.38$ leads to $p_s^* < \underline{p_s^*}$, a contradiction. Therefore, we need $\alpha < 6.38$. In summary, we have shown that α must lie in $(0.07, 6.38)$ when $k = 2$.

- Case 3 ($k = 3$). Taking $p = 1/(n-3)$ yields

$$\begin{aligned} p_s &= \frac{1}{n-3}(1-\frac{1}{n-3})^{n-3}(1-\frac{1}{n-3})^2 \\ &+ (n-2)\frac{1}{(n-3)^2}(1-\frac{1}{n-3})^{n-3}(1-\frac{3}{n-3}) \\ &+ \frac{n-2}{2}\frac{1}{(n-3)^2}(1-\frac{3}{n-3})^{n-3} \\ &\geq e^{-2}\frac{1}{n-3}(1-\frac{1}{n-3})^2 \\ &+ e^{-2}\left[\frac{n-2}{(n-3)^2}(1-\frac{1}{n-3}) + \frac{n-2}{2}\frac{1}{(n-3)^2}\right] \\ &= \frac{5n-18}{2e^2(n-3)^2}. \end{aligned}$$

Therefore, we derive a lower bound $\underline{p_s^*}$ as

$$\underline{p_s^*} = \frac{5n-18}{2e^2(n-3)^2}. \quad (21)$$

Note that $p_s^* < p^* = \alpha k/n = 3\alpha/n$. We have a contradiction, $p_s^* < 3\alpha/n \leq \underline{p_s^*}$ when $\alpha \leq \frac{n(5n-18)}{6e^2(n-3)^2}$. Since

$$\frac{n(5n-18)}{6e^2(n-3)^2} \geq \frac{5}{6e^2} \approx 0.11,$$

we need $\alpha > 0.11$.

On the other hand, similar to the case where $k \geq 4$, we have (18). We next derive the condition under which

$$p_s^* < \frac{\alpha k}{n} e^{-\frac{k(\alpha-1)^2}{2\alpha}} = \frac{3\alpha}{n} e^{-\frac{3(\alpha-1)^2}{2\alpha}} < \frac{5n-18}{2e^2(n-3)^2},$$

that is, the condition under which

$$\alpha e^{-\frac{3(\alpha-1)^2}{2\alpha}} < \frac{n(5n-18)}{6e^2(n-3)^2}.$$

The right hand side is larger than 0.11, while the left hand side is less than 0.11 when $\alpha \geq 4.17$. Therefore, $\alpha \geq 4.17$ leads to $p_s^* < \underline{p_s^*}$, a contradiction. Therefore, we need $\alpha < 4.17$. In summary, we have shown that α must lie in $(0.11, 4.17)$ when $k = 3$.

Summarizing the above three cases, we have finished the proof. □

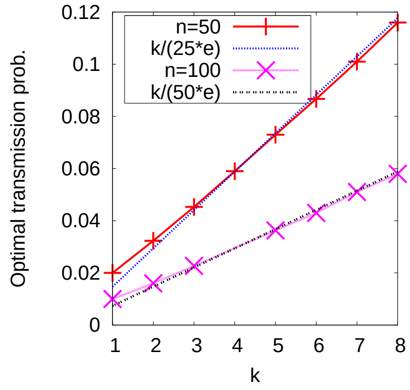

Figure 2: Numerical results: optimal transmission probability versus k when $n = 50$ or 100 under the MPR-k model in Section 4. The two dashed lines represent $2k/(ne)$.

B. NUMERICAL RESULTS

We next present numerical results on the optimal transmission probability under the MPR-k model in Section 4. Fig. 2 plots the optimal transmission probabilities under MPR-k as k increases from 2 to 8 when the clique size, n, is 50 and 100. We observe that the optimal transmission probability is approximately a linear function of k/n. Furthermore, for relatively large k, it is approximately $2k/(ne)$; and for small k, it is larger than $2k/(ne)$. Numerical results under other settings of n and k reveal a similar trend (figures omitted). This trend is consistent with Theorem 2 in Appendix A.