A Stability Study on First Order Neutral Systems with Three Rationally Independent Time Delays

Rifat Sipahi, Nejat Olgac, Dimitri Breda

***Abstract*— First order linear time invariant and time delayed dynamics of *neutral type* is taken into account with *three* rationally independent delays. There are two main contributions of this study: (a) It is the first complete treatment in the literature, on the stability analysis of systems with three delays. We use a recent procedure, the Cluster Treatment of Characteristic Roots (CTCR), for this purpose. This procedure results in an exact and exhaustive stability tableau in the domain of the three delays. (b) It provides a proof of a complex concept called the *delay-stabilizability* (also known as *strong stability*) as a by-product of CTCR. Furthermore we deploy a numerical method (infinitesimal generator approach) to approximate the dominant characteristic roots of this class of systems, which concur with the stability outlook generated by CTCR.**

I. Introduction

In this paper we study a class of linear time invariant (LTI) systems with three delays. These systems are in "neutral" category of time delay systems, as

$$\begin{aligned}\sum:\ & \dot{x}(t) + a\ \dot{x}(t-\tau_1) + b\ \dot{x}(t-\tau_2) + c\ x(t-\tau_1) \\ & + d\ x(t-\tau_2) + f\ x(t-\tau_3) + g\,x(t) = 0\end{aligned} \tag{1}$$

where *a, b, c, d, f, g* are all real scalars as well as the dependent variable $x(t)$, $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3) \in \Re^{3+}$ is the delay vector with three rationally independent elements. Notice that the neutral segment of the paper has two delays while a third delay appears in the retarded segment. First two delays can be construed inherent to the dynamics and the third one is from the feedback control law. The objective is to assess the stability robustness of this system against time delay uncertainties of $\{\boldsymbol{\tau}\} \in \Re^{3+}$.

The characteristic equation of this dynamics is transcendental (i.e., infinite dimensional)

$$\begin{aligned}CE(s, \tau_1, \tau_2, \tau_3) = s\ (1 + a\ e^{-\tau_1 s} + b e^{-\tau_2 s}) + c\ e^{-\tau_1 s} \\ + d\ e^{-\tau_2 s} + f\ e^{-\tau_3 s} + g = 0\end{aligned} \tag{2}$$

The question reduces to finding (τ_1, τ_2, τ_3) regions for which the spectrum of (2), $\sigma(\boldsymbol{\Sigma})$, remain in $\mathbf{C}^-$, the open left half of the complex plane. As a notational selection we represent the right half open plane by $\mathbf{C}^+$ and the imaginary axis by $\mathbf{C}^0$ in the rest of the text. Thus the entire complex plane is $\mathbf{C} = \mathbf{C}^- \cup \mathbf{C}^0 \cup \mathbf{C}^+$.

There are three motivations in this study: (i) To deploy a paradigm called the Cluster Treatment of Characteristic Roots (CTCR) on a three-delay problem, first time in the literature. CTCR was earlier examined over retarded time delay systems (TDS) [8], neutral TDS with a single delay [12, 19] and with two delays [26]. The presence of multiple and rationally independent delays substantially complicates the analysis especially for neutral case. This transition from single delay to multiple delays is quite dramatic from complexity perspective. By taking a three-delay setting we wish to amplify the capabilities of the CTCR routine. (ii) To prove a well-known *necessary condition* for stabilizability of neutral class of systems, interestingly as a by-product of CTCR. (iii) To cross-validate the findings of CTCR using a new numerical procedure.

Neutral time delay systems exhibit dramatically different behavior compared with the retarded type dynamics [11,13-15,19]. According to these investigations, it emanates from the characteristics of a difference equation

$$x + a\,x(t-\tau_1) + b\,x(t-\tau_2) = 0 \tag{3}$$

or the relevant discrete kernel operator [12, 19] in the neutral part of (2)

$$L(\tau_1, \tau_2) = 1 + a\,e^{-\tau_1 s} + b\,e^{-\tau_2 s} = 0 \tag{4}$$

It is proven via cumbersome mathematics that there is a necessary condition for $\boldsymbol{\Sigma}$ to be robustly stable in some regions of $\{\boldsymbol{\tau}\} \in \Re^{3+}$: $L(\tau_1, \tau_2)$ has to be stable for small delays. Interestingly, it is independent of τ_3. This necessary condition is also known as the 'small delay phenomenon' and it should hold when τ_1 and τ_2 transit from 0 to 0^+ (but still remaining rationally independent). We denote this transition as $\tilde{\tau}_{1,2} : \mathbf{0} \to \mathbf{0}^+$ in the entire text. Those dynamics that possess stable discrete kernel operator are called τ **-stabilizable** [12, 19]. On the other hand, if this condition is not satisfied, it is guaranteed that there exists no region of rationally independently varying delays in $\{\boldsymbol{\tau}\} \in \Re^{3+}$ where the dynamics may exhibit stability even if the non-delayed ($\boldsymbol{\tau} = \mathbf{0}$) dynamics is asymptotically stable. For such a case, during $\tilde{\tau}_{1,2} : \mathbf{0} \to \mathbf{0}^+$ transition, root continuity argument

Rifat Sipahi conducted this research at the University of Connecticut, Mechanical Engineering Department, CT 06269-3139. Currently at Northeastern University, Mechanical and Industrial Engineering Department, 321 Snell Engineering Center, Boston, MA 02115 USA (email: r.sipahi@neu.edu)

Nejat Olgac is with the University of Connecticut, Mechanical Engineering Department, CT 06269-3139 USA (email: olgac@engr.uconn.edu)

Dimitri Breda is with the University of Udine, Mathematics and Computer Science Department, ITALY (email: dbreda@dimi.uniud.it)

collapses and infinitely many unbounded characteristic roots appear in $\mathbf{C}^+$. This precludes the return of stability for any region of rationally independent delays. This "small delay phenomenon" and conditions guaranteeing root continuity in the transition of $\tilde{\tau}_{1,2} : \mathbf{0} \rightarrow \mathbf{0}^+$ has been extensively studied in the literature [11-19]. Briefly, the neutral dynamics in (1) should first be τ - stabilizable, before any stability robustness study is performed. It is also shown in [11] that for $\mathbf{\Sigma}$ this τ - stabilizability condition, in fact, reduces to

$$|a| + |b| < 1 \qquad (5)$$

The mission in this paper, however, is considerably different: we disregard the τ - stabilizability and simply attempt to deploy CTCR procedure to create a stability robustness tableau against uncertain delays. In this process, we prove the τ - stabilizability necessary condition as a by-product.

CTCR declares the aimed "stability robustness map" exhaustively, which implies that all the stable regions including or excluding the origin (0, 0, 0) are detected. This is a complex task and only a small number of investigations can be found which achieve this even on a simpler dynamics where $a = b = f = 0$. In effect, these selections reduce the system into "retarded" TDS with two delays. For example, [1] addresses this question to determine only the stability regions, which include the origin. [2] proposes some interesting lemmas for the location of the characteristic roots when additional conditions are imposed: $g = 0$, $\tau_1 = 1$, $1 < \tau_2 < 2$ and the study indicates the overwhelming difficulties arise for cases with $\tau_2 > 2$. [5, 6] treat the stability question on second order systems, which are resonant when delays are zero. All of these simplifications impose some critical restrictions on the problem. In [3, 22] the treatment entails the new concept (CTCR) for a very general class of retarded TDS cases with no such simplifications. This paper expands it to neutral systems including a third rationally independent delay in the retarded segment.

In the following section we present a review of the enabling paradigm, CTCR. In Section III, a brief discussion on 'small delay phenomenon' and a proof of the necessary condition (5) is given as a by-product of CTCR. In Section IV a numerical approximation method is described for the dominant characteristic roots. In Section V we present an example case study. Section VI contains some observations and comparisons.

II. Review of Cluster Treatment of Characteristic Roots (CTCR) and a Valuable By-Product

The highlights of the CTCR framework are presented here, which are borrowed from [3-4, 22]. We wish to restate that in [3, 22], the investigations are all for *retarded* type multiple delay systems. [4,8,12,19], however, are on deployability of CTCR on single delay systems of both retarded and neutral class. The only existing literature on neutral systems with two delays is [26]. This paper presents the first effort in the literature, showing the tracktability of the stability regions for three-delay cases.

Let us present first, some important characteristics of $\mathbf{\Sigma}$:

(a) The system, $\mathbf{\Sigma}$, is infinite dimensional. It is asymptotically stable if its spectra $\sigma(\mathbf{\Sigma}) \in \mathbf{C}^-$. Therefore the topology of $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ corresponding to imaginary root crossing, $\sigma(\mathbf{\Sigma}) \cap \mathbf{C}^0 \neq 0$, is crucial. For these points in $\boldsymbol{\tau} \in \Re^{3+}$, we denote the correspondence between (τ_1, τ_2, τ_3) and the crossing frequency ω by a simple notation, $< \boldsymbol{\tau}, \omega >$. Notice that both $\boldsymbol{\tau} \in \Re^{3+}$ and $\omega \in \Re$ vary continuously in their respective spaces while maintaining $< \boldsymbol{\tau}, \omega >$ feature. Let us denote the complete topologies of such $\boldsymbol{\tau}$ and ω as

$$\mathbf{\Omega} = \left\{ \omega \,\middle|\, CE(s = \omega i, \boldsymbol{\tau}) = 0,\ \boldsymbol{\tau} \in \Re^{3+},\ \omega \in \Re \right\} \qquad (6)$$

$$\mathbf{S}_\Omega = \{ s = \omega\, i \mid \omega \in \mathbf{\Omega} \}$$

We assume, for now, that the set, $\mathbf{\Omega}$, is completely known. It can be shown that for each $< \boldsymbol{\tau}, \omega >$ correspondence, there exist infinitely many others, which share the identical ω. It is defined by

$$< \boldsymbol{\tau} + \boldsymbol{\alpha} \frac{2\pi}{\omega}, \omega > , \quad \boldsymbol{\alpha} = (j, k, \ell)^T, \ j, k, \ell = 0, 1, 2, \ldots (7)$$

Furthermore, due to the root continuity, small perturbations in $\boldsymbol{\tau}$ at each of these infinitely many sets of delays yield small perturbations on ω, i.e.,

$$< \boldsymbol{\tau} + \boldsymbol{\alpha} \frac{2\pi}{\omega} + \boldsymbol{\varepsilon}, \omega + \varepsilon_c > \qquad (8)$$

Clearly, the elements of $\boldsymbol{\varepsilon}$ and ε_c are interdependent. Further elaboration on this point is kept outside the document to better streamline the discussions. One can detect infinitely many hypersurfaces from (8) in $\boldsymbol{\tau}$ domain, which traverse through these infinitely many mesh points earmarked by ω (as it varies). According to the D-subdivision method [9], these hypersurfaces continuously partition the $\boldsymbol{\tau}$ domain into again infinitely many closed regions in which the **number of unstable roots**, $NU(\boldsymbol{\tau})$, remains fixed. Clearly, this is a very complex formation of geometry in the space of delays. To overcome this complication and to achieve a complete stability map of $\mathbf{\Sigma}$, it is imperative that we group the $< \boldsymbol{\tau}, \omega >$ correspondence into "clusters" showing identical "clustering features". This leads us to the "Cluster Treatment of Characteristic Roots (CTCR)" paradigm. The underlying strengths of CTCR reside in the two intriguing features of

LTI-TDS both of which were recognized first time in the literature in [8, 12, 22]. They are presented next very briefly.
(b) First clustering feature: It is proven in [3, 22] that there is a **manageably small number of hypersurfaces** in $\boldsymbol{\tau}$ space, we call them the "*kernel*", $\wp_0(\boldsymbol{\tau}) = \wp_0(\tau_1, \tau_2, \tau_3)$, which characterizes the complete set of $\boldsymbol{\Omega}$. That is,

$$< \boldsymbol{\tau}, \omega > \Big|_{\boldsymbol{\tau} \in \wp_0(\boldsymbol{\tau})} \leftrightarrow \text{complete set } \omega \in \boldsymbol{\Omega} \qquad (9)$$

Therefore, if they exist, the "kernel hypersufrace set" is unique. The $\boldsymbol{\tau}$ points on the kernel hypersurfaces satisfy the condition

$$0 < \tau_k < \frac{2\pi}{\omega}, \quad k = 1,2,3 \ , \ \omega \in \boldsymbol{\Omega} \qquad (10)$$

This constraint implies that the points on the kernel hypersurface exhibit the smallest positive delay value for the respective ω. Notice that there are ∞^3 (3-dimensional infinite) candidate points defined by (7) in $\Re^{3+}$ resulting in the same root, $s = \omega\, i \in \mathbf{S}_\Omega$. Assume, for the moment, that all such points on the kernel hypersurface, call them generically as "kernel points" $\boldsymbol{\tau}_{\text{ker}}$, are already obtained for all possible $\omega \in \boldsymbol{\Omega}$. The following defines the complete kernel hypersurface:

$$\wp_0(\boldsymbol{\tau}) = \{\boldsymbol{\tau} \,|\, \langle \boldsymbol{\tau}, \omega \rangle, \ \boldsymbol{\tau} \in \Re^{3+}, \ \omega \in \boldsymbol{\Omega}, \quad 0 \le \tau_k \le \frac{2\pi}{\omega}, \ k = 1, 2, 3\} \qquad (11)$$

For an exclusive stability analysis, this "*kernel hypersurface*" set must be determined *completely* and *exhaustively*. That is, any and every delay set, $\boldsymbol{\tau}$, which results in a root $s \in \mathbf{S}_\Omega$ must be included.

For each point $\boldsymbol{\tau}$ on $\wp_0(\boldsymbol{\tau})$, one can create the "offspring" points corresponding to $j > 0$, $k > 0$ and $\ell > 0$ in (7). Let us denote them by $\wp_{jk\ell}(\boldsymbol{\tau})$ where *j*, *k* and ℓ identify the j^{th}, k^{th} and ℓ^{th} generation *offspring* in τ_1, τ_2 and τ_3, respectively. The union of *kernel* and *offspring* begets $\wp(\boldsymbol{\tau})$:

$$\wp(\boldsymbol{\tau}) = \wp_0(\boldsymbol{\tau}) \cup \bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty}\bigcup_{\ell=1}^{\infty} \wp_{jk\ell}(\boldsymbol{\tau}) \qquad (12)$$

Notice that all the infinitely many trajectories of $\wp_{jk\ell}(\boldsymbol{\tau})$ are created by the *kernel*, $\wp_0(\boldsymbol{\tau})$, via a non-linear point-wise shifting operation (by $2\pi / \omega$), as per (7). Obviously, each kernel point reflects its root crossing frequency ω, *identically* onto its offspring ($j > 0$, $k > 0$ and $\ell > 0$). Thus, ω remains *invariant* from kernel hypersurfaces to offspring hypersurfaces.

The *kernel* and the *offspring* constitute the complete (and exhaustive) distribution of (τ_1, τ_2, τ_3) points where the $\sigma(\boldsymbol{\Sigma})$ has at least one imaginary member. Furthermore, there is no point outside $\wp(\boldsymbol{\tau})$ where $< \boldsymbol{\tau}, \omega >$ correspondence holds. These are the only locations in $\boldsymbol{\tau}$ space where the system (1) could transit from stable to unstable posture (or vice versa). These hypersurfaces $\wp(\boldsymbol{\tau})$ must now be determined exhaustively. Since $\wp(\boldsymbol{\tau})$ is completely generated from the *kernel* $\wp_0(\boldsymbol{\tau})$, it is sufficient to determine the *kernel* itself exhaustively. We present a procedure for doing this later in this section.

(c) Second clustering feature: Invariance of the directional root tendency: The root tendency along τ_j, $j = 1, 2, 3$ axis at the respective crossing of $s = \omega\, i$ is defined by $RT_{s=\omega i}^{\tau_j} = \text{sgn}\left[\text{Re}\left(\frac{\partial s}{\partial \tau_j} \right)\Big|_{s=\omega i} \right]$. The root tendencies at the points $s \in \mathbf{S}_\Omega$ across the corresponding equidistant points of $\boldsymbol{\tau}$ on a *kernel* and its *offspring* parallel to the τ_1 axis (without loss of generality), remain unchanged so long as τ_2 and τ_3 are kept fixed (i.e., *k* and ℓ are fixed in (7), while *j* varies). This is proven for general LTI-TDS in [3, 22]. It constitutes the second clustering feature along the hypersurfaces defined by (12).

Exhaustive and complete determination of the *Kernel*

For this task we adopt the Rekasius substitution, which is first introduced in [7] and also utilized in [3, 8, 22]. It suggests the following representation for exponential terms in (2)

$$e^{-\tau_j s} = \frac{1 - T_j s}{1 + T_j s} \ , \ T_j \in \Re \ , \ j = 1, 2, 3 \qquad (13)$$

This representation becomes exact for $s = \omega\, i$, provided that relation between T_j and τ_j

$$\tau_j = \frac{2}{\omega}[\tan^{-1}(\omega\, T_j) + k\pi] \ , \ k = 0, 1, 2, ... \qquad (14)$$

holds. It represents a holographic mapping between T_j and τ_j. A given ω and a T_j value correspond to infinitely many equidistant τ_j's (with interval of $2\pi / \omega$). Notice that the grid size $2\pi / \omega$ is identical for τ_1, τ_2 and τ_3 sets as long as they correspond to the same ω. In the other direction, however, the mapping is one-to-one, that is, each τ_j and ω pair maps into a single T_j value.

This holographic mapping between $\tau_j \leftrightarrow T_j$ transforms the transcendental $CE(s, \boldsymbol{\tau})$ into a new characteristic equation $CE_T(s, \mathbf{T})$, $\mathbf{T} = (T_1, T_2, T_3)^{\text{T}}$, which is fractional polynomial type. Multiplying this equation by $(1 + T_1 s)(1 + T_2 s)(1 + T_3 s)$, one obtains

$$\overline{CE}(s, \mathbf{T}) = \sum_{j=0}^{4} b_j(\mathbf{T})\, s^j = 0 \qquad (15)$$

See Appendix for explicit formation of this equation.

Notice that $b_j(\mathbf{T})$ are some multinomials in three parameters T_1, T_2, T_3, and $b_0 = c+d+f+g$. An interesting relation between the infinite dimensional equation (2) and the fourth degree equation (15) is that they share the *same imaginary spectra completely*. This is a tremendous reduction in complexity of determining the imaginary spectra of (2). That is,

$$\begin{aligned} &\mathbf{\Omega}[\,\omega \,|\, CE(\omega i,\, \boldsymbol{\tau}) = 0\ ,\ \boldsymbol{\tau} \in \Re^{3+}\ ,\ \omega \in \Re] \\ &\equiv\ \overline{\mathbf{\Omega}}[\,\omega \,|\, \overline{CE}(\omega i,\, \mathbf{T}) = 0\ ,\ \mathbf{T} \in \Re^{3}\ ,\ \omega \in \Re] \end{aligned} \tag{16}$$

where $\overline{\mathbf{\Omega}}$ represents the complete imaginary root topology of $\overline{CE}$ for $\mathbf{T} \in \Re^3$.

The most beneficial point in transforming $CE(s, \boldsymbol{\tau})$ to $\overline{CE}(s, \mathbf{T})$ is obvious that the parametric equation (15) is much easier to study compared with (2). And its imaginary roots can be determined completely for the entire space of $\mathbf{T} \in \Re^3$ as ($s = \omega i$, $\omega \in \overline{\mathbf{\Omega}}$). As per the claim in (16) this set of imaginary roots is identical to $\mathbf{S}_\Omega$. Based on these observations, a structured procedure is followed as described next: One should first find the projection of the *kernel* in $\mathbf{T}$ space, call it the *core hypersurface* just to discriminate its domain $\mathbf{T}$ as opposed to the *kernel* in $\boldsymbol{\tau}$. And then one must transform the *core hypersurfaces* to the *kernel* hypersurfaces via (14) by using $\mathbf{T}$ and ω values. Determination of the *offspring*, next, is straightforward starting from the knowledge of the *kernel*. *Kernel* and *offspring* constitute the complete set of hypersurfaces in $\{\boldsymbol{\tau}\}$ where the possible stability transition can occur. This complete hypersurface set represents a dissection of $\boldsymbol{\tau} \in \Re^{3+}$ into regions where *NU* remains fixed. The root tendency invariance feature, then, reveals the variations in *NU*, exhaustively declaring the stable regions with *NU* = 0. This step completes the CTCR procedure.

To determine the imaginary characteristic roots of (15) exhaustively for all $\mathbf{T} \in \Re^3$, we use the Routh's array and its salient features [10], Table I. The imaginary roots of the fourth degree equation (15) are found at:

$$R_1(\mathbf{T}) = 0 \tag{17}$$

with the condition $$\Theta(\mathbf{T})\ b_0 > 0 \tag{18}$$

At every point $\mathbf{T}$ satisfying (17) and (18), there exists a crossing frequency

$$\omega = \mp \sqrt{b_0 / \Theta} \in \Re \tag{19}$$

Therefore, expressions (17) and (18) represent the *core hypersurface* completely. $R_1(\mathbf{T})$ expression is not given here to conserve space. One can verify that the formation of $R_1(\mathbf{T})$ is a multinomial in each of T_1, T_2, T_3 of degree 4 in each of them. Therefore, for a given value of T_2 and T_3, there can be at most 4 real T_1 values (note that T_1 is taken as a generic selection here thus it can be interchanged with T_2 or T_3). That is, the core hypersurfaces have at most 4 separate segments. Since there is one-to-one mapping between the *core* in $\mathbf{T}$ to *kernel* in $\boldsymbol{\tau}$, this system can possess at most four separate contours for the *kernel* in $\boldsymbol{\tau}$. This reverse mapping from core in $\mathbf{T}$ to kernel in $\boldsymbol{\tau}$ is achieved using (14) and the offspring via (18). This operation results in the complete representation of *kernel* and *offspring hypersurfaces*, $\wp(\boldsymbol{\tau})$.

TABLE I.
THE ROUTH'S ARRAY FOR $\overline{CE}(s, T_1, T_2)$, ARGUMENTS SUPPRESSED.

s^4	b_4	b_2	b_0
s^3	b_3	b_1	
s^2	$\Theta = \dfrac{b_2\, b_3 - b_1\, b_4}{b_3}$	b_0	
s^1	$\dfrac{\Theta\, b_1 - b_3\, b_0}{\Theta} = R_1$		
s^0	b_0		

III. MAIN RESULT: THE SMALL DELAY TREATMENT AND τ-STABILIZABILITY CONCEPT

In this section, we state the τ-stabilizability lemma and show that it is a direct by-product of CTCR.

τ-stabilizability lemma [13-15]: The conditions for 'τ-stabilizability' (described in Section I) are dependent *only* on *a* and *b* parameters, and for this, the difference equation in (3) (or so-called discrete kernel operator [12, 19]) has to be stable [11] as $\tilde{\tau}_{1,2} : \mathbf{0} \to \mathbf{0}^+$.

Some remarks: It has been proven, also in [11], that τ-stabilizability requires |*a*| + |*b*| < 1 as given in (5). Earlier studies [11, 13-15] show several implications of this feature:
(i) τ-stabilizability feature is independent of delays.
(ii) Therefore, it is simply investigated at infinitesimal delays, as $\tau \to 0^+$.
(iii) τ-nonstabilizable systems exhibit discontinuous variation of spectra during $\tilde{\tau}_{1,2} : \mathbf{0} \to \mathbf{0}^+$.

All of these interesting properties are proven via complex mathematical procedures [13-15]. As the main result of this study, we demonstrate that the CTCR paradigm generates τ-stabilizability condition as a natural by-product.

The imaginary spectra of the following equations are identical: The discrete characteristic equation

$$CE_D(s, \tau_1, \tau_2) = 1 + a\, e^{-\tau_1 s} + b\, e^{-\tau_2 s} = 0 \tag{20}$$

and its Rekasius transformed version

$$\overline{CE}_D(s, T_1, T_2) = (1+T_1 s)\,(1+T_2 s)\, CE_D\big|_{e^{-\tau_j s} = \frac{1-T_j s}{1+T_j s}} = 0 \tag{21}$$

Expanding the terms, we arrive at

$$\overline{CE}_D(s, \mathbf{T}) = c_1 T_1 T_2 s^2 + (T_1 c_2 + T_2 c_3)\ s + c_4 = 0 \tag{22}$$

where $c_1 = 1 - a - b$, $c_2 = 1 - a + b$, $c_3 = 1 + a - b$, $c_4 = 1 + a + b$.

Proof of the Lemma as a by-product of CTCR:

It is achieved as one determines the kernel and offspring hypersurfaces for $\boldsymbol{\tau} \in \Re^{3+}$ starting from $\boldsymbol{\tau} = \mathbf{0}$.

(i) For $\boldsymbol{\tau} = \mathbf{0}$, equation (2) becomes

$$s\,(1+a+b)+c+d+f+g=0 \qquad (23)$$

(ii) As $\boldsymbol{\tau} \to \boldsymbol{\varepsilon}$ transition occurs, with $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)^T$, $0 < \varepsilon_j << 1$ for j = 1, 2, 3, we wish to observe no imaginary root crossing. Otherwise such a root can only be found at $\omega \to \infty$ (the Riemann sphere concept, [15]), with unbounded norm. Since the repeated crossings take place with $2\pi/\omega \to 0$ periodicity, according to (7), all the kernel and offspring points collapse onto each other. That results in infinitely many unbounded unstable roots to appear for $\boldsymbol{\tau} \to \boldsymbol{\varepsilon}$. Obviously, to regain stability from this situation as $\boldsymbol{\tau}$ increases within finite $\boldsymbol{\tau} \in \Re^{3+}$, one needs some finite $\boldsymbol{\tau}$ points where stabilizing ($RT = -1$) crossings arise again with $\omega \to \infty$. This condition, as it is shown in [15, 24, 25], may happen, for neutral systems, but only at instances where the delays are rationally dependent with each other. In conclusion, if small delays cause an imaginary crossing, it is guaranteed that there will be *no three dimensional regions* with finite and rationally independent delays where the neutral system can be robustly stable. This is equivalent to τ-stabilizability condition mentioned earlier.

We, therefore, pursue with CTCR, to examine the imaginary crossings for $\boldsymbol{\tau} \to \boldsymbol{\varepsilon}$. For this, the Rekasius substitution is used as in (15) noting that, from (14)

$$\tan(\tau_j\,\omega/2) = T_j\,\omega \quad , \quad j = 1, 2 \qquad (24)$$

and in the limiting case $\mathbf{T} \to \boldsymbol{\varepsilon}/2$ corresponds to $\boldsymbol{\tau} \to \boldsymbol{\varepsilon}$. That is, if there is an imaginary root crossing of $CE(s,\boldsymbol{\tau})$ for $\boldsymbol{\tau} : \mathbf{0} \to \boldsymbol{\varepsilon}$, it will correspond to the same imaginary root crossing of $\overline{CE}(s,\mathbf{T})$ for $\mathbf{T} : \mathbf{0} \to \boldsymbol{\varepsilon}/2$. To simplify the analysis, two independent variables are introduced to relate T_1 to T_2 and T_3.

$$T_j = m_j T_1 \,, \quad j = 2, 3 \quad \text{and} \quad m_j \in \overline{Q}^{+} \qquad (25)$$

where $\overline{Q}^{+}$ is the set of positive irrational numbers. When a similar Routh's array is formed as in Table I with term-wise limits for small $\mathbf{T}$ (i.e., for instance by eliminating T_1T_2 terms in favor of T_1), one arrives at Table II. The term-wise limit on the terms $b_j(\mathbf{T})$ of equation (15), when necessary, are denoted by $\bar{b}_j(T_1)$. In this table, the variables form as in the following $\eta = (c_2 + m_2 c_3)(m_2 m_3 c_3 + m_2 c_1 + m_3^2 c_4 + m_3 c_2)$, h_j, j=1,2,3 (linear combinations of c_j, j=1…4), $h_1 = c_1 m_2 m_3$, $h_2 = c_2 + m_2 c_3 + m_3 c_4$, $h_3 = m_2 c_1 + m_3 c_2 + m_2 m_3 c_3$.

Three remarks on this array: (i) Block I is nothing but the Routh's array of (23), i.e., the characteristic equation (1) for $\boldsymbol{\tau} = \mathbf{0}$. $b_0 c_4$ >0 (<0) implies stable (unstable) non-delayed system . (ii) Block II is independent of c , d , f and g. (iii) Any sign change in Block II means occurrence of root crossings (to $\mathbf{C}^{+}$) which cause infinitely many unbounded unstable spectra, as discussed earlier. Obviously this is not desired for any combination of $m_j \in \Re^{+}$, $j = 2, 3$.

TABLE II.
ROUTH'S ARRAY OF EQ (15) WITH TERM-WISE LIMIT TAKEN ON ITS ELEMENTS

s^4	$b_4(T_1) = T_1^3 h_1$	$\bar{b}_2(T_1) = T_1 h_2$	$b_0 = c+d+f+g$
s^3	$\bar{b}_3(T_1) = T_1^2 h_3$	$\bar{b}_1(T_1) = c_4$	
s^2	$T_1(h_2 - h_1 c_4/h_3) = T_1\eta/h_3$	b_0	
s^1	c_4		
s^0	b_0		

Block I: rows s^1–s^0 (first column); Block II: rows s^4–s^1 (first column).

Closer inspection reveals the fact that in order for Block II elements not to have a sign change all four terms of c_j , j = 1, …, 4 should either be positive or negative. Negativity causes a logic violation as $c_1 + c_2 = 2 > 0$. Thus they all have to be positive

$$c_j > 0 \;,\; j = 1, \ldots, 4 \qquad (26)$$

It is trivial to show that these inequalities can be written in a compact form as $|a| + |b| < 1$, same as in (5). This restriction displays the confinement given in Figure 1.

On the other hand, when similar steps are conducted on the discrete operator (20) instead of (15), for the transition of $\mathbf{T} : \mathbf{0} \to \boldsymbol{\varepsilon}/2$, the coefficients of the quadratic form in (22) have to be positive. A quick inspection reveals that, this is only possible when $c_j > 0$, j = 1, …, 4 is guaranteed. These conditions are identical to those of (26), which are the conditions for τ-stabilizability. In other words, τ-stabilizability property is guaranteed as long as the discrete operator is stable. ♦

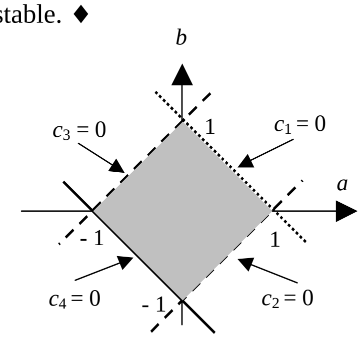

Figure 1. Depiction of conditions in (26) and τ- stabilizability region.

Notice that this result is obtained as a by-product of CTCR approach. Once τ-stabilizability property is assured, the CTCR continues further to determine the robust stability regions in $\boldsymbol{\tau}$ domain as explained in Section II. We will give example case studies in Section IV.

Observations:

- Using the steps of CTCR, we arrive at this very critical condition (5) via simple mathematical manipulations.
- As can be seen from Fig 1, τ - stabilizability is independent of $\boldsymbol{\tau}$ and it is only dictated by the parametric selection of (a, b) pairs.

IV. Case Studies

We present two example cases, which display all the features discussed in the text. The CTCR procedure is followed step-by-step to assist the reader.

Case 1: Take a = 0.2, b = 0.4, c = 2, d = 1, f = 6, g = 3 in (1). It is clear that this dynamics is τ - stabilizable since the condition in (5) is satisfied, i.e., $|a| + |b| < 1$. The characteristic equation is

$$CE(s,\tau_1,\tau_2,\tau_3) = s\,(1+0.2\,e^{-\tau_1 s}+0.4e^{-\tau_2 s}) + 2e^{-\tau_1 s} + e^{-\tau_2 s} + 6e^{-\tau_3 s} + 3 = 0 \qquad (36)$$

which has a single stable pole at s = – 7.5 for $\boldsymbol{\tau} = \mathbf{0}$. After Rekasius substitution it becomes

$$\begin{aligned}\overline{CE}\,(s, T_1, T_2, T_3) &= 0.4\,T_1\,T_2\,T_3\,s^4 \\ &(-6\,T_1\,T_2\,T_3 + 0.8\,T_2\,T_3 + 1.2\,T_1\,T_3 + 0.4\,T_1\,T_2)\,s^3 \\ &+ (6\,T_1\,T_2 - 2\,T_2\,T_3 + 0.8\,T_2 + 1.6T_3 - 4\,T_1 T_3 + 1.2T_1)\,s^2 \\ &+ (10\,T_2 + 1.6 + 8\,T_1)\,s + 12 = 0\end{aligned} \qquad (37)$$

We avoid the lengthy full-scale expressions to conserve space, and display only the small delay outlook of the array, i.e., for $\tilde{\mathbf{T}}_{1,2,3}$ (meaning all T_1, T_2 and T_3 are diminishing independently from each other). Again, by taking step wise limits and favoring the lowest degree terms in every step of building the Routh's array, (i.e. leaving only the lowest terms, and eliminating the others – as in Table II), we obtain Table III.

It is obvious that for all possible combinations of $0 < |T_1| < \varepsilon_1/2$, $\varepsilon_1 << 1$, $m_j \in \Re^+$, $j = 2, 3$, the first column elements display no sign change. This is a quick confirmation of the system being τ-stabilizable (as a by-product of CTCR deployment).

We then continue with the CTCR procedure, by generating the core hypersurfaces first, given by (17)-(18). The *kernel* and *offspring* hypersurfaces are then obtained using the mapping identity (14). This is a computationally intensive effort due to the 3-dimensional nature of the delay; however, CTCR still applies with its two fundamental propositions. Suppressing the numerical steps, we give the stability picture of the respective characteristic equation (36) with cross-sectional view of τ_3 = 0.5 in Figure 2, and τ_3 = 1 sec in Figure 3 (a straightforward CTCR implementation shows that the dynamics represented by the characteristic equation (36) is stable independent of any combination of τ_1 and τ_2 when τ_3 = 0, thus we suppress this plot). Notice the shaded stable regions and intriguing formation of some stability zones not including the origin. In fact these pictures can also be given in (τ_1, τ_2, τ_3) space, but for clarity we present the τ_3–constant cross-sections. It is important to point out that these stability displays of a three-delay dynamics are first in the literature.

TABLE III.
The limiting formation of Routh's array for Eq (37)

s^4	$0.4\,m_2\,m_3\,T_1^3$	$T_1(1.2 + 0.8\,m_2 + 1.6\,m_3)$	12
s^3	$\bar{b}_3(T_1) = T_1^2(0.4\,m_2 + 1.2\,m_3 + 0.8\,m_2\,m_3)$	1.6	
s^2	$T_1\,(1.92\,m_2 m_3 + 0.48\,m_2 + 0.32\,m_2^2 + 1.44 m_3 + 1.92 m_3^2 + 0.64 m_2^2 m_3 + 1.28 m_2 m_3^2)/(0.4\,m_2 + 1.2\,m_3 + 0.8\,m_2\,m_3)$	12	
s^1	1.6		
s^0	12		

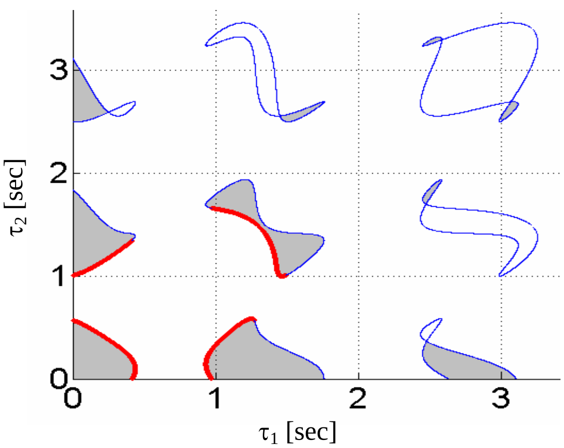

Figure 2. Case 1, stability regions (shaded) for τ_3 = 0.5 sec.

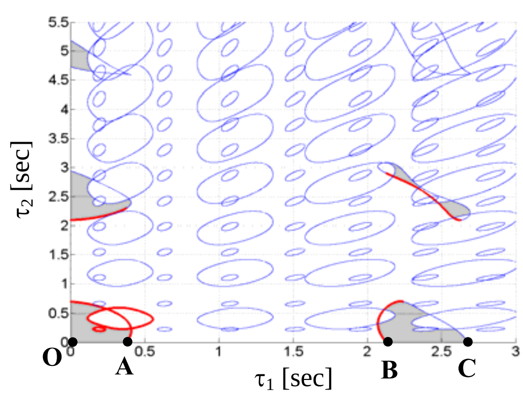

Figure 3. Case 1, stability regions (shaded) for τ_3 = 1 sec.

Furthermore, we include Figure 4, which represents a different view of the stability picture, this time the crosssection in (τ_1, τ_3) plane for τ_2 = 0. In order to establish the link between Figure 3 and 4, we label some stability switching points with O, A, B and C. It can be seen that the line segments $\overline{OA}$, $\overline{AB}$, $\overline{BC}$ are exactly the same on both figures.

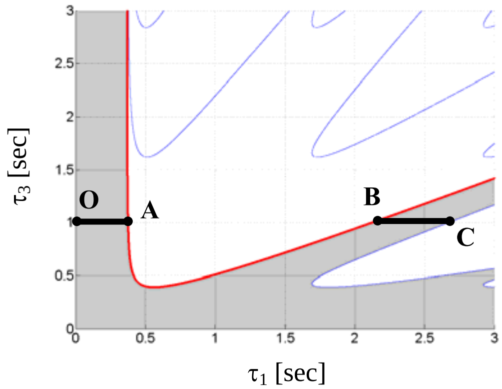

Figure 4. Case 1, stability regions (shaded) on (τ_1 ,τ_3) for τ_2 = 0.

The above presented outcome of CTCR is cross-validated using a numerical method, called pseudo-spectral differencing (PD), [20,21]. We provide some highlights of this methodology below, leaving the details to the cited references. (i) The method is a point-wise procedure for a given set of delays (τ_1, τ_2, τ_3), as it is the case for other comparable numerical studies [23, 27].

(ii) The dominant root is approximated using a discretization process, called infinitesimal generator.

Because of (i) this process needs to be executed over a carefully selected mesh to declare the stability outlook of the system, and this feature brings an inherent computational cost for similar dominant-root-finding procedures [23, 27]. Since CTCR declares implicitly the stability boundaries, especially with the help of Proposition I and II, the execution times and respective efficiencies are incomparably to the advantage of CTCR relevant to all of these numerical procedures.

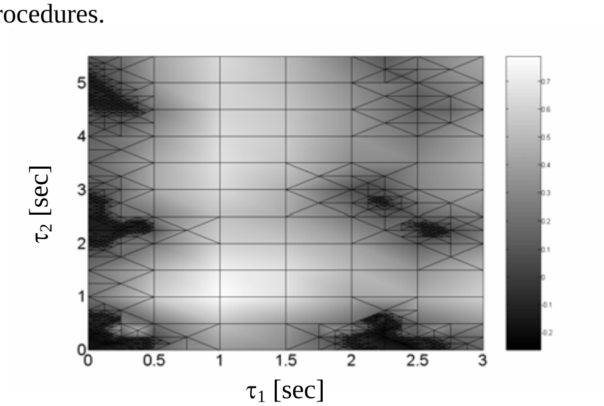

Figure 5. Case 1, distribution of real part of rightmost roots using pseudo-spectral differencing. for τ_3 = 1 sec.

Figure 5 shows the results of the PD/infinitesimal generator procedure. We determine the real part of the dominant roots of (36) and display it in gray scale for the entire region of $0 \le \tau_1 \le 3$, $0 \le \tau_2 \le 5.5$ when τ_3 = 1 sec. The stable zones are clearly visible (where real part is negative). A complete correspondence between Figures 3 and 5 is obvious.

Case 2: Take a = 0.7, b = 0.1, c = 2, d = 1, f = 4, g = 3 in (1). Notice that, dynamics in (1) is again τ - stabilizable for these numerical choices as it complies with $|a| + |b| < 1$. In the interest of space, we by-pass the intermediary steps of CTCR and present the results in Figure 6 and Figure 7. The corresponding right most root distribution of (1) obtained by PD/infinitesimal generator is presented in Figure 8 and Figure 9 in gray scale. For both cases, the stability boundaries are determined in full agreement with Figs. 6 and 7, respectively. Notice the varying mesh selections as it is optimized for a desired level of resolution. Some small nuances are missing in Figs. 8 and 9, due to this mesh selection. For instance, the unstable zone surrounding τ_1 = 0.9 and τ_2 = 0.5 is too small to detect (Fig. 7). Such small deviations, however, are routine for any numerical treatment.

Just to give a measure of the computational efficiency of CTCR, we state that computation time of each stability picture in Figure 6 and 7 is around 1.85 sec on a Pentium 4 CPU with 3 GHz processor speed and 2 GB RAM.

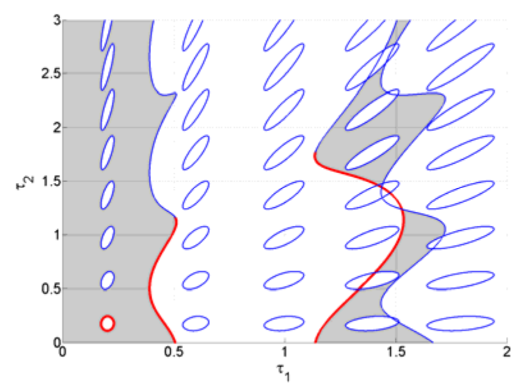

Figure 6. Case 2, stability regions (shaded) for τ_3 = 0.5 sec.

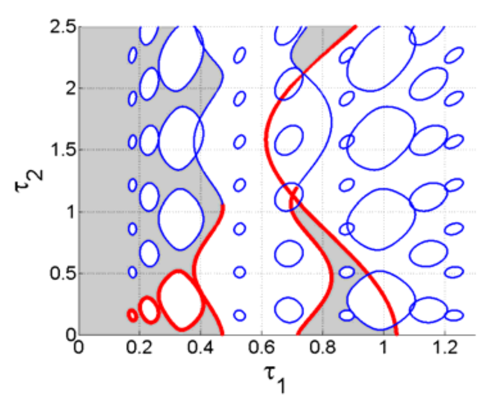

Figure 7. Case 2, stability regions (shaded) for τ_3 = 1.5 sec.

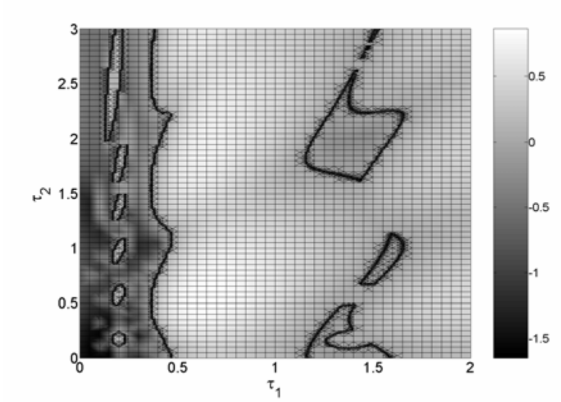

Figure 8. Case 2, distribution of real part of rightmost roots using infinitesimal generator for τ_3 = 0.5 sec.

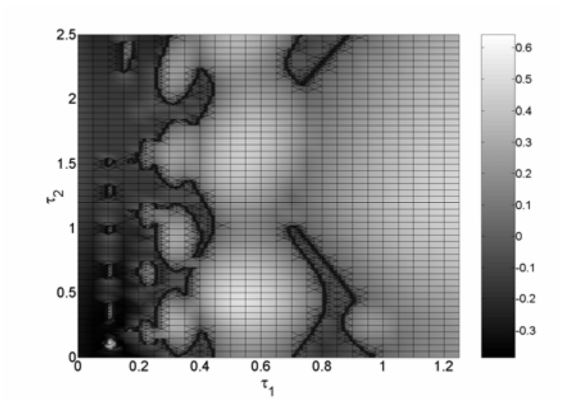

Figure 9. Case 2, distribution of real part of rightmost roots using infinitesimal generator for τ_3 = 1.5 sec.

V. Conclusion

A class of first order *neutral type* linear-time-invariant scalar dynamics with *three* independent time delays is studied, first time in the literature, for its robust stability against uncertain delays. We deploy a new framework, the Cluster Treatment of Characteristic Roots (CTCR) to handle the added complexities. Another contribution of the paper is the proof of a necessary condition, known as the τ-stabilizability, as a by-product of CTCR. CTCR offers a systematic and numerically very efficient procedure for the creation of complete stability maps of such systems, and its end results are cross-validated via a new numerical method over two case studies.

VII. Acknowledgement

This research has been supported in part by the awards from DoE (DE-FG02-04ER25656) and NSF (CMS-0439980) and (DMI-0522910).

Appendix

Take the most general form of equation (2) and deploy the Rekasius substitution to obtain $\overline{CE}(s,\mathbf{T})$ as in (15). To give a clearer picture to the reader, we present $b_j(\mathbf{T})$ multinomials below.

$$b_4 = T_1 T_2 T_3 (1 - a - b)$$

$$b_3 = (((-d - c + g - f)T_3 - a + 1 - b)\ T_2 + (1 - a + b)T_3)\,T_1 + (1 + a - b)T_3 T_2$$

$$b_2 = ((-d + f + g - c)T_2 + (g - c - f + d)T_3 - a + b + 1)T_1 + ((g - f - d + c)\ T_3 - b + a + 1)T_2 + (b + a + 1)T_3$$

$$b_1 = (-c + d + f + g)T_1 + (c - d + f + g)T_2 + (c + d - f + g)T_3 + 1 + a + b$$

$$b_0 = c + d + f + g$$

References

[1] J. K. Hale and W. Huang, "Global Geometry of the Stable Regions for Two Delay Differential Equations," *Journal of Mathematical Analysis and Applications*, vol. 178, pp. 344-362, 1993.

[2] R. Nussbaum, "Differential Delay Equations with Two Time Lags," *Memeoirs of American Mathematical Society*, vol. 205, Providence, 1978.

[3] R. Sipahi, N. Olgac, "A Unique Methodology for the Stability Robustness of Multiple Time Delay Systems," *Systems and Control Letters*, 55, 819-825, 2006.

[4] R. Sipahi and N. Olgac, "Degenerate Cases in Using the Direct Method", *ASME Journal of Dynamics Systems, Measurement & Control, Special Issue on Time Delayed Systems*, 125, 194-201, 2003.

[5] S.-I. Niculescu, "On Delay Robustness Analysis of a Simple Control Algorithm in High-Speed Networks," *Automatica*, vol. 38, pp. 885-889, 2002.

[6] G. Stepan, *Retarded Dynamical Systems: Stability and Characteristic Function*. New York: Longman Scientific & Technical, co-publisher John Wiley & Sons Inc., US., 1989.

[7] Z. V. Rekasius, "A Stability Test for Systems with Delays," Proc. Joint Automatic Control Conf., Paper No. TP9-A, 1980.

[8] N. Olgac and R. Sipahi, "An Exact Method for the Stability Analysis of Time Delayed LTI Systems," *IEEE Transactions on Automatic Control*, vol. 47, pp. 793-797, 2002.

[9] L. E. El'sgol'ts and S. B. Norkin, *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*. New York: Academic Press, 1973.

[10] B. C. Kuo, *Automatic Control Systems*: Prentice Hall, 1987.

[11] L. Cesari, J.K. Hale and J.P. LaSalle, *Dynamical Systems, An International Symposium, Volume 1*, New York: Academic Press, 1976.

[12] N. Olgac and R. Sipahi, "Direct Method for Analyzing the Stability of Neutral Type LTI-Time Delayed Systems", *Automatica*, 40(5), 847-853, 2004.

[13] Hale, J.K. & Verduyn Lunel, S.M., *An Introduction to Functional Differential Equations*. New York: Springer-Verlag, 1993.

[14] Hale, J.K. & Verduyn Lunel, S.M., "Effects of Small Delays on Stability and Control", *Operator Theory; Advances and Applications*, 122, 275-301, 2001.

[15] Hale, J.K. & Verduyn Lunel, S.M., "Strong Stabilization of Neutral Functional Differential Equations", *IMA J. Math. Control Inform.*, 19, 1-19, 2002.

[16] Chen, J., "On Computing the Maximal Delay Intervals for Stability of Linear Delay Systems", *IEEE Transactions on Automatic Controls*, 40(6), 1087-1092, 1995.

[17] Bellman, R. & Cooke, K., *Differential Difference Equations*. New York: Academic Press, 1963.

[18] Trefethen L., Spectral methods in MATLAB, SIAM, Software – Environment – Tools Series. Philadelphia, U.S.A., 2000.

[19] N. Olgac and R. Sipahi, "The Cluster Treatment of Characteristic Roots and the Neutral Type Time-Delayed Systems", *ASME Journal of Dynamic Systems, Measurement and Control*, 127, 88-97, 2005.

[20] D. Breda, S. Maset and R. Vermiglio, "Pseudospectral Differencing Methods for Characteristic Roots of Delay Differential Equations", *SIAM Journal of Scientific Computing*, 27(2), 482-495, 2006.

[21] D. Breda, S. Maset and R. Vermiglio, "Pseudospectral Approximation of Eigenvalues of Derivative Operators with Non-Local Boundary Conditions", *Special issue in Applied Numerical Mathematics,* 56(3-4), 318-331, 2005.

[22] R. Sipahi and N. Olgac, "Complete stability robustness of third order LTI multiple time-delay systems", *Automatica*, 41(9), 2005, 1413-1422.

[23] K. Engelborghs, T. Luzyanina and G. Samaey, "DDE-BIFTOOL v. 2.00: a Matlab package for bifurcation analysis of delay differential equations", Report TW330, Department of Computer science, K. U. Leuven, Belgium, 2001.

[24] C. Avellar, J. Hale, "On the Zeros of Exponential Polynomials", *Journal of Mathematical Analysis and Applications*, 73, 434-452, 1980.

[25] Michiels, W., Engelborghs, K., Roose, D. and Dochain, D., "Sensitivity to Infinitesimal Delays in Neutral Equations", *SIAM, J. Control and Optimization*, 40(4), 1134-1158, 2001.

[26] Sipahi R., Olgac, N., "Complete Stability Analysis of Neutral-Type First Order Two-Time-Delay Systems with Cross-Talking Delays", *SIAM Journal on Control and Optimization*, 45(3), 957-971, 2006.

[27] Vyhlídal, T., Zítek, P., "Quasipolynomial mapping based rootfinder for analysis of Time delay systems, " In Proc IFAC Workshop on Time-Delay Systems, TDS'03, Rocquencourt, France, 2003.