Kernel and Offspring Concepts for the Stability Robustness of Multiple Time Delayed Systems (MTDS)

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A novel treatment for the stability of linear time invariant (LTI) systems with rationally independent multiple time delays is presented in this paper. The independence of delays makes the problem much more challenging compared to systems with commensurate time delays (where the delays have rational relations). We uncover some wonderful features for such systems. For instance, all the imaginary characteristic roots of these systems can be found exhaustively along a set of surfaces in the domain of the delays. They are called the "kernel" surfaces (curves for two-delay cases), and it is proven that the number of the kernel surfaces is manageably small and bounded. All possible time delay combinations, which yield an imaginary characteristic root, lie either on this kernel or its infinitely many "offspring" surfaces. Another hidden feature is that the root tendencies along these surfaces exhibit an invariance property. From these outstanding characteristics an efficient, exact, and exhaustive methodology results for the stability assessment. As an added uniqueness of this method, the systems under consideration do not have to be stable for zero delays. Several example case studies are presented, which are prohibitively difficult, if not impossible to solve using any other peer methodology known to the authors. [DOI: 10.1115/1.2718235]

1 Introduction and the Problem Statement

A general class of linear time invariant multiple time delay system (LTI-MTDS) is taken into account:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \sum_{j=1}^{p} \mathbf{B}_j \mathbf{x}(t-\tau_j) \quad \text{with } \mathbf{x} \in \Re^{(n)}, \mathbf{A}, \mathbf{B}_j \in \Re^{(n\times n)} \tag{1}$$

$$\tau_j \geqslant 0 \quad j = 1, \ldots, p$$

This model can be construed as a representative dynamics of full state feedback control systems with multiple computational and actuation delays τ_j. The dynamics in Eq. (1) is also called "*retarded* time delay system," since the highest-order derivative terms are not affected by the delays. The characteristic equation of this LTI-MTDS is

$$\mathrm{CE}(s, \boldsymbol{\tau}) = \det\left(s\mathbf{I} - \mathbf{A} - \sum_{j=1}^{p} \mathbf{B}_j e^{-\tau_j s}\right) = 0 \tag{2}$$

The attribute "multiple time delay" is due to the delays, which are rationally unrelated to each other, such as

$$\frac{\tau_i}{\tau_j} \in \bar{Q} \quad 0 < i, \quad j \leqslant p \quad i \neq j \tag{3}$$

where $\bar{Q}$ is a set of irrational numbers. This condition enforces that τ_i and τ_j cannot be represented in (2) using a common divider. Thus the true complexity of p-delay dynamics is maintained.

It is clear that Eq. (2) contains terms in the form of $e^{-k\tau_j s}$, $j=1,\ldots,p$, $k=0,1,\ldots,n$ which represent k-toppled commensurate delay terms in τ_j; as well as $\exp(-\sum_{j=1}^{p} k_j \tau_j s)$ with $\sum_{j=1}^{p} k_j \leqslant n$, which display the *cross-talk* among the delay terms. These terms, in particular, add to the complexity of the equation considerably.

A common problem in the time delayed systems (TDS) literature is the stability assessment of the dynamics [1–6]. When there is only one single delay ($p=1$) the treatment, although quite complex, is manageable, and there is a large body of literature on the subject, see Refs. [1–10] as a small selection. Among the present methods, a recent development by the authors stands out with some unique features. It is called the Cluster Treatment of Characteristic Roots (CTCR) [7–10]. This methodology introduces a concept of "clustering" infinitely many characteristic roots of Eq. (2) into small (i.e., a manageable) number of groups with identical features. CTCR is based on two key propositions highlighting these clustering features. The new method is numerically very efficient. It results in an exact and exhaustive stability outlook of the dynamics.

With multiple delays, however, the assessment of general stability of Eq. (1) is an open problem in mathematics. The transition from single to multiple delays is nontrivial. Very limited current literature on MTDS points out that this stability problem becomes notoriously complex even for some simple forms of the dynamics [11–14]. In this paper we formulate a framework based on CTCR, which is applicable to the most general class of Eq. (1).

The imaginary axis, left and right half complex plane are denoted by $\mathbf{C}^0=\{s:\mathrm{Re}(s)=0\}$, $\mathbf{C}^-=\{s:\mathrm{Re}(s)<0\}$, $\mathbf{C}^+=\{s:\mathrm{Re}(s)>0\}$, respectively, where s is a complex number. By definition, the linear system in Eq. (1) is asymptotically stable if and only if all the infinitely many roots of the transcendental characteristic equation (2) are on left half complex plane, $s \in \mathbf{C}^-$. To explicitly determine the dominant root among the infinitely many, in terms of the delays (the only parameters in the system) is an impossible task [15]. Undoubtedly the multiplicity of the delays exacerbates this complexity further. It is clear from the literature that there exists no methodology today assessing the stability map of such complicated systems [11–14]. We consider the most general form of MTDS (1) in this work. Without loss of generality, however, we

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