An Improved Procedure in Detecting the Stability Robustness of Systems With Uncertain Delay

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Abstract—**A very efficient and reliable shortcut is explained here on the procedure, which resolves the stability robustness of LTI dynamics against an uncertain delay. This procedure reduces the numerical complexity of the solution by an order of magnitude. This note is in response to a number of user requests on the practicality of the mentioned methodology.**

Index Terms—**Clustering, linear time-invariant (LTI), robustness, robust stability, time delay.**

I. Discussions

This note is written with the intent of incorporating an important numerical improvement to an earlier contribution of the authors [1]. In that paper, we claimed to have introduced a systematic procedure to resolve precisely and exclusively the stability robustness tableau of the general LTI system with a single delay against uncertain delay. The process was named the "direct method (DM)" at the time, and appropriately renamed as the "cluster treatment of characteristic roots (CTCR)," after some practical application papers also appeared [2], [3]. In the DM procedure, we consider the linear time-invariant (LTI)-single delay systems in the form of

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-\tau), \quad \mathbf{x} \in \Re^n, \mathbf{A}, \mathbf{B} \in \Re^{n\times n}, \tau \in \Re^+ \tag{1}$$

where **A** and **B** are given matrices, τ is the uncertain but fixed delay. The characteristic equation of (1) is

$$\mathrm{CE}(s,\tau) = \det[s\,\mathbf{I} - \mathbf{A} - \mathbf{B}\,e^{-\tau s}] = 0. \tag{2}$$

The methodology follows the structured steps as given in [1, Sec. II]. We also state two enabling propositions with proofs (which are the key contributions of that paper).

Proposition I claims the "manageably smallness" of the number of possible imaginary roots of (2). And Proposition II claims the invariance of the direction of root crossings at these imaginary roots, for increasing τ values.

The method entails the following steps.

i) Determine all the possible imaginary roots of (2), say m such roots are found.
ii) Calculate the root tendencies at these m roots.
iii) Declare the complete stability tableau in $\tau \in \Re^+$.

Manuscript received July 12, 2005; revised February 6, 2006. Recommended by Associate Editor S. Tarbouriech. This work was supported in part by the U.S. Department of Energy under Grant DE-FG02-04ER25656, and in part by the National Science Foundation under Grants CMS-0439980 and DMI-0522910.

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Digital Object Identifier 10.1109/TAC.2006.878774

In step i), we suggested the use of the so-called Rekasius substitution

$$e^{-\tau s} = \frac{1 - T\,s}{1 + T\,s}, \qquad T \in \Re \tag{3}$$

and claimed that the resulting new equation is a simple polynomial in s which is parameterized in T [1, eq. (12)]

$$\sum_{k=0}^{2n} b_k(T)s^k = 0. \tag{4}$$

We further suggested that a Routh's array for (4) could generate a first column of which the number of sign changes is called NS. The variations of NS must be monitored for $-\infty < T < \infty$ for a crossing root, $s = \omega\, i$, to occur. We realized at the time, that this step (in fact, the first step of the methodology) was numerically very tedious and non-exclusive. Given the computational power at our disposal, however, we pushed forward and demonstrated by way of examples that it was successfully performed, only to invite numerous queries from the interested readers on this very point of practicability. We hereby rectify it proposing a numerically sound and much more efficient alternative procedure.

II. Highlight of this Note

The revised procedure over (4) still follows a Routh's array. We take advantage of the array's intriguing features, however, more effectively now. Notice that a proven and mathematically accepted rule of Routh's array is that: If there is a pair of imaginary roots of (4), the only term on the row corresponding to s^1, call it $R_1(T)$, must be zero ([5], p.543)

$$R_1(T) = 0 \tag{5}$$

This is a polynomial root finding problem, which is computationally simple and fast. These roots are evaluated exhaustively and precisely. Two important notes on that: a) One can show that degree$(R_1(T))$ is upperbounded (by a Fibonacci number as shown in [4]). b) We are interested in the real roots of (5) only, $T \in \Re$. The number of these real roots corresponds to the only possible stability switching points in τ.

Another useful property of the Routh's array is that the row corresponding to s^2 has two elements, $R_{21}(T)$ and $R_{22}(T)$. For a real root, say T_c, of (5) an auxiliary equation

$$R_{21}(T_c)\,s^2 + R_{22}(T_c) = 0 \tag{6}$$

forms a factor of the original CE in (2). In order for (6) to yield imaginary roots, $s = \mp\,\omega_c i$, the following additional condition has to be satisfied also [5, p. 543]:

$$R_{21}\;R_{22}\; > 0. \tag{7}$$

The corresponding root crossing is at

$$\omega_c = \sqrt{\frac{R_{22}(T_c)}{R_{21}(T_c)}}. \tag{8}$$

In summary, the first leg of the CTCR i) is accomplished by finding the discrete real solutions of (5) which satisfy (7), with the corresponding ω_c from (8). This process is the "lynch-pin" that removes the concern

IEEE Proof

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expressed by many colleagues over the process explained in [1]. We offer next an example deployment on the same problem given in the referenced paper

$$\begin{aligned}\mathrm{CE}(s,\tau) = s^3 + 6s^2 + 45.5s + 111 \\ + (0.9s^2 - 116.8s - 22.1)e^{-\tau s} \\ + (90.9s - 185.1)e^{-2\tau s} \\ + 119.4e^{-3\tau s} = 0. \end{aligned} \tag{9}$$

After Rekasius substitution, (9) becomes

$$\begin{aligned}\overline{\mathrm{CE}}(T,s) &= (1+Ts)^{\prime\prime}\mathrm{CE}(s,\tau)\big|_{e^{-\tau s}=\frac{1-Ts}{1+Ts}} \\ &= \sum_{k=0}^{2n} b_k(T)s^k = 0 \end{aligned} \tag{10}$$

Routh's array results in the $R_1(T)$ term as

$$\begin{aligned}R_1(T) = 0.400410^7T^9 - 541842.3962T^8 \\ - 0.106010^7T^7 - 78697.7127T^6 - 15015.6199T^5 \\ + 1216.0982T^4 + 401.1287T^3 \\ - 10.2507T^2 + 0.11978T - 0.112 = 0. \end{aligned} \tag{11}$$

Note that all numerical values in this section are truncated to conserve space. Only five of the nine roots of (11) are real, call them $T_{\mathrm{ck}}, k = 1,\ldots,5$

$$-0.4269, -0.1333, 0.0828, 0.0953, 0.6233.$$

We suppress the trivially calculated expressions of R_{21} and R_{22} here. Using (8), we evaluate the corresponding ω_{ck} values

$$15.5032, 0.8404, 3.0352, 2.9124, 2.1109.$$

The delays τ_{k0} are found as

$$0.222, 7.2105, 0.1623, 0.1859, 0.8725 \tag{12}$$

which are in full agreement with the earlier findings [1, Table I] with slight numerical errors. Needless to say, the set in (12) is more reliable. The CPU time is shortened remarkably. The algorithm in [1] takes 17 s on a PC with Pentium 4, 2.4-GHz processor and 512–MB RAM, while the shortcut presented here reduces it down to 2 s. Owing to this new procedure, the CTCR method can now be implemented automatically with no need of user intervention.

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