An Effective Method to Reduce Inventory in Job Shops

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Abstract — Inventory plays a major role in deciding the overall manufacturing costs, and a good scheduling system should balance the on-time delivery of products versus low work-in-process (WIP) inventory. In this paper, the “CONstant Work-In-Process” (CONWIP) concept is applied to job shop scheduling to effectively control WIP inventory. A new mathematical formulation of CONWIP-based job shop scheduling with a separable structure is presented. By using a synergistic combination of Lagrangian relaxation, dynamic programming, and heuristic methods, good schedules are obtained in a reasonable amount of computation time. Results show that the new method can directly control WIP levels while maintaining good on-time delivery performance.

I. Introduction

Inventory plays an important role in scheduling because it has major impact on overall manufacturing costs. Excessive inventory creates the needs for floor space, equipment, and manpower to transport, stock, and manage the inventory with no added value. Furthermore, defects are difficult to detect, creating high rework and scraps. This study was motivated by a request from Sikorsky Aircraft, one of our industrial partners, to better control WIP for job shops while maintaining good on-time delivery performance.

In most job shops, the “push-based” Material Requirement Planning (MRP) is used as a production planning and scheduling tool. The WIP of these shops, however, is usually high because of long lead-times of MRP to handle uncertainties, and the ignorance of machine capacities during the planning process. The “pull-based” just-in-time (JIT) production control manages WIP but is only applicable to high-volume/low-variety settings. Most other scheduling systems do not directly control inventory. Recently, the “CONstant Work-In-Process” (CONWIP) concept receives significant attention because of its simplicity in implementation, effectiveness in inventory control, and a few other advantages. Originally introduced for serial production lines, the CONWIP requires that the number of parts simultaneously in the system to be less than or equal to a certain constant – the maximum level of WIP allowed. It is argued that the CONWIP system is more effective than MRP, and is more widely applicable than JIT (Spearman, Woodruff, and Hopp, 1990).

For CONWIP-based serial production lines, the First-Come-First-Serve (FCFS) rule is usually used as the dispatching rule at work centers, and the key issue is to sequence the releases of raw materials to adequately utilize the machines and increase system throughput. Several heuristic methods have been presented in the literature. A heuristic method of releasing the highest priority job first for a multi-station network was presented in Wein (1992), and a static work balance sequencing rule was used in scheduling networks of queues in Duenyas (1993). CONWIP production lines was modeled as a tandem queuing system with a constant WIP level (number of containers) in Herer and Masin (1997), and the releasing sequence was obtained based on mean throughput and flow time using mean value analysis (MVA). The quality of schedules obtained by heuristic methods, however, is difficult to quantify and may be far from satisfaction.

The CONWIP concept can in principle be applied to other manufacturing settings (e.g., job shop). For CONWIP-based job shop scheduling, not only the releasing sequence of raw materials is important, but also the sequences of remaining operations on various machines are critical. Consequently, the problem is more complicated, and very little result has been reported. Based on our previous work (Luh, et al., 1993; Luh, et al., 1997), a new formulation for CONWIP-based job shop scheduling is presented in Section II. Unlike other existing models, the CONWIP constraints presented here are additive, and maintain the separability of the overall formulation. These CONWIP constraints are treated as extended machine capacity constraints, and are effectively handled by using Lagrangian relaxation as presented in Section III. To alleviate the difficulty caused by the increased number of Lagrangian multipliers, the newly developed surrogate subgradient method is used at the high level to speed up convergence. Numerical testing presented in Section IV shows that the method can directly control WIP level while maintaining good on-time delivery performance.
II. Problem Formulation

Extending the results of Luh, et al., (1993), the CONWIP-based job shop scheduling is formulated as an integer optimization problem. Assume that there are H machine types, each containing one or several identical machines. These machine types are indexed by \( h, h = 0, \ldots, H-1 \). There are I parts, and part \( i \), \( 0 \leq i \leq I-1 \), has due date \( d_i \) and priority \( \omega_i \) and consists of \( J_i \), non-preemptive sequential operations. Operation \( j, 0 \leq j \leq J_i-1 \), of part \( i \) is denoted as \((i, j)\), and requires a machine of type \( h \) belonging to a given set of eligible machine types \( H_i \) for a specified processing time \( p_{ih} \). The scheduling horizon consists of \( K \) time units, indexed by \( k (k = 0, \ldots, K-1) \). The constraints and the objective function of the formulation are presented below.

Machine Capacity Constraints

The machine capacity constraints state that the total number of operations assigned to machine type \( h \) must be less than or equal to the number of machines available at any time, \( i.e., \)

\[
\sum_{i=0}^{J_i-1} \sum_{j=0}^{J-1} \delta_{ijh} \leq M_{kh}, \ k = 0, \ldots, K-1; \ h \in H, \quad (2.1)
\]

where \( \delta_{ijh} \) is a 0-1 integer variable equal to 1 if operation \((i, j)\) is processed on machine type \( h \) at time \( k \), and 0 otherwise. That is,

\[
\delta_{ijh} = \begin{cases} 
1, & \text{if } b_{ij} \leq k \leq c_{ij}, \\
0, & \text{otherwise,}
\end{cases}
\]

where \( b_{ij} \) and \( c_{ij} \) are the beginning and completion times of operation \((i, j)\), respectively.

CONWIP Constraints

The CONWIP constrains are usually modeled by having a fixed number of “containers” or “cards” in the system, and new materials cannot be released into the system unless there is a vacant container or card. For job shops, there is no container, and there is no card. However, it is clear that a part is counted as WIP after it is released to the shop up to its completion, \( i.e., \) from the beginning of its first operation to the completion of its last operation. A set of 0-1 integer “inventory variables” \( \{\zeta_{ik}\} \) is therefore introduced. It equals 1 when part \( i \) is in the shop at time \( k \), and 0 otherwise, \( i.e., \)

\[
\zeta_{ik} = \begin{cases} 
1, & \text{if } b_{i0} \leq k < c_{iJ_i-1}, \\
0, & \text{otherwise.}
\end{cases}
\]

An inventory variable of part \( i \) with release time equal 2 and completion time 14 is illustrated in Figure 1.

![Figure 1. An illustration of an inventory variable](image)

The new CONWIP constraints require the number of parts simultaneously in the shop to be less than or equal to \( W \), the maximum WIP inventory allowed at any time, \( i.e., \)

\[
\sum_{i=0}^{I-1} \zeta_{ik} \leq W, \ k = 0, \ldots, K-1. \quad (2.2)
\]

Operation Precedence Constraints

An operation cannot be started until its preceding operation has been finished, \( i.e., \)

\[
c_{ij} + 1 \leq b_{i,j+1}, \quad \forall (i, j), \quad (2.3)
\]

where \( c_{ij} \) and \( b_{i,j+1} \) are the completion time of operation \((i, j)\) and the beginning time of operation \((i, j+1)\), respectively. The term “1” is required in (2.3) since operation \((i, j)\) is assumed to be completed at the end of time unit \( c_{ij} \), and operation \((i, j+1)\) is assumed to be started at the beginning of time unit \( b_{i,j+1} \). For the same reason, the term “1” also appears in the following processing time requirements.

Processing time Requirements

Each operation must be assigned the required amount of processing time on a selected machine type \( h \) belonging to the given set of eligible machine types \( H_{ij} \), \( i.e., \)

\[
c_{ij} = b_{ij} + p_{ijh} - 1, \quad \forall (i, j); \ h \in H_{ij}. \quad (2.4)
\]

Objective Function

The goals of on-time delivery and low work-in-process inventory are modeled as weighted penalties on tardiness and on releasing raw materials too early (the earliness penalties) in the objective function, \( i.e., \)

\[
J = \sum_{i=0}^{I-1} \omega_i T_i^2 + \sum_{i=0}^{I-1} \beta_i E_i^2. \quad (2.5)
\]

In the above, the tardiness weight \( \omega_i \) of part \( i \) reflects the importance of the part, and tardiness \( T_i \) is the amount of overdue time, \( i.e., \) \( max (c_i - d_i, 0) \) with \( c_i \) the part completion time (the completion time of the part’s last operation) and \( d_i \) its due date. Earliness \( E_i \) is defined as the amount that part beginning time (the beginning time of the part’s first operation) leads the desired release time following Czerwinski and Luh (1994). The purpose of tardiness penalties is to maintain good on-time delivery performance, while the purpose of earliness penalties is to prevent too much work-in-process inventory.
penalties is to prevent releasing raw materials too early, indirectly controlling WIP inventory.

The overall problem is to minimize the objective function (2.5) subject to constraints (2.1), (2.2), (2.3), and (2.4) by selecting the machine types \( \{h_{ij}\} \) and beginning times \( \{b_{ij}\} \) for all operations. Once \( \{b_{ij}\} \) and \( \{h_{ij}\} \) are selected, \( \{c_{ij}\}, \{T_i\}, \{E_i\}, \{\delta_{ijkh}\} \) and \( \{\zeta_{ik}\} \) can be easily derived. Since all constraints are linear and the objective function is additive, the problem is “separable,” which is essential for Lagrangian relaxation to be effective.

III. Solution Methodology

Machine capacity constraints and CONWIP constraints are first relaxed by using Lagrangian multipliers. The “relaxed problem” can then be decomposed into smaller and easier part subproblems. These subproblems are solved by using backward dynamic programming (BDP) with stages corresponding to operations, precedence constraints embedded in state transitions, and the “inventory” component of the part costs (WIP cost) efficiently calculated by treating CONWIP as an “extended” machine type. Since the new CONWIP constraints introduce additional multipliers, this causes slow convergence of multipliers. To alleviate the difficulty, the newly developed surrogate subgradient method (Zhao, Luh, and Wang, 1999) is used to iteratively update these multipliers at the high level to speed up convergence. Finally, a heuristic method is developed to generate a feasible schedule based on subproblem solutions.

III.1. Lagrangian Relaxation

By using Lagrangian multipliers \( \pi_{kh} \) to relax machine capacity constraints and \( \rho_k \) the CONWIP constraints, the following relaxed problem is obtained:

\[
\begin{align*}
\min L, \quad & \text{with } L = \sum_i (\omega_iT_i^2 + \beta_iE_i^2) \\
& + \sum_{i} \sum_{jkh} \pi_{kh}\delta_{ijkh} - \sum_{kh} \pi_{kh}M_{kh} \\
& + \sum_i \sum_{k} \rho_k\zeta_{ik} - \sum_k \rho_kW,
\end{align*}
\]

subject to (2.3) and (2.4). By regrouping relevant terms, the relaxed problem can be decomposed into the following part-level subproblems:

\[
\begin{align*}
\min_{\{b_{ij}, h_{ij}\}} L_{ij}, \quad & \text{with } L_{ij} = \omega_iT_i^2 + \beta_iE_i^2 \\
& + \sum_{j=0}^{J_i-1} \sum_{k=b_{ij}}^{c_{ij}} \pi_{kh} + \sum_{k=b_{i0}}^{\infty} \rho_k,
\end{align*}
\]

subject to (2.3) and (2.4). The above cost \( L_i \) includes tardiness penalty, earliness penalty, the cost for using machines, and the cost for contributing to WIP.

III.2. Dynamic Programming

In solving part subproblems, the backward DP is used, with stages corresponding to operations and precedence constraints embedded in state transitions. Since inventory variables \( \{\zeta_{ik}\} \) involve queue times as well as operation processing times, the calculation of the WIP cost within BDP is not easy.

From another point of view, CONWIP can be treated as an extended machine type with capacity qual to \( \mathbf{W} \). The major difference between this extended machine type and a regular machine type is that a “WIP” operation does not have a fixed processing time. It starts as a part is released to the shop, and completes when its last operation is finished as explained earlier. Since undetermined queue times are involved, the completion time cannot be uniquely calculated for a given beginning time in advance. Nevertheless, the following identity holds:

\[
\sum_{k=b_{i0}}^{b_{iJ_i-1}} \rho_k = \sum_{k=0}^{b_{i0}-1} \rho_k - \sum_{k=0}^{b_{iJ_i-1}} \rho_k.
\]

Consequently, the WIP cost can be allocated to the first and the last operation, and be easily calculated.

The BDP algorithm starts from the last stage with the following terminal cost:

\[
V_{i,J_i-1}(b_{iJ_i-1},h_{iJ_i-1}) = \omega_iT_i^2 + \sum_{k=b_{i0}}^{b_{iJ_i-1}} \pi_{kh} + V_{i,J_i-1}(b_{iJ_i-1},h_{iJ_i-1})
\]

The cumulative cost when moving backward is calculated recursively by

\[
V_{i,j}(b_{ij},h_{ij}) = \min_{\{b_{i,j+1},h_{i,j+1}\}} \left( \beta_iE_i^2 - \sum_{k=0}^{b_{i0}-1} \rho_k \right) \Delta_i + \sum_{k=b_{ij}}^{c_{ij}} \pi_{kh} + V_{i,j+1}(b_{i,j+1},h_{i,j+1})
\]

where

\[
\Delta_i = \left( \beta_iE_i^2 - \sum_{k=0}^{b_{i0}-1} \rho_k \right) + \sum_{k=b_{ij}}^{c_{ij}} \pi_{kh}
\]

and

\[
\sum_{k=b_{i0}}^{b_{iJ_i-1}} \rho_k = \sum_{k=0}^{b_{i0}-1} \rho_k - \sum_{k=0}^{b_{iJ_i-1}} \rho_k.
\]
\[ 0 \leq j \leq J_i - 2, \quad (3.5) \]

where \( \Delta_i \) is an integer variable equal to one if operation \((i, j)\) is the first operation of part \(i\), and zero otherwise. The optimal \( L_i^* \) is obtained as the minimal cumulative cost at the first stage, and the optimal beginning times and the corresponding machine types can be obtained by tracing forwards the stages.

### III.3. Dual Problem

Given the optimal subproblem costs \( \{L_i^*\} \), the high level dual problem is obtained as:

\[
\max \ D, \\
\{\pi_{kh}, \rho_k\}
\]

with \[
D = \sum_i L_i^* - \sum_{kh} \pi_{kh}M_{kh} - \sum_k \rho_k W. \quad (3.6)
\]

The iterative resolution of the dual problem requires the dual function to be evaluated many times. Since each dual function evaluation involves solving all the subproblems once, it is very expensive and thus causes slow convergence of multipliers especially for practical problems with hundreds of parts to be scheduled. The difficulty also exists in this problem since the relaxation of the new constraints introduces additional multipliers. To alleviate this slow convergence, the recently developed “surrogate subgradient method” is used to solve the dual problem. In this method, only approximate optimization of one subproblem is needed to get a proper “surrogate” subgradient direction to update multipliers, as opposed to solving all the subproblems precisely. The improvement of convergence will be illustrated in Example 3.

### III.4. Obtaining Feasible Solution

The solutions for part subproblems, when put together, may not satisfy the once relaxed constraints. A modified version of the list scheduling heuristics of Luh, et al., (1993) is thus developed to generate a feasible schedule based on subproblem solutions. In the heuristics, operations are first arranged in a list in the ascending order of their subproblem beginning times. They are then assigned to machines according to this list as machines become available. When operations have same beginning times, they are sorted in the list according to the incremental cost as defined in Luh, et al., (1993), and the operation with higher incremental cost will be scheduled first. In addition, the WIP level is computed for each time \(k\), and only when the WIP level is smaller than \(W\) can a new part be released. Otherwise, new parts are delayed to the next time slot. The process is repeated until all operations are assigned, and the cost for this feasible schedule can be computed. This heuristics can be run many times as the multipliers are updated, and the schedule with the lowest cost is chosen as the final schedule.

### IV. Numerical Results

The method has been implemented in C++ on a Pentium II-400 personal computer. For simplicity in implementation, the CONWIP constraints are approximately satisfied in the heuristics. The WIP level increases by one when a part is released to the shop, and decreases by one when the last operation of a part is scheduled to begin (as opposed to when the last operation is completed). This is consistent with shop floor practice that small violations of the CONWIP constraints are allowed. Three examples are presented below to demonstrate the method and to present insights obtained. In the testing, all the multipliers are initialized to zero, and the surrogate subgradient method is used to update multipliers, as opposed to solving all the subproblems once, it is very expensive and thus causes slow convergence of multipliers especially for practical problems with hundreds of parts to be scheduled. The difficulty also exists in this problem since the relaxation of the new constraints introduces additional multipliers. To alleviate this slow convergence, the recently developed “surrogate subgradient method” is used to solve the dual problem. In this method, only approximate optimization of one subproblem is needed to get a proper “surrogate” subgradient direction to update multipliers, as opposed to solving all the subproblems precisely. The improvement of convergence will be illustrated in Example 3.

#### Table 1. Numerical Results for Example 1

<table>
<thead>
<tr>
<th>Case</th>
<th>Earliness Penalty</th>
<th>CONWIP Constraints</th>
<th>Average WIP</th>
<th>Tardiness Cost</th>
<th>Feasible Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>N</td>
<td>N</td>
<td>8.48</td>
<td>166170</td>
<td>166170</td>
</tr>
<tr>
<td>2</td>
<td>N</td>
<td>Y(W=10)</td>
<td>6.88</td>
<td>194850</td>
<td>194850</td>
</tr>
<tr>
<td>3</td>
<td>N</td>
<td>Y(W=5)</td>
<td>4.55</td>
<td>206130</td>
<td>206130</td>
</tr>
<tr>
<td>4</td>
<td>N</td>
<td>Y(W=2)</td>
<td>3.17</td>
<td>383125</td>
<td>383125</td>
</tr>
<tr>
<td>5</td>
<td>Y</td>
<td>N</td>
<td>5.41</td>
<td>168535</td>
<td>197855</td>
</tr>
<tr>
<td>6</td>
<td>Y</td>
<td>Y(W=10)</td>
<td>5.36</td>
<td>178145</td>
<td>208118</td>
</tr>
<tr>
<td>7</td>
<td>Y</td>
<td>Y(W=5)</td>
<td>5.08</td>
<td>194815</td>
<td>223908</td>
</tr>
<tr>
<td>8</td>
<td>Y</td>
<td>Y(W=2)</td>
<td>2.84</td>
<td>412245</td>
<td>436163</td>
</tr>
</tbody>
</table>

For Cases 1 to 4, earliness penalties are not included in the objective function. It can be seen from the table that the tardiness cost becomes larger as \(W\) decreases, and the rate of increase is steep as \(W\) reduces below a certain threshold. This threshold, however, may not be apparent by simply looking at the problem, and the tradeoff between on-time delivery and low WIP inventory may not be easy without such results.

With small earliness penalties included for Cases 5 to 8, similar results are obtained. With \(\beta_i = 0.5\), the effects of having earliness penalties depend on \(W\). When \(W\) is large, earliness penalties have relatively large impact to reduce WIP inventory levels (Case 1 versus...
Case 5, and Case 2 versus Case 6). For a small W (Case 3 versus Case 7, and Case 4 versus Case 8), the impact is small since the CONWIP constraints dominate the release decisions. To graphically illustrate the results, the average WIP levels versus tardiness costs for various cases are depicted in Figure 2.

To graphically illustrate the results, the average WIP levels versus tardiness costs for various cases are depicted in Figure 2.

**Example 2**

This example is to illustrate the effects of having CONWIP constraints in optimization. There are 100 parts each with multiple (two to twelve) operations to be processed on 5 different machines. Two methods are tested: the one presented above, and the other with the CONWIP constraints only included in the heuristics. Both methods have earliness penalties with $\beta_i \equiv 0.1$. Results for different sets of parameters are presented in Table 2.

<table>
<thead>
<tr>
<th>Case</th>
<th>CONWIP Constr.</th>
<th>Max WIP</th>
<th>Average WIP</th>
<th>Dual Cost (x1000000)</th>
<th>Feasible Cost (x1000000)</th>
<th>Duality Gap</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Y</td>
<td>95</td>
<td>38.25</td>
<td>452.9</td>
<td>452.9</td>
<td>0%</td>
</tr>
<tr>
<td>2</td>
<td>Y</td>
<td>95</td>
<td>38.25</td>
<td>452.9</td>
<td>452.9</td>
<td>0%</td>
</tr>
<tr>
<td>3</td>
<td>Y</td>
<td>95</td>
<td>38.25</td>
<td>452.9</td>
<td>452.9</td>
<td>0%</td>
</tr>
<tr>
<td>4</td>
<td>Y</td>
<td>95</td>
<td>38.25</td>
<td>452.9</td>
<td>452.9</td>
<td>0%</td>
</tr>
<tr>
<td>5</td>
<td>Y</td>
<td>95</td>
<td>38.25</td>
<td>452.9</td>
<td>452.9</td>
<td>0%</td>
</tr>
<tr>
<td>6</td>
<td>Y</td>
<td>95</td>
<td>38.25</td>
<td>452.9</td>
<td>452.9</td>
<td>0%</td>
</tr>
</tbody>
</table>

Although the average WIP levels shown in the table are below the maximum allowed levels for all the cases, the maximum WIP levels many times do hit the ceilings. It is also clear that having the CONWIP constraints in optimization significantly reduces feasible costs (mostly tardiness costs), and the effect is drastic as W decreases.

**Example 3**

Given a snap shot of a factory, a practical data set is used in this example. There are 180 parts each with multiple operations (two to fifteen) with a total of 1504 operations, and 14 different machines. The initial WIP (parts already in the shop before scheduling) is 30, and ten parts are due very late (around 70 days after the scheduling beginning time). Three cases were tested to examine the effectiveness of CONWIP constraints in controlling WIP, and to show the improvement of the convergence by using the surrogate subgradient method instead of the subgradient method. The algorithm stops after 10 minutes of CPU time, and the results are presented in Table 3.

In Case 1, there were no CONWIP constraints, and the WIP over the scheduling horizon is depicted in Figure 3. It can be seen from this figure that the WIP levels were very high for the first 20 days. This is mainly caused by the early release of parts to fully utilize machines and to complete parts early, regardless the WIP level. The parts with very late due dates were not released in view of the existence of earliness penalties. The solution quality, as measured in terms of duality gap, however, is good in this case.
releasing of some parts to keep WIP below 40. However, since the CONWIP constraints were included in optimization, the schedule obtained maintains good on-time delivery performance.

![Figure 4. WIP distribution for Case 2 in Example 3](image)

V. Conclusions

A new separable formulation for CONWIP-based job shop scheduling is presented. Testing results demonstrated that it can generate schedules with controllable WIP levels. At the same time, good on-time delivery performance is obtained. The surrogate subgradient method has also been used to improve convergence.

The formulation can also be easily extended. For example, different part types may vary in size and value, and it may be more reasonable to have different weights for different part types in calculating WIP levels. This CONstant Weighted WIP concept can be easily formulated by having weights for parts in (2.2). Another example is that a job shop may want to limit WIP levels down to individual part types or families of part types as opposed to having one WIP level for the entire factory. This can be similarly modeled by having the CONWIP constraints for selected part types or families of part types. Further improvement of the method in terms of extended formulation and solution quality is under way.

ACKNOWLEDGMENT. This research was supported in part by the National Science Foundation under grants DMI-9500037, DMI-9813176, and the United Technologies Research Center. The authors wish to thank Mr. Yuanhui Zhang and Mr. Xing Zhao of the University of Connecticut for their assistance and invaluable suggestions.

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