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Intuitive robust stability metric for PID control of self-regulating processes

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Abstract

Published methods establish how plant-model mismatch in the process gain and dead time impacts closed-loop stability. However, these methods assume no plant-model mismatch in the process time constant. The work presented here proposes the robust stability factor metric, *RSF*, to examine the effect of plant-model mismatch in the process gain, dead time, and time constant. The *RSF* is presented in two forms: an equation form and a visual form displayed on robustness plots derived from the Bode and Nyquist stability criteria. This understanding of robust stability is reinforced through visual examples of how closed-loop performance changes with various levels of plant-model mismatch. One example shows how plant-model mismatch in the time constant can impact closed-loop stability as much as plant-model mismatch in the gain and/or dead time. Theoretical discussion shows that the impact is greater for small dead time to time constant ratios. As the closed-loop time constant used in Internal Model Control (IMC) tuning decreases, the impact becomes significant for a larger range of dead time to time constant ratios. To complete the presentation, the *RSF* is used to compare the robust stability of IMC-PI tuning to other PI, PID, and PID with Filter tuning correlations.

Keywords: Robustness plot; Robust stability; PID; Controller tuning; IMC

1. Introduction

When tuning controllers, it is critical to evaluate robust stability along with performance. This paper proposes an intuitive metric for robust stability. This metric, while general in nature, is derived here for Internal Model Control (IMC) [1] tuned Proportional-Integral (PI) control of First Order plus Dead Time (FOPDT) self-regulating processes. The FOPDT model is described by:

$$G_P(s) = \frac{K_P e^{-\theta_P s}}{\tau_P s + 1}. \tag{1}$$

In Eq. (1), K_P is the process gain, τ_P is the process time constant, and θ_P is the process dead time. A FOPDT model is a good approximation of most chemical processes for the purpose controller tuning [2,3]. In fact, a FOPDT model fit generally provides a conservative overestimation of θ_P that leads to more robust tuning. This work focuses upon the FOPDT model because the IMC tuning correlation used is based upon the FOPDT model, as are many other tuning correlations as listed in [4–6].

The PI controller has the following form:

$$G_C(s) = K_C \left[1 + \frac{1}{\tau_I s}\right]. \tag{2}$$

In Eq. (2), K_C is the controller gain and τ_I is the reset time. The IMC tuning correlations for PI control of an open-loop stable FOPDT process are as follows [7–9]:

$$K_C = \frac{\tau_P}{K_P\left(\tau_C + \theta_P\right)}; \quad \tau_I = \tau_P. \tag{3}$$

These IMC-PI tuning correlations are so named because they are based upon the equivalence between a PI controller and an approximation of an IMC controller. Implementation of these correlations produces a PI controller, not an IMC controller.

The closed-loop time constant, τ_C, is the single adjustable IMC tuning parameter that allows the desired levels of performance and robust stability to be specified. Increase τ_C for a more robustly stable controller or decrease τ_C for a less robustly stable controller.

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1.1. Literature review

Consider the plant-model mismatch between a FOPDT plant and a FOPDT model. The gain margin, *GM*, indicates how much plant-model mismatch in K_P alone (assuming no plant-model mismatch in τ_P and θ_P) may be tolerated for the plant to remain closed-loop stable. Similarly, the phase margin, *PM*, indicates how much plant-model mismatch in θ_P alone (assuming no plant-model mismatch in K_P and τ_P) may be tolerated for the plant to remain closed-loop stable. Neither *GM* nor *PM* indicates how much plant-model mismatch in τ_P may be tolerated.

Robustness plots [10–12] summarize the impact of plant-model mismatch upon closed-loop stability. In these plots, stable and unstable regions are clearly visualized on a plot of plant-model mismatch in K_P versus plant-model mismatch in θ_P. Various published examples of robustness plots [10–16] assume no plant-model mismatch in τ_P. In fact, [13] and [16] suggest that plant-model mismatch in τ_P generally does not significantly affect the robustness plot based upon simulation studies. Of these various published examples, only [12] states that robustness plots are based upon the Bode stability criterion.

Robustness plots are similar to the stability region plots presented in [17]. With stability region plots, the controller tuning parameters are varied without any plant-model mismatch. In contrast, robustness plots fix the controller tuning parameters and vary the level of plant-model mismatch.

Robustness plots provide a basis for quantitative metrics that describe robust stability with a single number. Such metrics have been proposed [11,15] for the two-dimensional case that assumes no plant-model mismatch in τ_P. The work presented here offers a quantitative metric that incorporates plant-model mismatch in τ_P along with plant-model mismatch in K_P and θ_P.

1.2. Novel contributions

The work presented here offers two novel contributions, with the second building upon the first. The derived robustness plot leads to the first novel contribution, the proposed robust stability factor metric, *RSF*. It has a simple, intuitive definition that is generally applicable to various controllers and plant model forms. The use of the *RSF* is demonstrated in an analysis of robust stability and controller performance throughout the operating range of a very nonlinear heat exchanger in Section 3.1.

With a single number, *RSF* clearly quantifies the impact upon robust stability of plant-model mismatch in two or more parameters (e.g., K_P, τ_P, θ_P). In Section 3.2, the *RSF* values with and without plant-model mismatch in τ_P are shown to be significantly different. This illustrative example supports the general results presented in Section 4 through dimensionless analysis of robust stability based upon fundamental stability theory. This leads to the novel conclusion that plant-model mismatch in τ_P can significantly impact the *RSF* and the robustness plot.

2. Development of robustness plot and *RSF*

The proposed metric, *RSF*, is developed here based upon a robustness plot derived for IMC-tuned PI control of a FOPDT (self-regulating) plant with quantified plant-model mismatch. This section builds upon [12] by outlining the general procedure for producing a robustness plot. This general procedure can be extended to other model and controller forms.

The following ratios quantify plant-model mismatch by relating the actual plant parameter values to the model parameter values used to tune the controller:

$$a = \frac{K_{P,\text{Plant}}}{K_{P,\text{Model}}}; \quad b = \frac{\tau_{P,\text{Plant}}}{\tau_{P,\text{Model}}}; \quad \text{and} \quad c = \frac{\theta_{P,\text{Plant}}}{\theta_{P,\text{Model}}}. \tag{4}$$

If $a = 2$, the actual K_P of the plant has a value twice that of the K_P of the model used to tune the controller. Similarly, $b = 0.5$ indicates that the actual τ_P of the plant has a value half that of the τ_P of the model used to tune the controller. The robustness plots presented in Figs. 2 and 4 plot a versus c with b held at a fixed value of 1.

These robustness plots are based upon the open-loop frequency response characteristics of the system. These characteristics include the amplitude ratio, $AR_{OL,\text{Plant}}$, and phase angle, $\varphi_{OL,\text{Plant}}$, which are defined as functions of the frequency, ω, by Eqs. (5) and (6):

$$AR_{OL,\text{Plant}}(\omega) = AR_C(\omega) AR_{P,\text{Plant}}(\omega) \tag{5}$$

$$\varphi_{OL,\text{Plant}}(\omega) = \varphi_C(\omega) + \varphi_{P,\text{Plant}}(\omega). \tag{6}$$

AR_C and $AR_{P,\text{Plant}}$ are the amplitude ratios of the controller and process, respectively. Similarly, φ_C and $\varphi_{P,\text{Plant}}$ are the phase angles of the controller and process, respectively.

The amplitude ratio, AR_C, and phase angle, φ_C, of a PI controller are defined by Eqs. (7) and (8):

$$AR_C(\omega) = \frac{K_C}{\omega\tau_I}\sqrt{1 + (\omega\tau_I)^2} \tag{7}$$

$$\varphi_C(\omega) = \tan^{-1}(\omega\tau_I) - \frac{\pi}{2}. \tag{8}$$

The amplitude ratio, $AR_{P,\text{Plant}}$, and phase angle, $\phi_{P,\text{Plant}}$, of a FOPDT plant are defined by Eqs. (9) and (10) with plant-model mismatch described by a, b, and c (as defined in Eq. (4)):

$$AR_{P,\text{Plant}}(\omega) = \frac{aK_{P,\text{Model}}}{\sqrt{1 + \left(b\omega\tau_{P,\text{Model}}\right)^2}} \tag{9}$$

$$\varphi_{P,\text{Plant}}(\omega) = -\tan^{-1}\left(b\omega\tau_{P,\text{Model}}\right) - c\omega\theta_{P,\text{Model}}. \tag{10}$$

Use the Bode stability criterion [18,19], if applicable, to evaluate closed-loop stability based upon the open-loop frequency response characteristics, $AR_{OL,\text{Plant}}$ and $\varphi_{OL,\text{Plant}}$. If the Bode stability criterion is not applicable, evaluate closed-loop stability using the Revised Bode stability criterion [19] or the Nyquist stability criterion [18,20]. The robustness plot is developed by evaluating closed-loop stability over the possible levels of plant-model mismatch. Plant-model mismatch may be due to a variety of factors that include plant nonlinearity (see the example in Section 3.1).

If the Bode stability criterion is applicable, evaluate closed-loop stability by first solving for the phase crossover frequency, ω_P:

$$\varphi_{OL,\mathrm{Plant}}(\omega_P) = -\pi. \quad (11)$$

Then, the plant is closed-loop stable if the following condition is met:

$$AR_{OL,\mathrm{Plant}}(\omega_P) \leq 1. \quad (12)$$

Apply Eqs. (11) and (12) to Eqs. (5) and (6), respectively, to produce the following analytically-derived region of stability for IMC-tuned PI control of a FOPDT plant:

$$\forall a, c : 0 < a \leq \frac{\pi}{2c}\left(\frac{\tau_C}{\theta_{P,\mathrm{Model}}} + 1\right), \quad c > 0, b = 1. \quad (13)$$

The robustness plots shown as Figs. 2a, b, and 4a are based upon Eq. (13).

This paper proposes the robust stability factor, *RSF*, to provide quantitative information about the robustness plot with a single number. The *RSF* is defined to describe a stable sub-region on the robustness plot. If plant-model mismatch is considered in only two dimensions (K_P and θ_P), the sub-region is a square (described by Eq. (14)) on a two-dimensional robustness plot. If plant-model mismatch is considered in three dimensions (K_P, τ_P and θ_P), the sub-region is a cube (described by Eq. (15)) on a three-dimensional robustness plot.

$$\forall a, c : \frac{1}{F_{2\mathrm{D}}} \leq a, c \leq F_{2\mathrm{D}}, b = 1 \quad (14)$$

$$\forall a, b, c : \frac{1}{F_{3\mathrm{D}}} \leq a, b, c \leq F_{3\mathrm{D}}. \quad (15)$$

RSF_{2D} and RSF_{3D} are defined as the maximum values of F_{2D} and F_{3D}, respectively, producing sub-regions lying completely within the stable region of the robustness plot. Note that RSF_{2D} and RSF_{3D} have minimal values of 1 and are undefined if the point ($a = 1$, $b = 1$, $c = 1$) lies outside the stable region of the robustness plot. Any difference between RSF_{2D} and RSF_{3D} indicates that plant-model mismatch in τ_P affects the amount of plant-model mismatch that may be tolerated in both K_P and θ_P.

The RSF_{2D} for IMC-tuned PI control of a FOPDT plant is analytically-derived to be the following:

$$RSF_{2\mathrm{D}}(\tau_C) = \sqrt{\frac{\pi}{2}\left(\frac{\tau_C}{\theta_{P,\mathrm{Model}}} + 1\right)}. \quad (16)$$

This converts into the following equation for selecting τ_C to achieve a desired value of RSF_{2D}:

$$\tau_C(RSF_{2\mathrm{D}}) = \left(\frac{2}{\pi}RSF_{2\mathrm{D}}^2 - 1\right)\theta_{P,\mathrm{Model}}. \quad (17)$$

It is not possible to analytically derive the RSF_{3D} for IMC-tuned PI control of a FOPDT plant. Therefore, software tools (e.g. Excel, Matlab) are used to solve for RSF_{3D}. The following examples illustrate the use and meaning of the RSF_{2D} and RSF_{3D} metrics.

Table 1
IMC-PI tuning of heat exchanger (based upon FOPDT model: $K_P = -0.59$ °C/%CO, $\tau_P = 1.12$ min, $\theta_P = 0.85$ min)

	K_C (%CO/°C)	τ_I (min)	$RSF_{2\mathrm{D}}$	$RSF_{3\mathrm{D}}$
Tuning (a) ($\tau_C = 0.77$ min)	−1.17	1.12	1.73	1.62
Tuning (b) ($\tau_C = 2.87$ min)	−0.51	1.12	2.62	2.57

3. Illustrative examples

3.1. Example 1 — Heat exchanger

This first example examines PI control of a heat exchanger simulation based upon a first-principles (not FOPDT) model [2,3,21]. The heat exchanger cools a stream of hot liquid flowing through the "tube" part of the heat exchanger with cooling water flowing through the "shell" part. The PI controller regulates the exit temperature of this hot liquid by manipulating the flow of the cooling water into the jacket. Before entering the heat exchanger, the hot liquid mixes with a stream of warm liquid. The warm liquid flow rate acts as a disturbance to the process. Increasing the warm liquid flow from its normal operating level of 10 L/min increases the flow rate of the mixed stream while decreasing its temperature.

The heat exchanger was tuned using the procedure outlined in [2,3]. First, the design level of operation was determined to be an exit temperature of 140 °C with a warm liquid flow rate of 10 L/min. Second, dynamic data was collected from the process from a test around this design level of operation. Third, Loop-Pro™ [21] fit a FOPDT model (listed below the caption of Table 1) to this dynamic data. Fourth, [21] applied this FOPDT model to produce the IMC tuning correlations for the desired level of performance and robust stability (determined by the selection of τ_C). Table 1 lists the PI tuning parameters produced by the IMC tuning correlation for two values of τ_C (0.77 and 2.87 min).

The RSF_{2D} and RSF_{3D} values listed in Table 1 are greater for the larger τ_C value (2.87 min) than for the smaller τ_C value (0.77 min). This agrees with the general rule that robust stability (namely *RSF*) increases with increasing τ_C. The RSF_{2D} and RSF_{3D} values are only slightly different. This indicates that, for this case, plant-model mismatch in τ_P does not significantly impact the *RSF*. This is expected since the θ_P is nearly as large as the τ_P.

Fig. 1 demonstrates the impact of τ_C upon closed-loop performance for a step in the set point from 135 to 145 °C followed by a step in the warm liquid flow rate from 10 to 15 L/min. A $\tau_C = 0.77$ min produces a response that rises to the set point quickly but overshoots by over 10%. In contrast, a $\tau_C = 2.87$ min produces a response that rises to the set point with less speed but does so without any overshoot. The smaller τ_C (0.77 min), however, provides superior disturbance rejection performance in response to the change in warm liquid flow rate that occurs at a time of about 30 min.

However, one should consider robust stability along with performance when selecting a controller. The *RSF* values listed in Table 1 quantify the robust stability of both sets of tuning

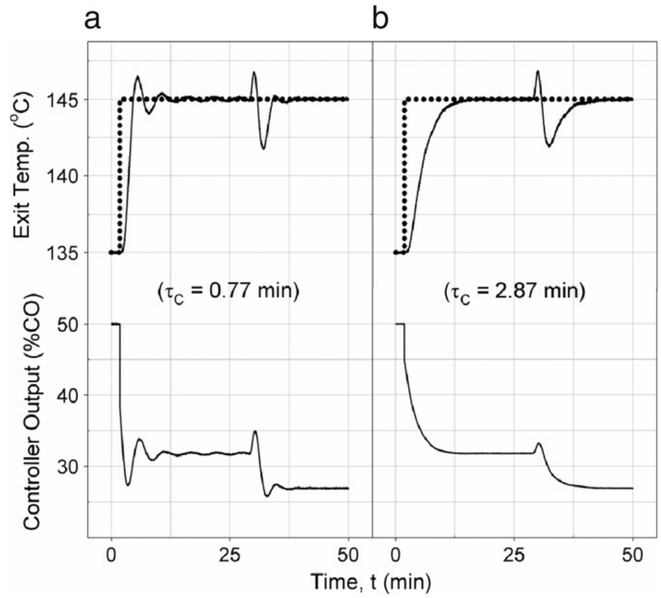

Fig. 1. Effect of τ_C upon performance around the Design Level (140 °C): $\tau_C = 0.77$ min (a) responds quickly but overshoots set point while $\tau_C = 2.87$ min (b) responds less quickly but does not overshoot set point.

parameters. The robustness plots presented in Fig. 2 provide a corresponding visual representation of robust stability. The RSF_{2D} values are depicted graphically by the squares shown on the plots. This illustrates the definition of RSF_{2D} presented in Eq. (14). As described in the definition, the square shown is the largest square around the design point ($a = 1$, $c = 1$) remaining entirely within the stable region. The design point is depicted by a dot in Fig. 2a and 2b.

Both Fig. 2a and 2b assume no plant-model mismatch in τ_P ($b = 1$). Therefore, both plots are based upon Eq. (13). While Fig. 2 is presented for the heat exchanger system, these two robustness plots (along with the corresponding values of RSF_{2D}) are generally applicable. Fig. 2a and 2b apply when

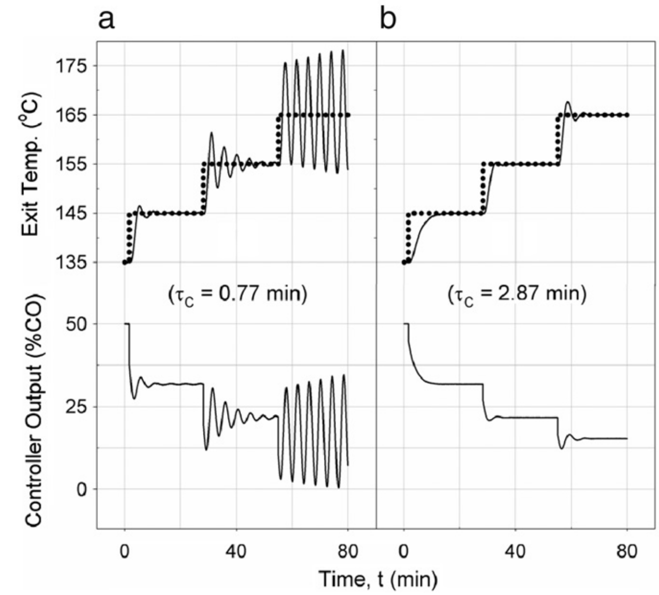

Fig. 3. Effect of τ_C upon performance away from the Design Level (140 °C): smaller τ_C (0.77 min) (a) responds quickly around 140 °C but becomes unstable around 165 °C while larger τ_C (2.87 min) (b) responds less quickly around 140 °C but remains robustly stable around 165 °C.

tuning a PI controller for any FOPDT plant with τ_C-to-$\theta_{P,\mathrm{Model}}$ ratios of 0.9 and 3.4, respectively.

The stability boundary curve of a robustness plot is the boundary between the stable and unstable region. The gain margin, GM, is the point ($a = GM$, $c = 1$) on the stability boundary curve where there is only plant-model mismatch in K_P. Similarly, a dead time margin, DM, may be defined as the point ($a = 1$, $c = DM$) on the stability boundary curve where there is only plant-model mismatch in θ_P.

The K_p and θ_P of the nonlinear heat exchanger vary with exit temperature [3]. This is depicted in Fig. 2a and 2b with a line beginning at the design point (defined for 140 °C) and

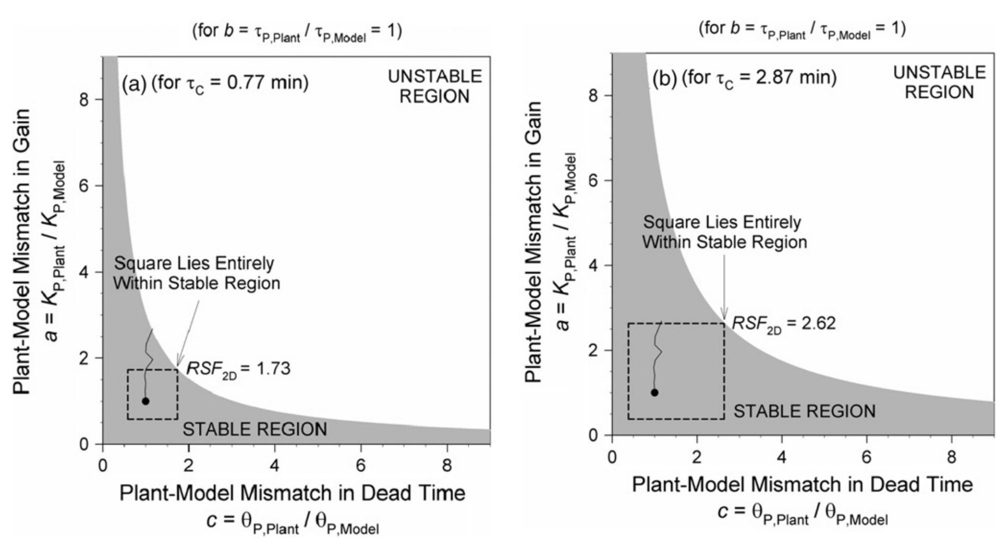

Fig. 2. Robustness plots show increase in robust stability with increasing τ_C (with plant-model mismatch due to nonlinearity shown by solid line from the dot at the design level (140 °C) to 160 °C).